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Statistical properties of ideal photons in a two-dimensional dye-filled spherical cap cavity

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Within the framework of quantum statistical mechanics, we have proposed an exact analytical solution to the problem of Bose–Einstein condensation (BEC) of harmonically trapped two-dimensional (2D) ideal photons. We utilize this analytical solution to investigate the statistical properties of ideal photons in a 2D dye-filled spherical cap cavity. The results of numerical calculation of the analytical solution agree completely with the foregoing experimental results in the BEC of harmonically trapped 2D ideal photons. The analytical expressions of the critical temperature and the condensate fraction are derived in the thermodynamic limit. It is found that the 2D critical photon number is larger than the one-dimensional (1D) critical photon number by two orders of magnitude. The spectral radiance of a 2D spherical cap cavity has a sharp peak at the frequency of the cavity cutoff when the photon number exceeds the critical value determined by a temperature.

Keywords: rigorous results in statistical mechanics, Bose–Einstein condensation, 2D photons

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1. Introduction

Bose–Einstein condensation (BEC) is a common quantum property of many-particle systems in which the number of particles is conserved. In 1995, the three groups observed the BEC in several trapped gases of ultracold alkali atoms.[1–3] Recently there has been an intense interest of research into the BEC of photons in low-dimensional optical microcavities. Number-conserving thermalization has been observed experimentally[4] for a two-dimensional (2D) photon gas in a dye-filled optical microcavity, which acts as a 'white-wall' box. In the presence of thermalization processes that conserve photon number, Weitz and colleagues have observed the BEC of 2D photons in a dye-filled optical microcavity.[5,6] Kirton and Keeling have established a nonequilibrium model of photonic BEC in the dye-filled microcavity.[7]

Recent progresses in research into BEC of photons have been made both theoretically and experimentally. In theoretical sides, Fischer and Weill have presented a classical laser light condensation phenomenon.[8] Stoof and colleagues have proposed a Schwinger–Keldysh theory for BEC of photons in a dye-filled optical microcavity.[9] Stoof and colleagues have still investigated the effect of interactions on number fluctuations in a BEC of light.[10] Chiocchetta and Carusotto have developed a quantum Langevin model for nonequilibrium condensation of light.[11] Snoke and Girvin have indicated that a system of bosons interacting only with incoherent phonons can undergo Bose condensation.[12] Based on the hierarchical maximum entropy principle, Sob'yanin has presented a general theory of BEC of light in a dye-filled optical microcavity.[13] Kirton and Keeling have examined the mechanisms behind thermalization and BEC of photons in a dye-filled microcavity.[14] In experimental sides, Weitz and colleagues have observed grand-canonical number statistics in a photon BEC in a dye-filled microcavity.[15] Nyman and colleagues have observed that the system of photons in a pumped dye-filled microcavity can be tuned between BEC, multimode condensation, and lasing.[16] A BEC experiment with 2D ideal photons has revealed a cusp singularity of the specific heat.[17] Most recently, Stein and Pelster have investigated the thermodynamic properties of trapped ideal photons in dimensional crossover from 2D case to one-dimensional (1D) case.[18] Schmitt has reviewed the BEC of 2D photons in a dye-filled optical microcavity.[19] Weitz and colleagues have observed a non-Hermitian phase transition in an optical quantum gas.[20] Schmitt and colleagues have observed nonlinear response and Onsager regression in a photon Bose–Einstein condensate.[21] In this paper, we investigate the statistical properties of ideal photons in a 2D dye-filled spherical cap cavity. The statistical properties of the BEC state of an ideal 2D photons are expounded in the following.

The 2D systems of many particles have fascinated physicists for more than forty years now.[22] The physics in 2D systems is more interesting than that in 1D and 3D systems. The 2D character leads to incredibly rich physics. Some exact analytical solutions in 2D systems can be obtained using specific methods. The BEC in such effectively 2D systems demontrates new features: well below the transition temperature T_c the equilibrium state is a true condensate, whereas at intermediate temperatures $T < T_c$ one has a quasicondensate (condensate with fluctuating phase). The Kosterlitz–Thouless theory associates this phase transition with the emergence of a

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topological order. The Kosterlitz–Thouless transition has been observed in an ultracold Bose atomic gas in a 2D magneto-optical trap.[23]

For simplicity, we neglect interactions between the photons. Therefore, the object of study is an ideal photon gas moving in a 2D harmonic potential. The topic of BEC of a noninteracting boson gas in a 2D harmonic potential is treated in all the references in the following manners: (i) the whole study is carried out numerically by a computer, (ii) the thermodynamic limit is used, and (iii) the quasiclassical approximation that the discrete level structure is approximated by a continuous density of states is adopted. In the present paper, we derive an exact analytical solution of BEC of an ideal boson gas in a 2D harmonic potential. This solution is valid for arbitrary temperature and particle number. The solution is of great significance owing to the following reasons. Firstly, the noninteracting BEC in a 2D harmonic trap is a starting point for other various researches in the presence of interactions and disorders. Secondly, the form of this solution is so beautiful that the solution can be used for purposes of tuition. Thirdly, the method and concept used in this study can be applied in the whole field of statistical physics. We utilize this analytical solution to investigate the statistical properties of ideal photons in a 2D dye-filled spherical cap cavity. The results of numerical calculation of the analytical solution agree completely with the foregoing experimental results in the BEC of harmonically trapped 2D ideal photons. The exact analytical solution of BEC of an ideal boson gas in a 2D harmonic trap can be verified in present-day physics laboratories.

The remainder of this paper is organized as follows. Section 2 describes the characters of a 2D spherical cap cavity. In Section 3, we describe an microscopic theory of ideal photons in a 2D isotropic harmonic potential. In Section 4, we derive an exact analytical solution to the BEC of a finite number of ideal photons in a 2D isotropic harmonic potential. In Section 5, we derive an exact analytical solution to the BEC of 2D ideal photons in the thermodynamic limit. Section 6 describes the radiation properties of a 2D spherical cap cavity. The comprehensive discussion is given in Section 7.

2. Characters of a 2D spherical cap cavity

Now consider a 2D spherical cap cavity whose interior is filled by a dye solution. The dye solution consists of the dye molecules and the solvent (methanol) and acts as a heat bath that is kept at a constant temperature T. The resonator surface is made of two high-reflecting spherical cap mirrors. The two spherical cap mirrors are hollow and have the same size. The spherical caps have the radius R (say $R = 1$ m) and are symmetrical with respect to the origin. The z axis (cavity axis) passes through the centroids of two hollow spherical caps. The spherical cap cavity has the maximum mirror separation d_0 (say $d_0 = 1.46$ μm). Now we need to introduce a 2D position vector $\boldsymbol{r} = (x, y)$. The mirror separation at distance r from the cavity axis is given by

$$d(r) = d_0 - 2\left(R - \sqrt{R^2 - r^2}\right). \tag{1}$$

An energy spectrum of pump irradiation is restricted to values far above the thermal energy. Because of repeated absorption and re-emission of photons by the dye molecules, the photons thermalize to the temperature of the dye solution (room temperature).

