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Effective regulation of the interaction process among three optical solitons

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The interaction between three optical solitons is a complex and valuable research direction, which is of practical application for promoting the development of optical communication and all-optical information processing technology. In this paper, we start from the study of the variable-coefficient coupled higher-order nonlinear Schrödinger equation (VCHNLSE), and obtain an analytical three-soliton solution of this equation. Based on the obtained solution, the interaction of the three optical solitons is explored when they are incident from different initial velocities and phases. When the higher-order dispersion and nonlinear functions are sinusoidal, hyperbolic secant, and hyperbolic tangent functions, the transmission properties of three optical solitons before and after interactions are discussed. Besides, this paper achieves effective regulation of amplitude and velocity of optical solitons as well as of the local state of interaction process, and interaction-free transmission of the three optical solitons is obtained with a small spacing. The relevant conclusions of the paper are of great significance in promoting the development of high-speed and large-capacity optical communication, optical signal processing, and optical computing.

Keywords: optical solitons, solitons interactions, nonlinear Schrödinger equation, higher-order dispersion and nonlinear effects

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1. Introduction

Optical soliton is a special kind of light wave envelope with significant scientific implications and potential applications in optical communication, information processing, and precision measurement.[1–4] The advancement of nonlinear optics has been significantly aided by the study of optical solitons, especially in terms of understanding their propagation properties in optical fibers, which provides a rich research content and experimental phenomena for nonlinear optics.[5–8] In those research directions, by investigating the interactions between optical solitons, the development of some optical devices is promoted, such as optical switches, optical couplers, and high efficiency optical frequency converter, which have potential applications in optical information processing and optical computing.[9–12] In addition, the interactions between optical solitons, such as attraction and repulsion, are similar to the interactions between particles, which provide new perspectives and experimental platforms for the study of fundamental physical phenomena.[13–16]

During the study of optical soliton interactions, the complexity of soliton dynamics, including the phenomena of soliton fusion, splitting, and energy exchange, can be understood more deeply through examining the three optical solitons' interaction.[17–20] The investigation of three optical solitons' interaction contributes to optimize the design of optical communication systems and improve transmission capacity and efficiency, especially in multi-channel parallel transmission and encoding and decoding of optical soliton signals.[21–26] In addition, under certain conditions, the study of three-soliton interactions can maintain good stable propagation characteristics, which is of great significance in guiding the formation of three-soliton arrays and array wave-guides.[27–29]

Based on variable-coefficient coupled higher-order nonlinear Schrödinger equation (VCHNLSE), the interaction characteristics of three optical solitons are examined in this work. The specific forms of the equation are shown below:[30]

$$\begin{aligned}
&\mathrm{i}q_{1,t}+q_{1,xx}+2\sigma(|q_1|^2+2|q_2|^2)q_1\\
&+2\sigma q_1^*q_2^2+\mathrm{i}\varepsilon(t)q_{1,xxx}\\
&+6\mathrm{i}\varepsilon(t)\sigma(|q_1|^2+|q_2|^2)q_{1,x}+6\mathrm{i}\varepsilon(t)\sigma(q_1q_2^*+q_1^*q_2)q_{2,x}=0,\\
&\mathrm{i}q_{2,t}+q_{2,xx}+2\sigma(2|q_1|^2+|q_2|^2)q_2\\
&+2\sigma q_2^*q_1^2+\mathrm{i}\varepsilon(t)q_{2,xxx}+6\mathrm{i}\varepsilon(t)\sigma(|q_1|^2+|q_2|^2)q_{2,x}\\
&+6\mathrm{i}\varepsilon(t)\sigma(q_1q_2^*+q_1^*q_2)q_{1,x}=0.
\end{aligned}\tag{1}$$

In Ref. [30], the studies on Eq. (1) have mainly focused on the cases of one-soliton transport and two-soliton interactions, but no research has been done on the interactions of three optical solitons. And the theory of nonlinear optics can benefit from the investigation of three optical solitons's interactions, especially in understanding the propagation properties of optical

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solitons in optical fibers and nonlinear dynamical behavior of optical solitons. This paper aims to examine three optical solitons's interaction and discuss their transmission properties before and after interactions. By effectively regulating the interactions of the three solitons, it will provide theoretical support and experimental guidance for the development of all-optical communication technology.

The structure of this paper is as follows. In Section 2, the three-soliton solution will be given. In Section 3, the effective regulation of the interaction process among three optical solitons will be demonstrated. In Section 4, this paper will be summarized.

2. Three-soliton solutions

According to Ref. [30], we rewrite the Hirota bilinear forms of Eq. (1) as

$$\begin{aligned}
&\left[\mathrm{i}\mathrm{D}_t+\mathrm{D}_x^2+\mathrm{i}\varepsilon(t)\mathrm{D}_x^3\right]G\cdot F=\lambda_1 GF,\\
&\left[\mathrm{i}\mathrm{D}_t+\mathrm{D}_x^2+\mathrm{i}\varepsilon(t)\mathrm{D}_x^3\right]H\cdot F=\lambda_2 HF,\\
&\mathrm{D}_x^2F\cdot F=2\sigma\left(|G|^2+|H|^2\right),\\
&-2\sigma(GH^*+G^*H)\left[3\mathrm{i}\varepsilon(t)\mathrm{D}_xH\cdot F+HF\right]=\lambda_1 GF^3,\\
&-2\sigma(GH^*+G^*H)\left[3\mathrm{i}\varepsilon(t)\mathrm{D}_xG\cdot F+GF\right]=\lambda_2 HF^3, \quad (2)
\end{aligned}$$

with $\lambda_1=\lambda_2=1$. In order to obtain the three-soliton solutions, we can expand G, H, and F as $G=\eta g_1+\eta^3 g_3+\eta^5 g_5$, $H=\eta h_1+\eta^3 h_3+\eta^5 h_5$, and $F=1+\eta^2 f_2+\eta^4 f_4+\eta^6 f_6$, and set

$$\begin{aligned}
g_1 &= \alpha_1\mathrm{e}^{\theta_1}+\alpha_2\mathrm{e}^{\theta_2}+\alpha_3\mathrm{e}^{\theta_3},\\
g_3 &= m_1(t)\mathrm{e}^{\theta_1+\theta_2+\theta_1^*}+m_2(t)\mathrm{e}^{\theta_1+\theta_2+\theta_2^*}\\
&\quad+m_3(t)\mathrm{e}^{\theta_1+\theta_2+\theta_3^*}+m_4(t)\mathrm{e}^{\theta_1+\theta_3+\theta_1^*}\\
&\quad+m_5(t)\mathrm{e}^{\theta_1+\theta_3+\theta_2^*},+m_6(t)\mathrm{e}^{\theta_1+\theta_3+\theta_3^*}\\
&\quad+m_7(t)\mathrm{e}^{\theta_2+\theta_3+\theta_1^*}+m_8(t)\mathrm{e}^{\theta_2+\theta_3+\theta_2^*}\\
&\quad+m_9(t)\mathrm{e}^{\theta_2+\theta_3+\theta_3^*},\\
g_5 &= p_1(t)\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_2^*}+p_2(t)\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_3^*}\\
&\quad+p_3(t)\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_2^*+\theta_3^*},\\
h_1 &= \beta_1\mathrm{e}^{\theta_1}+\beta_2\mathrm{e}^{\theta_2}+\beta_3\mathrm{e}^{\theta_3},\\
h_3 &= k_1(t)\mathrm{e}^{\theta_1+\theta_2+\theta_1^*}+k_2(t)\mathrm{e}^{\theta_1+\theta_2+\theta_2^*}+k_3(t)\mathrm{e}^{\theta_1+\theta_2+\theta_3^*}\\
&\quad+k_4(t)\mathrm{e}^{\theta_1+\theta_3+\theta_1^*}+k_5(t)\mathrm{e}^{\theta_1+\theta_3+\theta_2^*},\\
&\quad+k_6(t)\mathrm{e}^{\theta_1+\theta_3+\theta_3^*}+k_7(t)\mathrm{e}^{\theta_2+\theta_3+\theta_1^*}\\
&\quad+k_8(t)\mathrm{e}^{\theta_2+\theta_3+\theta_2^*}+k_9(t)\mathrm{e}^{\theta_2+\theta_3+\theta_3^*},\\
h_5 &= r_1(t)\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_2^*}+r_2(t)\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_3^*}\\
&\quad+r_3(t)\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_2^*+\theta_3^*},\\
f_2 &= n_1(t)\mathrm{e}^{\theta_1+\theta_1^*}+n_2(t)\mathrm{e}^{\theta_1+\theta_2^*}+n_3(t)\mathrm{e}^{\theta_1+\theta_3^*}\\
&\quad+n_4(t)\mathrm{e}^{\theta_2+\theta_1^*}+n_5(t)\mathrm{e}^{\theta_2+\theta_2^*},+n_6(t)\mathrm{e}^{\theta_2+\theta_3^*}\\
&\quad+n_7(t)\mathrm{e}^{\theta_3+\theta_1^*}+n_8(t)\mathrm{e}^{\theta_3+\theta_2^*}+n_9(t)\mathrm{e}^{\theta_3+\theta_3^*},\\
f_4 &= q_1(t)\mathrm{e}^{\theta_1+\theta_2+\theta_1^*+\theta_2^*}+q_2(t)\mathrm{e}^{\theta_1+\theta_2+\theta_1^*+\theta_3^*}\\
&\quad+q_3(t)\mathrm{e}^{\theta_1+\theta_2+\theta_2^*+\theta_3^*}+q_4(t)\mathrm{e}^{\theta_1+\theta_3+\theta_1^*+\theta_2^*}\\
&\quad+q_5(t)\mathrm{e}^{\theta_1+\theta_3+\theta_1^*+\theta_3^*},+q_6(t)\mathrm{e}^{\theta_1+\theta_3+\theta_2^*+\theta_3^*}\\
&\quad+q_7(t)\mathrm{e}^{\theta_2+\theta_3+\theta_1^*+\theta_2^*}+q_8(t)\mathrm{e}^{\theta_2+\theta_3+\theta_1^*+\theta_3^*}\\
&\quad+q_9(t)\mathrm{e}^{\theta_2+\theta_3+\theta_2^*+\theta_3^*},\\
f_6 &= l_1(t)\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_2^*+\theta_3^*},
\end{aligned}$$