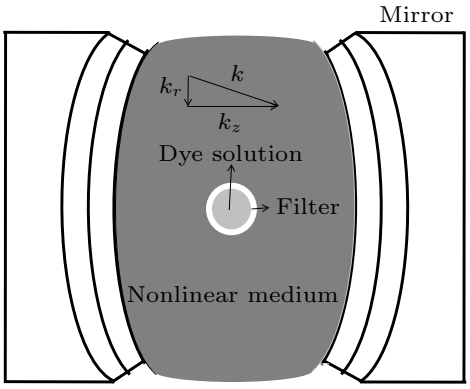


Fig. 1. A spherical cap cavity imposes boundary conditions which quantize the allowed values of k_z, where z is the axis normal to the mirrors. For a plane-wave mode which propagates at a small angle with respect to the z axis, there arises an effective nonrelativistic dispersion relation for a 2D photon, whose effective mass is $m_{\rm ph} = \hbar k_z(0)/v$.

As shown in Fig. 1, the wave vector $\boldsymbol{k}$ of photons in a spherical cap cavity can be resolved into the sum of transversal wave vector $\boldsymbol{k}_r$ and longitudinal wave vector $\boldsymbol{k}_z$: $\boldsymbol{k} = \boldsymbol{k}_r + \boldsymbol{k}_z$. It is useful to separate the fields into components parallel to and transverse to the z axis

$$\boldsymbol{E} = \boldsymbol{E}_z + \boldsymbol{E}_r, \quad \boldsymbol{B} = \boldsymbol{B}_z + \boldsymbol{B}_r. \tag{2}$$

It is assumed that the dye solution has a linear index of refraction n_0. $v = c/n_0$ is the speed of light in the medium and c is the speed of light in the vacuum. The frequency ω of electromagnetic field in the cavity depends on the transverse wave vector $\boldsymbol{k}_r$ through the relation $\omega = v\sqrt{\boldsymbol{k}_r^2 + \boldsymbol{k}_z^2}$. For a given k_r, only certain ω values are allowed in a typical resonant cavity situation. Since the boundary conditions on $\boldsymbol{E}$ and $\boldsymbol{B}$ are different, the eigenvalues will in general be different. The fields thus naturally divide themselves into three distinct categories: (i) transverse magnetic (TM) waves have $B_z = 0$ everywhere and the boundary condition is $E_x|_{\rm S} = E_y|_{\rm S} = 0$; (ii) transverse electric (TE) waves have $E_z = 0$ everywhere and the boundary condition is $B_z|_{\rm S} = 0$; (iii) transverse electromagnetic (TEM) waves have $E_z = B_z = 0$ everywhere and the boundary condition is $E_x|_{\rm S} = E_y|_{\rm S} = 0$. The TEM mode has $\boldsymbol{k}_r = 0$ and thus is the fundamental oscillation of the cavity.[24] If the spherical cap cavity has a mirror separation $d(r)$ at distance r from the

cavity axis, then the local resonance frequencies in the spherical cap cavity are given by

$$\omega_{\boldsymbol{k}_r q} = v\sqrt{\boldsymbol{k}_r^2 + \frac{q^2\pi^2}{d^2(r)}}, \tag{3}$$

where $q = 0, 1, 2, \ldots$.

At this moment, we want to derive the dispersion relation for the photon in a spherical cap cavity. Generally speaking, this relation can be written as

$$E(\boldsymbol{k}) = \hbar v\sqrt{\boldsymbol{k}_r^2 + \boldsymbol{k}_z^2}, \tag{4}$$

where $\hbar$ is Planck's constant reduced. This is a relativistic equation. If the maximum mirror separation d_0 is much smaller than the base radius r_0 of two spherical caps ($d_0 \ll r_0$), the transverse wave number k_r is much smaller than the longitudinal wave number k_z ($k_r \ll k_z$). This is the paraxial approximation in a slab-like cavity.[4,5] Under the paraxial approximation, we make a Maclaurin expansion of the relativistic relation and obtain a nonrelativistic dispersion relation for the 2D ideal photons inside the spherical cap cavity,

$$E(\boldsymbol{k}_r) = \hbar v\left(k_z + \frac{\boldsymbol{k}_r^2}{2k_z}\right), \tag{5}$$

where $k_z = q\pi/d$. Under the paraxial approximation of $r \ll R$, the substitution of Eq. (1) into $k_z(r) = q\pi/d(r)$ yields an expression of $k_z(r)$

$$k_z(r) = k_z(0) + k_z(0)\left(\frac{r}{\sqrt{d_0 R}}\right)^2, \tag{6}$$

where $k_z(0) = q\pi/d_0$. The substitution of Eq. (6) into Eq. (5) results in

$$E(\boldsymbol{k}_r) = m_{\rm ph}v^2 + \frac{(\hbar\boldsymbol{k}_r)^2}{2m_{\rm ph}} + \frac{1}{2}m_{\rm ph}\Omega^2 r^2. \tag{7}$$

It can be seen that the spherical cap cavity maps the three-dimensional (3D) photons onto the 2D photons. For paraxial propagation, the 2D photons are the nonrelativistic massive particles that possess an effective mass $m_{\rm ph} = \hbar k_z(0)/v = \hbar\omega_{\rm c}/v^2$, where $\omega_{\rm c} = k_z(0)v$ is the lower cutoff frequency of 2D photons. If the pump laser beam has a frequency ω_0, the excited longitudinal-mode number q of the spherical cap cavity is determined by the relation $\omega_0 = q\pi v/d_0$. Nonrelativistic massive photons move in the 2D transverse resonator plane under harmonic confinement with trapping frequency $\Omega = v/\sqrt{d_0 R/2}$.

3. Microscopic theory of ideal photons in a 2D isotropic harmonic potential

If there is an interaction between photons, the photon condensate of density n in the spherical cap cavity is identified by four length scales: the transverse radius R_r, the axial radius R_z, the scattering length a which denotes the strength of the two-body interaction, and the healing length $\xi = (4\pi na)^{-1/2}$. In the BEC of photons,[4–6] both the radii and lengths are fixed by the interaction between the photons, and thus, $R_r > \xi > R_z > a$. In this regime, the photon condensate is two-dimensional and is well described by the so-called Thomas–Fermi approximation.[25] New phenomena that have been observed in this regime are, for example, quasicondensates with a fluctuating phase and the Kosterlitz–Thouless phase transition.[23] A qualitatively different behavior of a photon condensate is expected when $R_z > \xi > R_r > a$ since then the condensate becomes restricted to one dimension. In this paper, we investigate a disk-shaped 2D condensate with $R_r > \xi > R_z > a$.

The present paper investigates a 2D gas of ideal photons with spin one ($a = 0$). As the healing length of a noninteracting gas diverges, the Thomas–Fermi approximation is invalid. $m_{\rm ph}$ signifies the mass of photons. The system under study consists of N ideal photons moving in a 2D isotropic harmonic potential. The 2D photons possess a transverse momentum $\boldsymbol{p}_r = \hbar\boldsymbol{k}_r$. In nonrelativistic quantum mechanics, the canonical momentum of 2D photons is given by the operator

$$\boldsymbol{p}_r = -{\rm i}\hbar\left(\boldsymbol{e}_x\frac{\partial}{\partial x} + \boldsymbol{e}_y\frac{\partial}{\partial y}\right), \tag{8}$$

where $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are the unit vectors along the x and y axes, respectively. The Hamiltonian of 2D photons under an isotropic harmonic potential is given by

$$H_{\rm ph} = -\frac{\hbar^2}{2m_{\rm ph}}\nabla_r^2 + \frac{1}{2}m_{\rm ph}\Omega^2 r^2, \tag{9}$$

where ∇_r^2 is the 2D Laplace operator defined by

$$\nabla_r^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

and Ω is the transverse trapping frequency.