where $\theta_j=A_jx+B_j(t)+C_j$, $\alpha_j=\alpha_j^r+\mathrm{i}\alpha_j^i$, and $\beta_j=\beta_j^r+\mathrm{i}\beta_j^i$, $j=1,2,3$, α_j^r, α_j^i, β_j^r, and β_j^i are real numbers, A_j and C_j are complex numbers. The three soliton solutions are

$$q_1=\frac{g_1+g_3+g_5}{1+f_2+f_4+f_6},\quad q_2=\frac{h_1+h_3+h_5}{1+f_2+f_4+f_6}, \qquad (3)$$

where

$$\begin{aligned}
&\beta_1^r=-\frac{\alpha_1^i\beta_1^i}{\alpha_1^r},\quad \beta_2^r=-\frac{\alpha_2^i\beta_2^i}{\alpha_2^r},\quad \beta_3^r=-\frac{\alpha_3^i\beta_3^i}{\alpha_3^r},\\
&\beta_2^i=\frac{\alpha_2^r\beta_1^i}{\alpha_1^r},\quad \beta_3^i=\frac{\alpha_3^r\beta_1^i}{\alpha_1^r},\\
&B_1(t)=\int\left[-A_1^2(\varepsilon(t)A_1-\mathrm{i})\right]\mathrm{d}t,\\
&B_2(t)=\int\left[-A_2^2(\varepsilon(t)A_2-\mathrm{i})\right]\mathrm{d}t,\\
&B_3(t)=\int\left[-A_2^3(\varepsilon(t)A_3-\mathrm{i})\right]\mathrm{d}t,\\
&n_1(t)=\frac{\sigma(|\alpha_1|^2+|\beta_1|^2)}{(A_1+A_1^*)^2},\quad n_2(t)=\frac{\sigma(\alpha_1\alpha_2^*+\beta_1\beta_2^*)}{(A_1+A_2^*)^2},\\
&n_3(t)=\frac{\sigma(\alpha_1\alpha_3^*+\beta_1\beta_3^*)}{(A_1+A_3^*)^2},\quad n_4(t)=\frac{\sigma(\alpha_2\alpha_1^*+\beta_2\beta_1^*)}{(A_2+A_1^*)^2},\\
&n_5(t)=\frac{\sigma(|\alpha_2|^2+|\beta_2|^2)}{(A_2+A_2^*)^2},\quad n_6(t)=\frac{\sigma(\alpha_2\alpha_3^*+\beta_2\beta_3^*)}{(A_2+A_3^*)^2},\\
&n_7(t)=\frac{\sigma(\alpha_3\alpha_1^*+\beta_3\beta_1^*)}{(A_3+A_1^*)^2},\quad n_8(t)=\frac{\sigma(\alpha_3\alpha_2^*+\beta_3\beta_2^*)}{(A_3+A_2^*)^2},\\
&n_9(t)=\frac{\sigma(|\alpha_3|^2+|\beta_3|^2)}{(A_3+A_3^*)^2},
\end{aligned}$$

$$\begin{aligned}
m_j(t) &= \mathrm{e}^{-Q_j}\int\Bigg\{\Big[(n_j(t)\alpha_2+n_{j+3}(t)\alpha_1)\frac{\mathrm{d}}{\mathrm{d}t}B_j^*(t)+(n_j(t)\alpha_2-n_{j+3}(t)\alpha_1)\frac{\mathrm{d}}{\mathrm{d}t}B_1(t)+(-n_j(t)\alpha_2+n_{j+3}(t)\alpha_1)\frac{\mathrm{d}}{\mathrm{d}t}B_2(t)\\
&\quad+\alpha_2\frac{\mathrm{d}}{\mathrm{d}t}n_j(t)+\alpha_1\frac{\mathrm{d}}{\mathrm{d}t}n_{j+3}(t)\Big]+\alpha_2[\mathrm{i}+(A_1-A_2+A_j^*)\varepsilon(t)](A_1-A_2+A_j^*)^2n_j(t)\\
&\quad+\alpha_1[\mathrm{i}+(-A_1+A_2+A_j^*)\varepsilon(t)](A_1-A_2-A_j^*)^2n_{j+3}(t)\Bigg\}\mathrm{e}^{Q_j}\mathrm{d}t,
\end{aligned}$$

$$
\begin{aligned}
m_{j+3}(t) &= \mathrm{e}^{-Q_{j+3}} \int \Big\{ \Big[(n_j(t)\alpha_2 + n_{j+6}(t)\alpha_1) \frac{\mathrm{d}}{\mathrm{d}t} B_j^*(t) + (n_j(t)\alpha_2 - n_{j+6}(t)\alpha_1) \frac{\mathrm{d}}{\mathrm{d}t} B_1(t) + (-n_j(t)\alpha_2 + n_{j+6}(t)\alpha_1) \frac{\mathrm{d}}{\mathrm{d}t} B_3(t) \\
&\quad + \alpha_3 \frac{\mathrm{d}}{\mathrm{d}t} n_j(t) + \alpha_1 \frac{\mathrm{d}}{\mathrm{d}t} n_{j+6}(t) \Big] + \alpha_3 [\mathrm{i} + (A_1 - A_3 + A_j^*)\varepsilon(t)](A_1 - A_3 + A_j^*)^2 n_j(t) \\
&\quad + \alpha_1 [\mathrm{i} + (-A_1 + A_3 + A_j^*)\varepsilon(t)](A_1 - A_3 - A_j^*)^2 n_4(t) \Big\} \mathrm{e}^{Q_{j+3}} \mathrm{d}t, \\
m_{j+6}(t) &= \mathrm{e}^{-Q_{j+6}} \int \Big\{ \Big[(n_{j+3}(t)\alpha_2 + n_{j+6}(t)\alpha_1) \frac{\mathrm{d}}{\mathrm{d}t} B_j^*(t) + (n_{j+3}(t)\alpha_2 - n_{j+6}(t)\alpha_1) \frac{\mathrm{d}}{\mathrm{d}t} B_1(t) + (-n_{j+3}(t)\alpha_2 + n_{j+6}(t)\alpha_1) \frac{\mathrm{d}}{\mathrm{d}t} B_3(t) \\
&\quad + \alpha_3 \frac{\mathrm{d}}{\mathrm{d}t} n_{j+3}(t) + \alpha_1 \frac{\mathrm{d}}{\mathrm{d}t} n_{j+6}(t) \Big] + \alpha_3 [\mathrm{i} + (A_2 - A_3 + A_j^*)\varepsilon(t)](A_2 - A_3 + A_j^*)^2 n_{j+3}(t) \\
&\quad + \alpha_1 [\mathrm{i} + (-A_2 + A_3 + A_j^*)\varepsilon(t)](A_2 - A_3 - A_j^*)^2 n_{j+6}(t) \Big\} \mathrm{e}^{Q_{j+6}} \mathrm{d}t, \\
k_j(t) &= \mathrm{e}^{-Q_j} \int \Big\{ \Big[(n_j(t)\beta_2 + n_{j+3}(t)\beta_1) \frac{\mathrm{d}}{\mathrm{d}t} B_j^*(t) + (n_j(t)\beta_2 - n_{j+3}(t)\beta_1) \frac{\mathrm{d}}{\mathrm{d}t} B_1(t) + (-n_j(t)\beta_2 + n_{j+3}(t)\beta_1) \frac{\mathrm{d}}{\mathrm{d}t} B_2(t) \\
&\quad + \beta_2 \frac{\mathrm{d}}{\mathrm{d}t} n_j(t) + \beta_1 \frac{\mathrm{d}}{\mathrm{d}t} n_{j+3}(t) \Big] + \beta_2 [\mathrm{i} + (A_1 - A_2 + A_j^*)\varepsilon(t)](A_1 - A_2 + A_j^*)^2 n_j(t) \\
&\quad + \beta_1 [\mathrm{i} + (-A_1 + A_2 + A_j^*)\varepsilon(t)](A_1 - A_2 - A_j^*)^2 n_{j+3}(t) \Big\} \mathrm{e}^{Q_j} \mathrm{d}t, \\
k_{j+3}(t) &= \mathrm{e}^{-Q_{j+3}} \int \Big\{ \Big[(n_j(t)\beta_2 + n_{j+6}(t)\beta_1) \frac{\mathrm{d}}{\mathrm{d}t} B_j^*(t) + (n_j(t)\beta_2 - n_{j+6}(t)\beta_1) \frac{\mathrm{d}}{\mathrm{d}t} B_1(t) + (-n_j(t)\beta_2 + n_{j+6}(t)\beta_1) \frac{\mathrm{d}}{\mathrm{d}t} B_3(t) \\
&\quad + \beta_3 \frac{\mathrm{d}}{\mathrm{d}t} n_j(t) + \beta_1 \frac{\mathrm{d}}{\mathrm{d}t} n_{j+6}(t) \Big] + \beta_3 [\mathrm{i} + (A_1 - A_3 + A_j^*)\varepsilon(t)](A_1 - A_3 + A_j^*)^2 n_j(t) \\
&\quad + \beta_1 [\mathrm{i} + (-A_1 + A_3 + A_j^*)\varepsilon(t)](A_1 - A_3 - A_j^*)^2 n_4(t) \Big\} \mathrm{e}^{Q_{j+3}} \mathrm{d}t, \\