The wave function $\Psi(\boldsymbol{r})$ of a photon at position $\boldsymbol{r}$ satisfies the stationary Schrödinger equation

$$\left(-\frac{\hbar^2}{2m_{\rm ph}}\nabla_r^2 + \frac{1}{2}m_{\rm ph}\Omega^2 r^2\right)\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r}), \tag{10}$$

where E is the energy eigenvalue of photons. The Hamiltonian given by Eq. (9) is that of a 2D isotropic harmonic oscillator. Because of the symmetry of the problem, we utilize the polar coordinate system in which the position vector $\boldsymbol{r}$ is given by $\boldsymbol{r} = (r, \varphi)$. The Schrödinger equation in polar coordinates is

$$\left[-\frac{\hbar^2}{2m_{\rm ph}}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{1}{2}m_{\rm ph}\Omega^2 r^2\right] \times \Psi(r,\varphi) = E\Psi(r,\varphi). \tag{11}$$

To separate the angular and radial parts of the wave function $\Psi(r, \varphi)$, we try writing

$$\Psi(r,\varphi) = {\rm e}^{{\rm i}m\varphi}R(r), \quad m = 0, \pm1, \pm2, \ldots. \tag{12}$$

Substituting Eq. (12) into the Schrödinger equation (11) yields a differential equation for the radial part of the wave function $R(r)$:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}-\frac{m^2}{\xi^2}+(2e-\xi^2)\right]R(\xi)=0, \tag{13}$$

where

$$k=\sqrt{\frac{m_{\mathrm{ph}}\Omega}{\hbar}},\quad \xi=kr,\quad e=\frac{E}{\hbar\Omega}.$$

When $\xi\to 0$, equation (13) becomes

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}-\frac{m^2}{\xi^2}\right)R(\xi)=0. \tag{14}$$

It can be verified that the solution finite for $\xi=0$ is

$$R(\xi)\propto\xi^{|m|}.$$

When $\xi\to\infty$, equation (13) becomes

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}-\xi^2\right)R(\xi)=0. \tag{15}$$

It can be verified that the solution finite as $\xi\to\infty$ is

$$R(\xi)\propto\exp(-\xi^2/2).$$

It is therefore natural to make in Eq. (13) the substitution

$$R(\xi)=\xi^{|m|}\exp(-\xi^2/2)u(\xi). \tag{16}$$

The insertion of Eq. (16) into Eq. (13) yields the differential equation for the function $u(\xi)$:

$$\begin{aligned}&\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}+\left(\frac{2|m|+1}{\xi}-2\xi\right)\frac{\mathrm{d}u}{\mathrm{d}\xi}\\&+[2e-2(|m|+1)]u(\xi)=0.\end{aligned} \tag{17}$$

If we introduce an independent variable $\rho=\xi^2$, then equation (17) reduces to

$$\rho\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2}+(|m|+1-\rho)\frac{\mathrm{d}u}{\mathrm{d}\rho}-\left(\frac{|m|+1}{2}-\frac{e}{2}\right)u(\rho)=0. \tag{18}$$

The last equation is just the differential equation satisfied by the confluent hypergeometric function $F(\alpha,\gamma;\rho)$. The corresponding parameters take the values:

$$\alpha=\frac{|m|+1}{2}-\frac{e}{2},\quad \gamma=|m|+1. \tag{19}$$

The series of a confluent hypergeometric function terminates if α is equal to a negative integer or to zero. Concomitantly, we demand that

$$\frac{|m|+1}{2}-\frac{e}{2}=-n_r,\quad n_r=0,1,2,\ldots. \tag{20}$$

As a result, the energy eigenvalue of a 2D isotropic harmonic oscillator is given by

$$E_n=\hbar\Omega(n+1),\quad n=2n_r+|m|=0,1,2,\ldots. \tag{21}$$

Correspondingly, the wave function of a 2D isotropic harmonic oscillator is determined by

$$\Psi_{n_rm}(\xi,\varphi)\propto \mathrm{e}^{\mathrm{i}m\varphi}\xi^{|m|}\exp(-\xi^2/2)F(-n_r,|m|+1;\xi^2). \tag{22}$$

It is easily shown that the degree of degeneracy of an energy level E_n is $f_n=n+1$.

4. Quantum statistical properties of a finite number of ideal photons

Because the number of ideal photons in a 2D isotropic harmonic potential is conserved, the 2D photon gas possesses a nonzero chemical potential μ. The projection of the spin operator along the momentum direction is called the helicity σ and spin-1 photons have $\sigma=\pm1$. At temperature T, $N_{n\sigma}$ represents the thermal population of the photon number occupying oscillator level n and helicity σ. In the conventional way, we can find that

$$N_{n\sigma}=\frac{1}{\mathrm{e}^{(E_n-\mu)/k_{\mathrm{B}}T}-1}, \tag{23}$$

where k_{B} is Boltzmann's constant. Equation (23) is the well-known Bose–Einstein distribution. The chemical potential μ is determined by the condition of conservation of the photon number N

$$\sum_{n\sigma}f_nN_{n\sigma}=N, \tag{24}$$

where $f_n=n+1$ is the degree of degeneracy for a 2D isotropic harmonic potential. The phenomenon of BEC for ideal photons is wholly depicted by Eqs. (23) and (24). The remaining task is to determine the chemical potential as a function of N and T.

For this goal, one first rewrites Eq. (21) as

$$E_n=n\hbar\Omega+\hbar\Omega. \tag{25}$$

By substituting the last equation into Eq. (23), we can rewrite Eq. (23) as follows:

$$N_{n\sigma}=\frac{z\mathrm{e}^{-\beta n\hbar\Omega}}{1-z\mathrm{e}^{-\beta n\hbar\Omega}}, \tag{26}$$

where $\beta=1/(k_{\mathrm{B}}T)$. The fugacity z can be given by the definition

$$z=\exp(\beta\mu^*), \tag{27}$$

where we have introduced an effective chemical potential $\mu^*=\mu-\hbar\Omega$. The ground state of the system corresponds to $n=0$. Because we have moved the zero-point energy into the effective chemical potential, the energy of the ground state has been taken to be zero. From Eq. (26), the number of photons

in the ground state is given by $N_0 = 2z/(1-z)$. The ground-state population diverges as $z \to 1$, and hence the BEC occurs at $z = 1$. For this reason, we call N_0 the photon number of condensate. When one splits the ground state population off from Eq. (24), one can rewrite Eq. (24) as

$$\frac{2z}{1-z} + \sum_{n\neq 0,\sigma} f_n N_{n\sigma} = N. \tag{28}$$

In the next step, we put Eq. (26) into Eq. (28) and complete the summation over n, then we obtain

$$\frac{2z}{1-z} + 2\sum_{j=1}^{\infty} \frac{z^j q^j}{(1-q^j)} + 2\sum_{j=1}^{\infty} \frac{z^j q^j}{(1-q^j)^2} = N, \tag{29}$$

where $q = \exp(-\beta\hbar\Omega)$. When deriving the third term of Eq. (29), we have employed the method developed in Ref. [26]. In the above summation, we use the degeneracy factor 2 to account for the two possible polarizations. In Eq. (29), the first infinite series converges and its analytical expression can be acquired as

$$F_q(x) = \sum_{j=1}^{\infty} \frac{z^j q^j}{(1-q^j)} = \frac{\ln(1-q) + \psi_q(x)}{\ln q}, \tag{30}$$

where we have ushered a new physical quantity x by the definition

$$x = 1 - \mu^*/(\hbar\Omega). \tag{31}$$

In Eq. (30), we have ushered a new function $\psi_q(x)$. $\psi_q(x)$ is called the q-digamma function, which is defined by

$$\psi_q(x) = \mathrm{d}[\ln\Gamma_q(x)]/\mathrm{d}x. \tag{32}$$

Here $\Gamma_q(x)$ is called the q-gamma function, which is defined by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}, \tag{33}$$

when $|q| < 1$ and $x \neq 0, -1, -2, \ldots$. At first Jackson introduced the q-gamma function[27] and then Krattenthaler and Srivastava introduced the q-digamma function.[28] In recent decades the q-gamma function and the q-polygamma function have paid a great role in science and technology.[29]