k_{j+6}(t) &= \mathrm{e}^{-Q_{j+6}} \int \Big\{ \Big[(n_{j+3}(t)\beta_2 + n_{j+6}(t)\beta_1) \frac{\mathrm{d}}{\mathrm{d}t} B_j^*(t) + (n_{j+3}(t)\beta_2 - n_{j+6}(t)\beta_1) \frac{\mathrm{d}}{\mathrm{d}t} B_1(t) + (-n_{j+3}(t)\beta_2 + n_{j+6}(t)\beta_1) \frac{\mathrm{d}}{\mathrm{d}t} B_3(t) \\
&\quad + \beta_3 \frac{\mathrm{d}}{\mathrm{d}t} n_{j+3}(t) + \beta_1 \frac{\mathrm{d}}{\mathrm{d}t} n_{j+6}(t) \Big] + \beta_3 [\mathrm{i} + (A_2 - A_3 + A_j^*)\varepsilon(t)](A_2 - A_3 + A_j^*)^2 n_{j+3}(t) \\
&\quad + \beta_1 [\mathrm{i} + (-A_2 + A_3 + A_j^*)\varepsilon(t)](A_2 - A_3 - A_j^*)^2 n_{j+6}(t) \Big\} \mathrm{e}^{Q_{j+6}} \mathrm{d}t, \\
Q_j &= (A_1 + A_2 + A_j^*)^3 \int \varepsilon(t) \mathrm{d}t + B_1(t) + B_2(t) + B_j^*(t) - \big[\mathrm{i}A_1^2 + (2\mathrm{i}A_2 + 2\mathrm{i}A_j^*)A_1 + \mathrm{i}A_2^2 + 2\mathrm{i}A_j^* A_2 + \mathrm{i}A_j^{*2} \big] t, \\
Q_{j+3} &= (A_1 + A_3 + A_j^*)^3 \int \varepsilon(t) \mathrm{d}t + B_1(t) + B_3(t) + B_j^*(t) - \big[\mathrm{i}A_1^2 + (2\mathrm{i}A_3 + 2\mathrm{i}A_j^*)A_1 + \mathrm{i}A_3^2 + 2\mathrm{i}A_j^* A_3 + \mathrm{i}A_j^{*2} \big] t, \\
Q_{j+6} &= (A_2 + A_3 + A_j^*)^3 \int \varepsilon(t) \mathrm{d}t + B_2(t) + B_3(t) + B_j^*(t) - \big[\mathrm{i}A_1^2 + (2\mathrm{i}A_3 + 2\mathrm{i}A_j^*)A_2 + \mathrm{i}A_2^3 + 2\mathrm{i}A_j^* A_3 + \mathrm{i}A_j^{*2} \big] t, \\
q_1(t) &= (A_1 + A_1^* + A_2 + A_2^*)^{-2} \{ -n_1(t) n_5(t) (A_1 - A_2 + A_1^* - A_2^*)^2 - n_2(t) n_4(t) (A_1 - A_2 - A_1^* + A_2^*)^2 \\
&\quad + \sigma [k_1(t)\beta_2^* + k_2(t)\beta_1^* + m_1(t)\alpha_2^* + m_2(t)\alpha_1^* + k_1^*(t)\beta_2 + k_2^*(t)\beta_1 + m_1^*(t)\alpha_2 + m_2^*(t)\alpha_1] \}, \\
q_2(t) &= (A_1 + A_1^* + A_2 + A_3^*)^{-2} \{ -n_1(t) n_6(t) (A_1 - A_2 + A_1^* - A_3^*)^2 - n_3(t) n_4(t) (A_1 - A_2 - A_1^* + A_3^*)^2 \\
&\quad + \sigma [k_1(t)\beta_3^* + k_3(t)\beta_1^* + m_1(t)\alpha_3^* + m_3(t)\alpha_1^* + k_4^*(t)\beta_2 + k_5^*(t)\beta_1 + m_4^*(t)\alpha_2 + m_5^*(t)\alpha_1] \}, \\
q_3(t) &= (A_1 + A_2^* + A_2 + A_3^*)^{-2} \{ -n_2(t) n_6(t) (A_1 - A_2 + A_2^* - A_3^*)^2 - n_3(t) n_5(t) (A_1 - A_2 - A_2^* + A_3^*)^2 \\
&\quad + \sigma [k_2(t)\beta_3^* + k_3(t)\beta_2^* + m_2(t)\alpha_3^* + m_3(t)\alpha_2^* + k_7^*(t)\beta_2 + k_8^*(t)\beta_1 + m_7^*(t)\alpha_2 + m_8^*(t)\alpha_1] \}, \\
q_4(t) &= (A_1 + A_1^* + A_3 + A_2^*)^{-2} \{ -n_1(t) n_8(t) (A_1 - A_3 + A_1^* - A_2^*)^2 - n_2(t) n_7(t) (A_1 - A_3 - A_1^* + A_2^*)^2 \\
&\quad + \sigma [k_4(t)\beta_2^* + k_5(t)\beta_1^* + m_4(t)\alpha_2^* + m_5(t)\alpha_1^* + k_1^*(t)\beta_3 + k_3^*(t)\beta_1 + m_1^*(t)\alpha_3 + m_3^*(t)\alpha_1] \}, \\
q_5(t) &= (A_1 + A_3^* + A_1 + A_3^*)^{-2} \{ -n_1(t) n_9(t) (A_1 - A_3 + A_1^* - A_3^*)^2 - n_3(t) n_7(t) (A_1 - A_3 - A_1^* + A_3^*)^2 \\
&\quad + \sigma [k_4(t)\beta_3^* + k_6(t)\beta_1^* + m_4(t)\alpha_3^* + m_6(t)\alpha_1^* + k_4^*(t)\beta_3 + k_6^*(t)\beta_1 + m_4^*(t)\alpha_3 + m_6^*(t)\alpha_1] \}, \\
q_6(t) &= (A_1 + A_2^* + A_3 + A_3^*)^{-2} \{ -n_2(t) n_9(t) (A_1 - A_3 + A_2^* - A_3^*)^2 - n_3(t) n_8(t) (A_1 - A_3 - A_2^* + A_3^*)^2 \\
&\quad + \sigma [k_5(t)\beta_3^* + k_6(t)\beta_2^* + m_5(t)\alpha_3^* + m_6(t)\alpha_2^* + k_7^*(t)\beta_3 + k_9^*(t)\beta_1 + m_7^*(t)\alpha_3 + m_9^*(t)\alpha_1] \}, \\
q_7(t) &= (A_2 + A_1^* + A_3 + A_2^*)^{-2} \{ -n_4(t) n_8(t) (A_2 - A_3 + A_1^* - A_2^*)^2 - n_5(t) n_7(t) (A_2 - A_3 - A_1^* + A_2^*)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sigma[k_7(t)\beta_2^*+k_8(t)\beta_1^*+m_7(t)\alpha_2^*+m_8(t)\alpha_1^*+k_2^*(t)\beta_3+k_3^*(t)\beta_2+m_2^*(t)\alpha_3+m_3^*(t)\alpha_2]\},\\
q_8(t) &= (A_2+A_1^*+A_3+A_3^*)^{-2}\{-n_4(t)n_9(t)(A_2-A_3+A_1^*-A_3^*)^2-n_6(t)n_7(t)(A_2-A_3-A_1^*+A_3^*)^2\\
&+\sigma[k_7(t)\beta_3^*+k_9(t)\beta_1^*+m_7(t)\alpha_3^*+m_9(t)\alpha_1^*+k_5^*(t)\beta_3+k_6^*(t)\beta_2+m_5^*(t)\alpha_3+m_6^*(t)\alpha_2]\},\\
q_9(t) &= (A_2+A_2^*+A_3+A_3^*)^{-2}\{-n_5(t)n_9(t)(A_2-A_3+A_2^*-A_3^*)^2-n_6(t)n_8(t)(A_2-A_3-A_2^*+A_3^*)^2\\
&+\sigma[k_8(t)\beta_3^*+k_9(t)\beta_2^*+m_8(t)\alpha_3^*+m_9(t)\alpha_2^*+k_8^*(t)\beta_3+k_9^*(t)\beta_2+m_8^*(t)\alpha_3+m_9^*(t)\alpha_2]\},\\
p_1(t) &= \mathrm{e}^{-R_1}\int\Big\{\big[\mathrm{i}+(A_1-A_2+A_3+A_1^*+A_2^*)\varepsilon(t)\big](A_1-A_2+A_3+A_1^*+A_2^*)^2\alpha_2q_4(t)\\
&+\big[\mathrm{i}+(A_1+A_2-A_3+A_1^*+A_2^*)\varepsilon(t)\big](A_1+A_2-A_3+A_1^*+A_2^*)^2\alpha_3q_1(t)\\
&+\big[\mathrm{i}+(-A_1+A_2+A_3+A_1^*+A_2^*)\varepsilon(t)\big](A_1-A_2-A_3-A_1^*-A_2^*)^2\alpha_1q_7(t)\\
&+\big[\mathrm{i}+(A_1-A_2-A_3-A_1^*+A_2^*)\varepsilon(t)\big](A_1-A_2-A_3-A_1^*+A_2^*)^2n_2(t)m_7(t)\\
&+\big[\mathrm{i}+(-A_1-A_2+A_3+A_1^*-A_2^*)\varepsilon(t)\big](-A_1-A_2+A_3+A_1^*-A_2^*)^2n_7(t)m_2(t)\\
&+\big[\mathrm{i}+(-A_1-A_2+A_3-A_1^*+A_2^*)\varepsilon(t)\big](A_1+A_2-A_3+A_1^*-A_2^*)^2n_8(t)m_1(t)\\
&+\big[\mathrm{i}+(-A_1+A_2-A_3-A_1^*+A_2^*)\varepsilon(t)\big](A_1-A_2+A_3+A_1^*-A_2^*)^2n_5(t)m_4(t)\\
&+\big[\mathrm{i}+(A_1-A_2-A_3+A_1^*-A_2^*)\varepsilon(t)\big](A_1-A_2-A_3+A_1^*-A_2^*)^2n_1(t)m_8(t)\\
&+\big[\mathrm{i}+(-A_1+A_2-A_3+A_1^*-A_2^*)\varepsilon(t)\big](A_1-A_2+A_3-A_1^*+A_2^*)^2n_4(t)m_5(t)+\alpha_1\frac{\mathrm{d}}{\mathrm{d}t}q_7(t)+\alpha_2\frac{\mathrm{d}}{\mathrm{d}t}q_4(t)+\alpha_3\frac{\mathrm{d}}{\mathrm{d}t}q_1(t)\\
&-n_4(t)\frac{\mathrm{d}}{\mathrm{d}t}m_5(t)-n_8(t)\frac{\mathrm{d}}{\mathrm{d}t}m_1(t)-n_7(t)\frac{\mathrm{d}}{\mathrm{d}t}m_2(t)-n_5(t)\frac{\mathrm{d}}{\mathrm{d}t}m_4(t)-n_1(t)\frac{\mathrm{d}}{\mathrm{d}t}m_8(t)-n_2(t)\frac{\mathrm{d}}{\mathrm{d}t}m_7(t)\\