According to Eqs. (32) and (33), we can derive an expression of the q-digamma function

$$\psi_q(x) = -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{n+x}}{(1-q^{n+x})}. \tag{34}$$

In mathematics,[29,30] one can easily prove that equation (34) can be reduced into the form

$$\psi_q(x) = -\ln(1-q) + \ln q \sum_{j=1}^{\infty} \frac{q^{jx}}{(1-q^j)}. \tag{35}$$

By use of Eqs. (27), (31), and (35), we immediately find that equation (30) sets up. At this point, we explain in detail how the q-digamma function is introduced. In Eq. (30), the function $F_q(x)$ represents the photon number of noncondensate. As shown in Eq. (30), the function $F_q(x)$ consists of two parts $\ln(1-q)/\ln q$ and $\psi_q(x)/\ln q$. The first part $\ln(1-q)/\ln q$ denotes the noncondensed photon number excited by nonzero temperarure T. The second part $\psi_q(x)/\ln q$ stands for the noncondensed photon number excited by nonzero chemical potential μ^*. As a result, the physical implication of the q-digamma function is that it signifies the noncondensed photon number excited by nonzero chemical potential μ^*.

Since the second infinite series in Eq. (29) converges, we can derive its approximate analytical expression. At first we note that the second infinite series in Eq. (29) can be converted into a q-series:

$$G_q(x) = \sum_{j=1}^{\infty} \frac{q^{jx}}{(1-q^j)^2} = \sum_{k=0}^{\infty}\sum_{j=1}^{\infty} \frac{q^{j(x+k)}}{(1-q^j)}. \tag{36}$$

With the help of Eq. (30), the approximate analytical expression can be acquired as

$$G_q(x) = \frac{(k_c+1)\ln(1-q)}{\ln q} + \frac{1}{\ln q}\sum_{k=0}^{k_c} \psi_q(x+k), \tag{37}$$

where we replace the upper limit ∞ of summation with an upper cutoff k_c and in practice one can set $k_c = 200$. Through numerical calculation, it is found that the upper cutoff $k_c = 200$ is sufficient for a high precision calculation. The q-digamma function (in $F_q(x)$) and the q-digamma function series (in $G_q(x)$) are used to derive the properties of BEC of 2D ideal photons. In the above, we give the mathematical transformation process of $F_q(x)$ and $G_q(x)$.

Inasmuch as the temperature T appears in the definition $z = \exp(\beta\mu^*)$, the fugacity z does not replace the chemical potential μ^* independently and so z is not a good physical quantity. The quantity x can replace the chemical potential μ^* independently and so x is a good physical quantity. As a result, the quantity x is called the reduced chemical potential. In the same way, the quantity q can replace the temperature T independently and so q is a good physical quantity. Why is the reduced chemical potential x more suitable for describing the 2D photon BEC problem than the fugacity z? The answer is simple. If the fugacity z were used, the 2D photon BEC problem would not be solved analytically. In terms of the good physical quantities x and q, equation (29) is cast into an equation of state

$$\frac{2q^{x-1}}{1-q^{x-1}} + 2H_q(x) = N, \tag{38}$$

$$H_q(x) = F_q(x) + G_q(x), \tag{39}$$

where $2H_q(x)$ represents the photon number of noncondensate. From Eq. (38), we can determine the reduced chemical potential x numerically. x is a function of temperature T and photon number N. When x is acquired, the number of

photons in the ground state can be obtained from the relation $N_0 = 2q^{x-1}/(1-q^{x-1})$. To meet Eq. (38), one requires that $x \geq 1$. As $x = 1$, a 2D photon gas is in the state of BEC. The analytical function $H_q(x)$ carries all the information of BEC of a 2D photon gas.

In order to undertake a numerical calculation, we notice that the linear refractive index for methanol is $n_0 = 1.33$. From the formula $\Omega = v/\sqrt{d_0 R/2}$, the trapping frequency of the harmonic potential is obtained as $\Omega = 2.6382 \times 10^{11}$ s^{-1} or $\Omega/2\pi = 4.1988 \times 10^{10}$ Hz. The wavelength of the incident laser beam is $\lambda_0 = 532$ nm and its frequency is $\omega_0 = 2\pi c/\lambda_0$. From the relation $\omega_0 = q\pi v/d_0$, the excited longitudinal-mode number q of the spherical cap cavity is calculated as $q = 7$. With the relation $\omega_c = q\pi v/d_0$, the lower cutoff frequency of the spherical cap cavity is calculated as $\omega_c = 3.3952 \times 10^{15}$ s^{-1} or $\omega_c/(2\pi) = 5.4036 \times 10^{14}$ Hz. From the formula $m_{ph} = \hbar\omega_c/v^2$, the effective mass of 2D photons is computed as $m_{ph} = 7.0471 \times 10^{-36}$ kg. We define a 2D condensation temperature T_{2D} by $k_B T_{2D} = \hbar\Omega$. Thus we have $T_{2D} = 2.0151$ K. In the above statistical mechanics, we introduce the concept of temperature of a 2D photon gas. The temperature is meaningless when the number of photons is smaller than several tens, and it is meaningful when N is larger than dozens. According to Eq. (38), the variation of the reduced chemical potential x with the photon number N is given in Fig. 2 for various T. It is interesting to note that at $T = 2$ K, $x \approx 1$ for all N. Namely, when $\Omega/(2\pi) = 4.1988 \times 10^{10}$ Hz and $T = 2$ K, a 2D photon gas is always in the state of BEC no matter how large the photon number N is. Note that when $T \leq 300$ K and $N \geq 7 \times 10^4$, $x = 1$. Namely, when $T \leq 300$ K and $N \geq 7 \times 10^4$, a 2D photon gas is always in the state of BEC. Furthermore, figure 2 shows that when $N \to \infty$, $x = 1$ for all T. Namely, when $\Omega/(2\pi) = 4.1988 \times 10^{10}$ Hz and $N \to \infty$, a 2D photon gas is always in the state of BEC no matter how large the temperature T is. According to Eq. (38), the variation of the reduced chemical potential x with the temperature T is given in Fig. 3 for various N. It is interesting to note that at $N = 10^4$, $x = 1$ for $T \leq 100$ K. For a fixed N, x is a monotonically increasing function of temperature T.

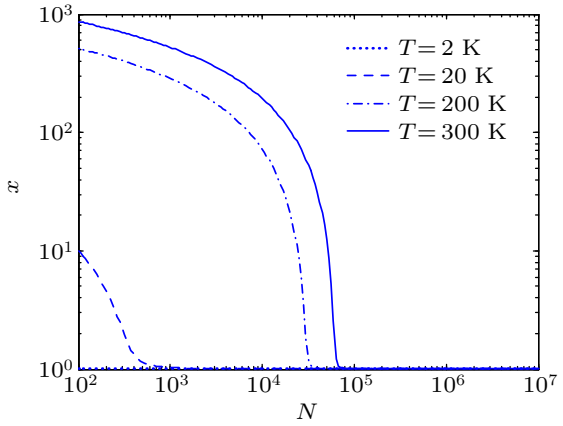


Fig. 2. The variation of the reduced chemical potential x with the photon number N in a 2D spherical cap cavity.

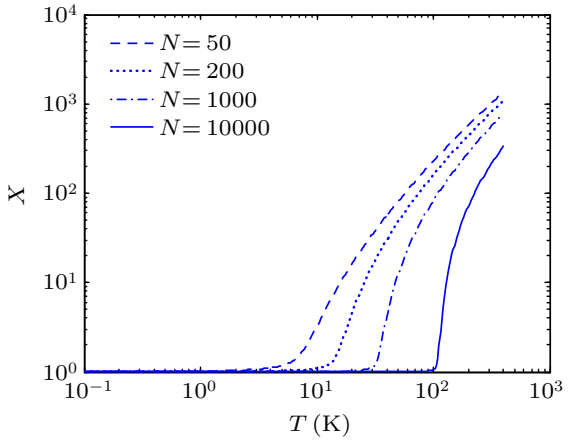


Fig. 3. The variation of the reduced chemical potential x with the temperature T in a 2D spherical cap cavity.

In the next step, we must calculate the condensate fraction N_0/N, which is given by

$$N_0/N = \frac{2q^{x-1}}{N(1-q^{x-1})}. \tag{40}$$

To garner the condensate fraction, one need to put the reduced chemical potential into Eq. (40). N_0/N is a function of temperature T and photon number N. As calculating N_0/N, one must combine Eq. (38) with Eq. (40). According to Eq. (40), the variation of the condensate fraction N_0/N with the temperature T is given in Fig. 4 for various N. Figure 4 shows that for a finite number of 2D photons, there is no exact transition temperature T_c. For a fixed N, N_0/N decreases smoothly to zero when T is near T_c. The phenomenon of phase transitions becomes clearer and clearer as the photon number becomes very large. Namely, when $N \geq 10^3$, there is an approximate transition temperature T_c. As $T \geq T_c$, $N_0/N = 0$. According to Eq. (40), the variation of the condensate fraction N_0/N with the photon number N is given in Fig. 5 for various T. It is interesting to note that at $T = 2$ K, $N_0/N \approx 1$ for all N. Figure 5 shows that for a fixed T, there is a critical number of photons N_c, above which a 2D photon gas is in the state of BEC and below which $N_0/N = 0$. A nice feature of the exact results in Eqs. (38) and (40) is that they are valid for arbitrary T and N.

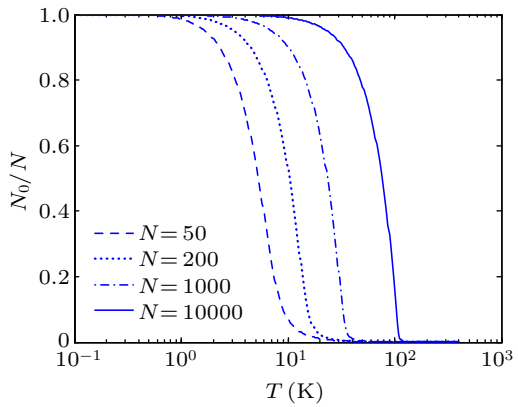


Fig. 4. In a 2D spherical cap cavity, the variation of the condensate fraction N_0/N with the temperature T.

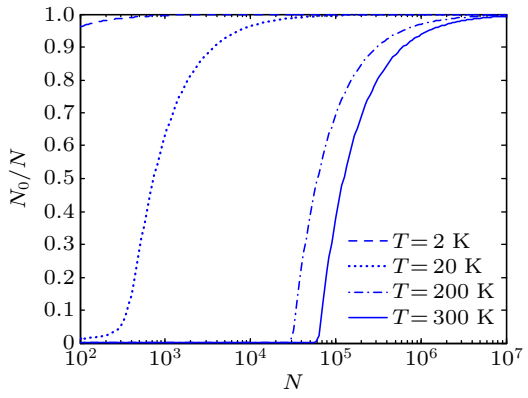


Fig. 5. In a 2D spherical cap cavity, the variation of the condensate fraction N_0/N with the photon number N.

5. Quantum statistical properties of ideal photons in the thermodynamic limit

The most interesting phenomenon of BEC of ideal photons happens in the thermodynamic limit when $N \to \infty$. For large N, the density of particle number should be kept constant, as achieved by decreasing the trap frequency Ω. Consequently, the thermodynamic limit in the 2D trap corresponds to taking $N \to \infty$ and $\Omega \to 0$ with the product $N\Omega^2$ kept constant.[25] Next, one can rewrite Eq. (38) in the form

$$N_0 + 2H_q(x) = N. \tag{41}$$

We have known that when $N \to \infty$, $x = 1$. The critical temperature T_c can now be acquired by setting $N_0 = 0$ and $x = 1$ in Eq. (41). This operation leads to the following equation for the critical temperature:

$$2H_{q_c}(1) = N, \tag{42}$$

where $q_c = \exp(-\hbar\Omega/(k_B T_c))$. Based on Eq. (39), we have found that $H_{q_c}(1)$ has the rigorous analytical expression

$$h(q_c) = H_{q_c}(1) = \frac{\ln(1-q_c) + \psi_{q_c}(1)}{\ln q_c} + \frac{\psi'_{q_c}(1)}{(\ln q_c)^2}, \tag{43}$$

where $\psi'_q(x) = \mathrm{d}[\psi_q(x)]/\mathrm{d}x$ is the q-trigamma function. For comparison, in the following we give the expression of a critical temperature in a 1D barrel cavity[31]

$$N = 2\frac{\ln(1-q_c) + \psi_{q_c}(1)}{\ln q_c}. \tag{44}$$

The 2D critical temperature is obviously smaller than the 1D critical temperature.

Equation (41) has an analytical solution in the limit as $N \to \infty$. We can obtain this solution as

$$x = \begin{cases} 1, & T \le T_c, \\ \text{the root of } 2H_q(x) = N, & T > T_c. \end{cases} \tag{45}$$

With the help of Eq. (42), from Eq. (41) one can find that the condensate fraction of photons is given by

$$\frac{N_0}{N} = \begin{cases} 1 - \frac{h(q)}{h(q_c)}, & T \le T_c, \\ 0, & T > T_c, \end{cases} \tag{46}$$

where $h(q)$ and $h(q_c)$ are given by Eq. (43) and $h(q)/h(q_c)$ represents the noncondensate fraction. The condensate fraction N_0/N given by Eq. (46) is a monotonically decreasing function of temperature T. $N_0/N = 1$ at $T = 0$ K and $N_0/N = 0$ at $T = T_c$.