&+m_4(t)\frac{\mathrm{d}}{\mathrm{d}t}n_5(t)+m_8(t)\frac{\mathrm{d}}{\mathrm{d}t}n_1(t)+m_7(t)\frac{\mathrm{d}}{\mathrm{d}t}n_2(t)+m_5(t)\frac{\mathrm{d}}{\mathrm{d}t}n_4(t)+m_1(t)\frac{\mathrm{d}}{\mathrm{d}t}n_8(t)+m_2(t)\frac{\mathrm{d}}{\mathrm{d}t}n_7(t)\\
&+(-n_4(t)m_5(t)+n_2(t)m_7(t)+n_1(t)m_8(t)-n_5(t)m_4(t)-n_7(t)m_2(t)-n_8(t)m_1(t)\\
&-\alpha_1q_7(t)+\alpha_2q_4(t)+\alpha_3q_1(t))\frac{\mathrm{d}}{\mathrm{d}t}B_1(t)+(n_4(t)m_5(t)-n_2(t)m_7(t)-n_1(t)m_8(t)+n_5(t)m_4(t)\\
&-n_7(t)m_2(t)-n_8(t)m_1(t)+\alpha_1q_7(t)-\alpha_2q_4(t)+\alpha_3q_1(t))\frac{\mathrm{d}}{\mathrm{d}t}B_2(t)\\
&+(-n_4(t)m_5(t)-n_2(t)m_7(t)-n_1(t)m_8(t)-n_5(t)m_4(t)+n_7(t)m_2(t)+n_8(t)m_1(t)\\
&+\alpha_1q_7(t)+\alpha_2q_4(t)+\alpha_3q_1(t))\frac{\mathrm{d}}{\mathrm{d}t}B_3(t)+(n_4(t)m_5(t)-n_2(t)m_7(t)+n_1(t)m_8(t)-n_5(t)m_4(t)\\
&+n_7(t)m_2(t)-n_8(t)m_1(t)+\alpha_1q_7(t)+\alpha_2q_4(t)+\alpha_3q_1(t))\frac{\mathrm{d}}{\mathrm{d}t}B_1^*(t)+(-n_4(t)m_5(t)+n_2(t)m_7(t)\\
&-n_1(t)m_8(t)+n_5(t)m_4(t)-n_7(t)m_2(t)+n_8(t)m_1(t)+\alpha_1q_7(t)+\alpha_2q_4(t)+\alpha_3q_1(t))\frac{\mathrm{d}}{\mathrm{d}t}B_2^*(t)\Big\}\mathrm{e}^{R_1}\,\mathrm{d}t,\\
p_2(t) &= \mathrm{e}^{-R_2}\int-\Big\{\big[\mathrm{i}+(A_1-A_2+A_3+A_1^*+A_3^*)\varepsilon(t)\big](A_1-A_2+A_3+A_1^*+A_3^*)^2\alpha_2q_5(t)\\
&+\big[\mathrm{i}+(A_1+A_2-A_3+A_1^*+A_3^*)\varepsilon(t)\big](A_1+A_2-A_3+A_1^*+A_3^*)^2\alpha_3q_2(t)\\
&+\big[\mathrm{i}+(A_1-A_2-A_3-A_1^*-A_3^*)\varepsilon(t)\big](A_1-A_2-A_3-A_1^*-A_3^*)^2\alpha_1q_8(t)\\
&+\big[\mathrm{i}+(A_1-A_2-A_3+A_1^*-A_3^*)\varepsilon(t)\big](A_1-A_2-A_3+A_1^*-A_3^*)^2n_1(t)m_9(t)\\
&+\big[\mathrm{i}+(-A_1-A_2+A_3+A_1^*-A_3^*)\varepsilon(t)\big](A_1+A_2-A_3-A_1^*+A_3^*)^2n_7(t)m_3(t)\\
&+\big[\mathrm{i}+(-A_1+A_2-A_3-A_1^*+A_3^*)\varepsilon(t)\big](A_1-A_2+A_3+A_1^*-A_3^*)^2n_6(t)m_4(t)\\
&+\big[\mathrm{i}+(-A_1+A_2-A_3+A_1^*-A_3^*)\varepsilon(t)\big](A_1-A_2+A_3-A_1^*+A_3^*)^2n_4(t)m_6(t)\\
&+\big[\mathrm{i}+(-A_1-A_2+A_3-A_1^*+A_3^*)\varepsilon(t)\big](A_1+A_2-A_3+A_1^*-A_3^*)^2n_9(t)m_1(t)\\
&+\big[\mathrm{i}+(A_1-A_2-A_3-A_1^*+A_3^*)\varepsilon(t)\big](A_1-A_2-A_3-A_1^*+A_3^*)^2n_3(t)m_7(t)+\alpha_1\frac{\mathrm{d}}{\mathrm{d}t}q_8(t)+\alpha_2\frac{\mathrm{d}}{\mathrm{d}t}q_5(t)+\alpha_3\frac{\mathrm{d}}{\mathrm{d}t}q_2(t)\\
&-n_7(t)\frac{\mathrm{d}}{\mathrm{d}t}m_3(t)-n_9(t)\frac{\mathrm{d}}{\mathrm{d}t}m_1(t)-n_3(t)\frac{\mathrm{d}}{\mathrm{d}t}m_7(t)-n_1(t)\frac{\mathrm{d}}{\mathrm{d}t}m_9(t)-n_4(t)\frac{\mathrm{d}}{\mathrm{d}t}m_6(t)-n_6(t)\frac{\mathrm{d}}{\mathrm{d}t}m_4(t)\\
&+m_7(t)\frac{\mathrm{d}}{\mathrm{d}t}n_3(t)+m_9(t)\frac{\mathrm{d}}{\mathrm{d}t}n_1(t)+m_3(t)\frac{\mathrm{d}}{\mathrm{d}t}n_7(t)+m_1(t)\frac{\mathrm{d}}{\mathrm{d}t}n_9(t)+m_4(t)\frac{\mathrm{d}}{\mathrm{d}t}n_6(t)+m_6(t)\frac{\mathrm{d}}{\mathrm{d}t}n_4(t)
\end{aligned}
$$