Despite Eqs. (42) and (46) are achieved in the case of $N \to \infty$, they are efficacious for arbitrary T and large N ($N \ge 10^3$). As $10^2 \le N < 10^3$, equations (42) and (46) are applicable too. Actually, we first employ Eq. (42) to determine the transition temperature T_c and then utilize Eq. (46) to determine the condensate fraction N_0/N. Qualitatively, the phase transition of 2D photons can happen at an exact transition temperature T_c, clearly indicating that the behavior of 2D phase transitions is more vivid than that of 1D phase transitions. According to Eq. (42), the variation with N of 2D transition temperature T_c is shown in Fig. 6. According to Eq. (44), the variation with N of 1D transition temperature T_c is also shown in Fig. 6. The critical temperature T_c is a monotonically increasing function of photon number N. The 1D transition temperature is larger than the 2D transition temperature by an order of magnitude or more. Since the 2D transition temperature is smaller than the 1D transition temperature, the 2D Bose–Einstein condensate is stabler than the 1D Bose–Einstein condensate. Note that the scaling of T_c with N reported in the paper is a finite-size effect. According to Eq. (46), figure 7 shows the condensate fraction N_0/N versus the temperature T for $N = 10^2$, 10^3, 10^4, and 10^5. From Eq. (42), we find that at $N = 10^2$, 10^3, 10^4, and 10^5, $T_c = 10.09$ K, 33.40 K, 109.44 K, and 400.00 K, respectively. Figure 7 demonstrates a clear behavior of phase transitions of BEC in a 2D case. From Fig. 7, one sees that for a fixed $T < T_c$, the condensate fraction N_0/N is a monotonically increasing function of the photon number N. According to Eq. (45), the variation with temperature T of x is given in Fig. 8. For a fixed N and at $T > T_c$, the reduced chemical potential x increases steeply with temperature T. Many properties of BEC in harmonically trapped 2D photons are exhibited in the present paper.

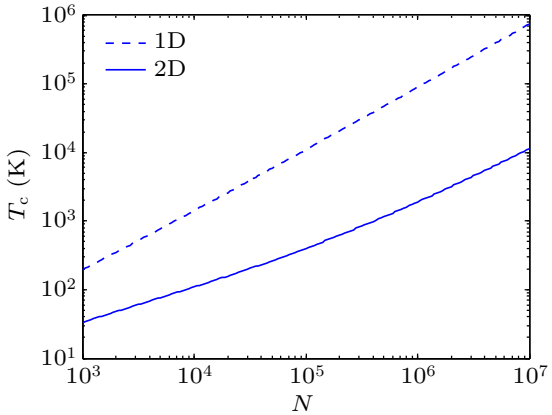


Fig. 6. In 1D and 2D harmonic potentials, the variation of the transition temperature T_c with the photon number N.

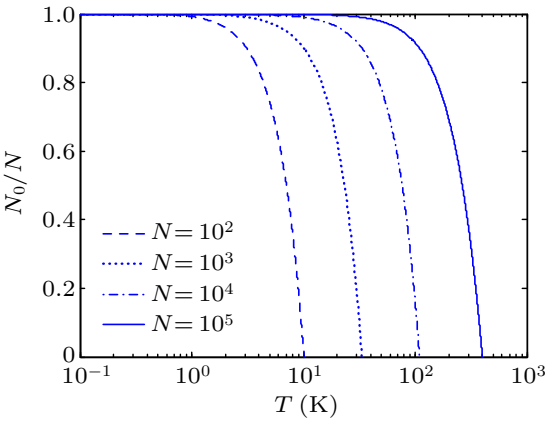


Fig. 7. The condensate fraction for the photon number N in a 2D spherical cap cavity versus the temperature. Plots are shown for $N = 10^2$, 10^3, 10^4, and 10^5.

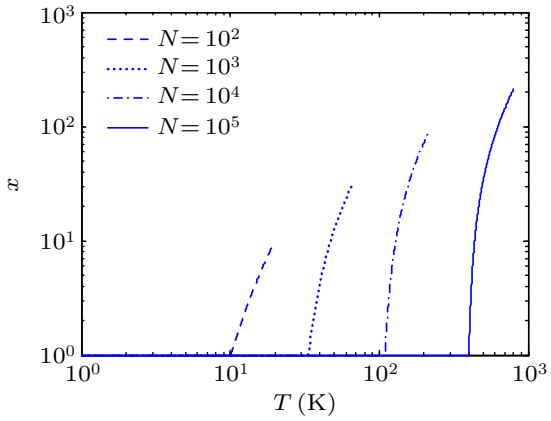


Fig. 8. The reduced chemical potential x for the photon number N in a 2D spherical cap cavity versus the temperature. Plots are shown for $N = 10^2$, 10^3, 10^4, and 10^5.

Inasmuch as the required experiments for this theory have been carried out at room temperature (say $T = 300$ K), the transition temperature T_c must be larger than the room temperature. Here one can set $T_c = 990.16$ K, which corresponds to the total photon number $N = 4 \times 10^5$. At $T = 300$ K and $N = 4 \times 10^5$, from Eq. (40) we find that the condensation fraction is $N_0/N = 0.845$. The temperature is meaningful only when $N > 40$. This requires that the power of pumping light should exceed a threshold. On the other hand, if the cavity losses are too high, we will not be able to assign a temperature. The photon system will be out of thermal equilibrium when the power of pumping light exceeds a saturation. In order for a photonic BEC to be observed, the power P of pumping light must be in the interval:[4,5] 1 W$\lesssim P \lesssim$ 25 W. At a fixed temperature T, there is a critical photon number N_c, which is the precise onset of BEC in this harmonically trapped 2D system. According to the above statistical mechanics, the critical photon number N_c is determined by

$$N_c = 2h(q), \tag{47}$$

where $h(q)$ is given by Eq. (43). At room temperature ($T =$ 300 K), the calculation of Eq. (47) gives $N_c = 62093$. For comparison, in the following we give the expression of a critical photon number in a 1D barrel cavity:[31]

$$N_c = 2\frac{\ln(1-q)+\psi_q(1)}{\ln q}. \tag{48}$$

At room temperature ($T = 300$ K), the calculation of Eq. (48) gives $N_c = 1662$. It is interesting to note that the critical number N_c of a 1D photon gas is smaller than that of a 2D photon gas by two orders of magnitude. In the seminal experiment,[5] the critical photon number at $T = 300$ K is $N_c = 77000$. This specific datum directly shows the consistency between our theory and the experiment. In Ref. [32], we have employed the exact analytical solution in the present paper to calculate the spatial density of ideal Bose atoms in a 2D isotropic harmonic trap. When $N < N_c$, the density profile exhibits a 2D dark spot in the center and a bright ring outside. When $N > N_c$, the density profile exhibits a 2D sharp peak, which is well described by a 2D Gaussian distribution. These results are consistent with Fig. 2(b) in Ref. [5]. In Ref. [33], we have employed the exact analytical solution in the present paper to calculate the thermodynamic properties of ideal Bose atoms in a 2D isotropic harmonic trap. Near the critical temperature, we find a cusp singularity of the specific heat. This demonstrates a second-order phase transition of BEC in a 2D case. This critical behavior is analogous to the λ-transition of liquid helium. These results are consistent with Fig. 3(b) in Ref. [17].

There is a critical optical power P_c, which is directly proportional to the critical photon number N_c. The experiment in a curved-mirror microresonator has demonstrated that the critical optical power P_c grows linearly with the mirror radius of curvature R and possesses the form

$$P_c = N_c\hbar\omega_c\Omega^2 R/(4v). \tag{49}$$

We assume that equation (49) holds also in a 1D barrel cavity. According to Eqs. (47) and (49), the variation with T of 2D critical optical power P_c is shown in Fig. 9. According to Eqs. (48) and (49), the variation with T of 1D critical optical power P_c is also shown in Fig. 9. The critical optical power P_c is a monotonically increasing function of temperature T. At room temperature ($T = 300$ K), the statement that the critical power P_c of a 1D photon gas is smaller than that of a 2D photon gas by two orders of magnitude is to be proven. When the power P of pumping light is larger than 25 W, a 2D photon gas will be in a normal state, in which the 2D photon gas possesses a nonzero chemical potential μ^* ($x > 1$). The chemical potential μ^* is determined by measuring the power P_{out} of light transmitted by an area of the cavity mirror, which corresponds to a photon number N in the cavity, and solving Eq. (38) for $x = 1 - \mu^*/(\hbar\Omega)$ and $q = \exp(-\hbar\Omega/(k_B T))$.

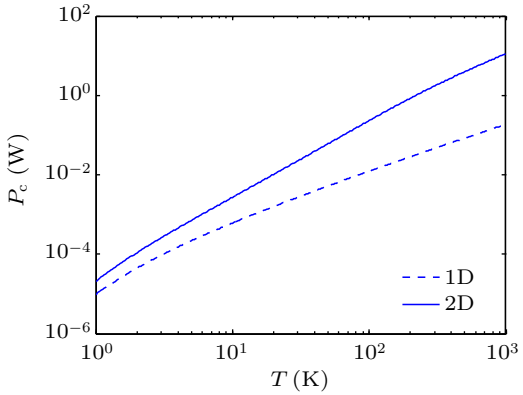


Fig. 9. In 1D and 2D harmonic potentials, the variation of the critical optical power P_c with the temperature T.

6. Radiation properties of a 2D spherical cap cavity

The phenomenon of BEC of ideal photons should influence the radiation properties of a 2D spherical cap cavity. In this section we depict the electromagnetic radiation emitted by a 2D spherical cap cavity in thermal equilibrium at a definite temperature. First of all, one should remember that the 2D spherical cap cavity is a quasi-2D system. The 2D spherical cap cavity has a specific spectral radiance that depends on the wavelength, the temperature and the photon number. In order to derive this spectral radiance, we rewrite the oscillator levels of photons as $E_n = \hbar\omega_{\boldsymbol{k}} - \hbar\omega_c$, where $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|$ is the frequency of a photon with 3D wave vector $\boldsymbol{k}$. If $N_{\boldsymbol{k}\sigma}$ signifies the thermal population of the photon number occupying wave vector $\boldsymbol{k}$ and helicity σ, from Eq. (23) we have its expression

$$N_{\boldsymbol{k}\sigma} = \frac{1}{\mathrm{e}^{(\hbar\omega_{\boldsymbol{k}} - \hbar\omega_c - \mu^*)/k_B T} - 1}. \tag{50}$$

The total energy E of photons in a 2D spherical cap cavity is the main thermodynamic quantity which we are interested in. In the conventional way, the total energy E of photons is given by

$$E = \sum_{\boldsymbol{k}\sigma} \hbar\omega_{\boldsymbol{k}} N_{\boldsymbol{k}\sigma}. \tag{51}$$

Because the temperature concerned is much larger than the level spacing, the wave vector $\boldsymbol{k}$ is quasi-continuous. In the usual way we can alter the summation to an integration in 3D momentum space. Since the trapping potential or the shape of the cavity is incorporated into the chemical potential μ^* in Eq. (50), the 2D spherical cap cavity can be regarded as uniform in 3D momentum space. Consequently, in the integral we can use the density of states $V/(2\pi)^3$ for a 3D box with volume V. When equation (50) is substituted into Eq. (51), one immediately achieves

$$E = \frac{\hbar c V}{\pi^2} \int_0^\infty \frac{k^3}{\mathrm{e}^{(\hbar c k - \hbar\omega_c - \mu^*)/k_B T} - 1} \mathrm{d}k. \tag{52}$$

Through the relation $k = 2\pi/\lambda$, the integration variable can be changed from k into λ, where λ is the wavelength of light. Consequently, we can simplify Eq. (52) as

$$E = V \int_0^\infty \rho(\lambda, T, N)\, \mathrm{d}\lambda\ , \tag{53}$$

where $\rho(\lambda, T, N)$ stands for the spectral energy density of a 2D spherical cap cavity and is given by

$$\rho(\lambda, T, N) = \frac{8\pi hc}{\lambda^5} \frac{1}{\mathrm{e}^{(hc/\lambda - hc/\lambda_c - \mu^*)/k_B T} - 1}, \tag{54}$$

where λ_c is the wavelength of the cavity cutoff defined by the relation $\omega_c = 2\pi c/\lambda_c$ and the computation gives $\lambda_c = 554.8$ nm. Equation (54) is called Planck's law.

Inasmuch as the radiation is isotropic and propagates at the speed of light c, the spectral radiance of light transmitted by an area of the cavity mirror is defined by

$$\begin{aligned} I(\lambda, T, N) &= \frac{\rho(\lambda, T, N) c}{4\pi} \\ &= \frac{2hc^2}{\lambda^5} \frac{1}{\mathrm{e}^{(hc/\lambda - hc/\lambda_c - \mu^*)/k_B T} - 1}\ , \end{aligned} \tag{55}$$

where equation (54) is used. Because the chemical potential μ^* is a function of temperature T and photon number N, the spectral radiance I is a function of wavelength λ, temperature T and photon number N. The spectral radiance of the cavity is measured in terms of the power emitted per unit area of the cavity mirror, per unit solid angle, per unit wavelength. The SI units of the spectral radiance I are $\mathrm{W{\cdot}sr^{-1}{\cdot}m^{-3}}$.

In the next step, we must calculate the the spectral radiance I, which is given by Eq. (55). The chemical potential μ^* is related to the reduced chemical potential x by the relation $\mu^* = \hbar\Omega(1-x)$. I is a function of wavelength λ, temperature T and photon number N. As we calculate I, we must combine Eq. (38) with Eq. (55). According to Eq. (55), the variation of the spectral radiance I with the wavelength λ is given in Fig. 10 for various N at $T = 300$ K. From Eq. (42), one can find that at $T = 300$ K, $N_c = 62093$. Figure 10 exhibits that for a fixed T and N, the spectral radiance I is a monotonically increasing function of the wavelength λ. From Fig. 10, one can see that the spectral radiance of a 2D spherical cap cavity has a sharp peak at the wavelength of the cavity cutoff when $N > N_c$. This is a signature of BEC of photons when $N > N_c$. According to Eq. (55), the variation of the spectral radiance I with the temperature T is given in Fig. 11 for various λ at $N = 8800$. From Eq. (42), one can find that at $N = 8800$, $T_c = 102.38$ K. When $\lambda = 524.8$ nm and 534.8 nm, the spectral radiance I is a monotonically increasing function of the temperature T. When $\lambda = 544.8$ nm, the spectral radiance I has a maximum at $T = 343$ K. From Fig. 11, one can see that the spectral radiance of a 2D spherical cap cavity has a sharp peak at the wavelength of the cavity cutoff when $T < T_c$. This is also a signature of BEC of photons when $T < T_c$. According to

Eq. (55), the variation of the spectral radiance I with the photon number N is given in Fig. 12 for various λ at $T = 300$ K. Figure 12 reveals that for a fixed T, the spectral radiance I is a slowly increasing function of the photon number N when $\lambda < \lambda_c$. Further, figure 12 shows that for a fixed T, the spectral radiance I has a sharp peak near $N_c = 74500$ when $\lambda = \lambda_c$. A nice feature of the exact results in Eq. (55) is that they are valid for arbitrary λ, T, and N. The present theory can address the anomaly of the spectral radiance at the cutoff frequency of a 2D spherical cap cavity. In fact, the anomaly of the spectral radiance at the cutoff frequency of a 2D spherical cap cavity has been observed and analyzed in Refs. [4,5]. As is well known, this anomaly is associated with the properties of resonance fluorescence near the cutoff (or a bandedge).[34] Figure 10 here is consistent with Fig. 2(a) in Ref. [5].

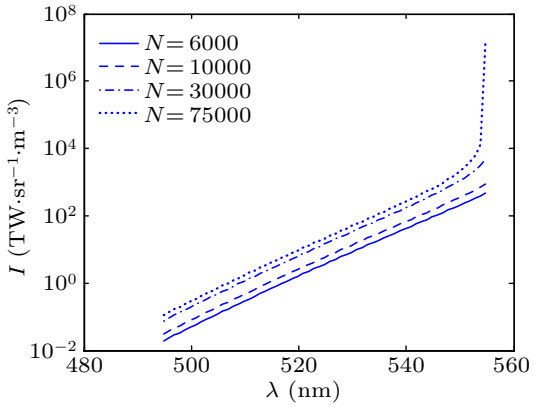


Fig. 10. The spectral radiance of N photons in a 2D spherical cap cavity versus the wavelength. Plots are shown for $N = 6000$, 10000, 30000, 75000, and $T = 300$ K.

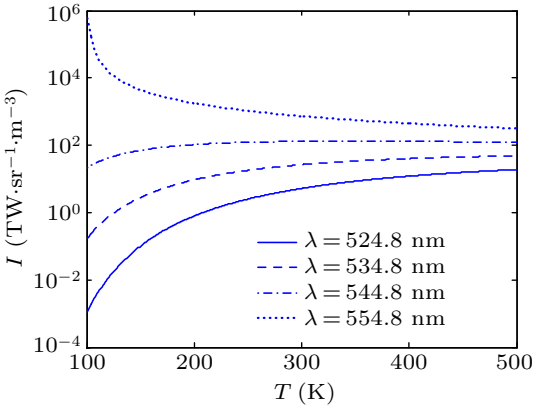


Fig. 11. The spectral radiance of N photons in a 2D spherical cap cavity versus the temperature. Plots are shown for $\lambda = 524.8$ nm, 534.8 nm, 544.8 nm, 554.8 nm, and $N = 8800$.

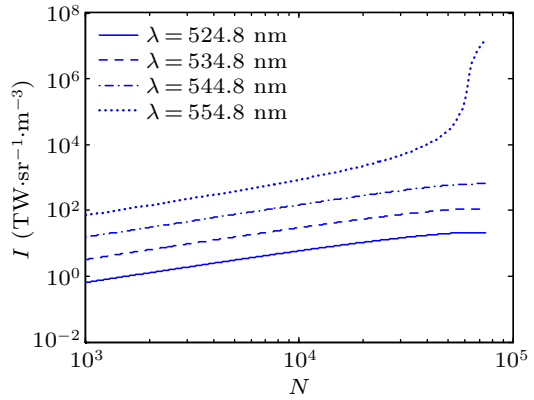


Fig. 12. The spectral radiance of N photons in a 2D spherical cap cavity versus the photon number. Plots are shown for $\lambda = 524.8$ nm, 534.8 nm, 544.8 nm, 554.8 nm, and $T = 300$ K.

7. Discussion

In this paper our first aim is to point out that most BEC properties of photons in a dye-filled optical microcavity[5,6] can be explained by a quantum statistical theory of ideal photons in a 2D harmonic potential. Our second aim is to point out that the problem of BEC of harmonically trapped 2D photons can be solved analytically. The analytical solution is associated with an analytical function, which involves a series of q-digamma functions. The q-digamma function was introduced in mathematics twenty years ago and now finds a variety of applications in science and technology. In this study we introduce a new physical quantity that is called the reduced chemical potential to replace the fugacity. In the problem of BEC of harmonically trapped 2D photons, the fugacity z is not a good physical quantity, but the reduced chemical potential x is a good physical quantity. Concretely speaking, the analytical solution of the problem of BEC of harmonically trapped 2D photons is associated with the following series:

$$G_q(x) = \sum_{k=0}^{\infty} F_q(x+k), \quad |q| < 1, \tag{56}$$

where $F_q(x)$ involves a q-digamma function and is defined by Eq. (30). We have found that the above series converges very quickly. Consequently, the upper limit ∞ of summation can be replaced by an upper cutoff k_c and in practice we set $k_c = 200$. The numerical calculation demonstrates that the upper cutoff $k_c = 200$ is sufficient for a high precision calculation. By contrast, the analytical solution of the problem of BEC of harmonically trapped 1D photons is associated with a single q-digamma function.[31] This contrast illustrates that the analytical solution of harmonically trapped 2D photons is more complicated than that of harmonically trapped 1D photons. Further, this contrast leads to the fact that the 2D critical temperature is smaller than the 1D critical temperature.

We employ the analytical solution of harmonically trapped 2D photons to investigate the BEC properties of ideal photons in a 2D harmonic potential. 2D photons in a spherical cap cavity possess an effective mass m_{ph} and a trapping frequency Ω. As a result, the BEC properties of a 2D photon gas are very similar to those of a 2D atom gas. Theoretical results obtained with this analytical solution agree completely with the experimental results observed in a dye-filled optical microcavity.[5,6] From the present research, we can draw the three main conclusions: (i) optical nonlinearity is secondary

for the BEC of harmonically trapped 2D photons; (ii) it is plausible that the critical power P_c of a 1D photon gas is smaller than that of a 2D photon gas by two orders of magnitude; and (iii) most BEC properties of photons in a dye-filled optical microcavity can be described by the analytical solution of ideal photons in a 2D harmonic potential. The off-diagonal long-range order in the one-particle density matrix in the BEC theory is associated with the macroscopic condensation of photons. However, we do not consider the interaction between atoms and cavity. In this aspect, non-Abelian Josephson effect between two $F = 2$ spinor Bose–Einstein condensates in double optical traps is investigated,[35] Josephson effect for photons in two weakly linked microcavities is predicted,[36] quantum magnetic dynamics of polarized light in arrays of microcavities is explored.[37]

There are two questions which should be addressed in the following. Firstly, we must distinguish between normal Bose–Einstein condensation (NBEC) and generalized Bose–Einstein condensation (GBEC).[38,39] One can classify the BEC in an ideal Bose gas into the three types. In type I there is a discrete number of quantum states each having macroscopic occupation; type II has condensation into a continuous band of states, with each state having macroscopic occupation; in type-III each state is microscopically occupied while the entire condensate band is macroscopically occupied. NBEC has a single condensed state and is a special case of type-I condensation. Either of type-II and type-III condensations is called a GBEC. The specific characteristic of NBEC is that it possesses a single critical temperature T_c. The photon BEC of the spherical cap cavity is an NBEC. The specific characteristic of GBEC of harmonically trapped bosons is that it possesses two critical temperatures T_m and T_c, where $T_m < T_c$. The photon BEC of the ellipsoidal cap cavity is a GBEC. Secondly, we must distinguish between a strictly 2D system and a quasi-2D system. The strictly 2D system has a single atomic layer while the quasi-2D system possesses the thickness of micron and nanometre order. The photon BEC in a quasi-2D system can reveal the quantum size effect. The strictly 2D photon system does not exist in the nature. The present paper is only relevant to quasi-2D photon systems, which are recently studied experimentally.

To sum up, we have proposed the BEC theory of a 2D photon gas in a spherical cap cavity filled with a dye solution. We have proposed an analytical solution to the problem of BEC in harmonically trapped 2D ideal photons. We utilize this analytical solution to investigate the statistical properties of ideal photons in a 2D dye-filled spherical cap cavity. The numerical calculation of the analytical solution gives many interesting results. The analytical expressions of the critical temperature and the condensate fraction are derived in the thermodynamic limit. We successfully address the anomaly of the spectral radiance at the frequency of the cavity cutoff.

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