$$
\begin{aligned}
&+(-n_4(t)m_6(t)+n_3(t)m_7(t)+n_1(t)m_9(t)-n_6(t)m_4(t)-n_7(t)m_3(t)-n_9(t)m_1(t)\\
&-\alpha_1 q_8(t)+\alpha_2 q_5(t)+\alpha_3 q_2(t))\frac{\mathrm{d}}{\mathrm{d}t}B_1(t)+(n_4(t)m_6(t)-n_3(t)m_7(t)-n_1(t)m_9(t)+n_6(t)m_4(t)\\
&-n_7(t)m_3(t)-n_9(t)m_1(t)+\alpha_1 q_8(t)-\alpha_2 q_5(t)+\alpha_3 q_2(t))\frac{\mathrm{d}}{\mathrm{d}t}B_2(t)\\
&+(-n_4(t)m_6(t)-n_3(t)m_7(t)-n_1(t)m_9(t)-n_6(t)m_4(t)+n_7(t)m_3(t)+n_9(t)m_1(t)\\
&+\alpha_1 q_8(t)+\alpha_2 q_5(t)-\alpha_3 q_2(t))\frac{\mathrm{d}}{\mathrm{d}t}B_3(t)+(n_4(t)m_6(t)-n_3(t)m_7(t)+n_1(t)m_9(t)-n_6(t)m_4(t)\\
&+n_7(t)m_3(t)-n_9(t)m_1(t)+\alpha_1 q_8(t)+\alpha_2 q_5(t)+\alpha_3 q_2(t))\frac{\mathrm{d}}{\mathrm{d}t}B_1^*(t)\\
&+(-n_4(t)m_6(t)+n_3(t)m_7(t)-n_1(t)m_9(t)+n_6(t)m_4(t)-n_7(t)m_3(t)+n_9(t)m_1(t)\\
&+\alpha_1 q_8(t)+\alpha_2 q_5(t)+\alpha_3 q_2(t))\frac{\mathrm{d}}{\mathrm{d}t}B_3^*(t)\Big\}\mathrm{e}^{R_2}\,\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
p_3(t) \;=\; &\mathrm{e}^{-R_3}\int-\Big\{\big[\mathrm{i}+(A_1-A_2+A_3+A_2^*+A_3^*)\varepsilon(t)\big](A_1-A_2+A_3+A_2^*+A_3^*)^2\alpha_2 q_6(t)\\
&+\big[\mathrm{i}+(A_1+A_2-A_3+A_2^*+A_3^*)\varepsilon(t)\big](A_1+A_2-A_3+A_2^*+A_3^*)^2\alpha_3 q_3(t)\\
&+\big[\mathrm{i}+(-A_1+A_2+A_3+A_2^*+A_3^*)\varepsilon(t)\big](A_1-A_2-A_3-A_1^*-A_3^*)^2\alpha_1 q_9(t)\\
&+\big[\mathrm{i}+(-A_1+A_2-A_3-A_2^*+A_3^*)\varepsilon(t)\big](A_1-A_2+A_3+A_2^*-A_3^*)^2 n_6(t)m_5(t)\\
&+\big[\mathrm{i}+(A_1-A_2-A_3-A_2^*+A_3^*)\varepsilon(t)\big](A_1-A_2-A_3-A_2^*+A_3^*)^2 n_3(t)m_8(t)\\
&+\big[\mathrm{i}+(A_1-A_2-A_3+A_2^*-A_3^*)\varepsilon(t)\big](A_1-A_2-A_3+A_2^*-A_3^*)^2 n_2(t)m_9(t)\\
&+\big[\mathrm{i}+(-A_1-A_2+A_3+A_2^*-A_3^*)\varepsilon(t)\big](A_1+A_2-A_3-A_2^*+A_3^*)^2 n_8(t)m_3(t)\\
&+\big[\mathrm{i}+(-A_1-A_2+A_3-A_2^*+A_3^*)\varepsilon(t)\big](A_1+A_2-A_3+A_2^*-A_3^*)^2 n_9(t)m_2(t)\\
&+\big[\mathrm{i}+(-A_1+A_2-A_3+A_2^*-A_3^*)\varepsilon(t)\big](A_1-A_2+A_3-A_2^*+A_3^*)^2 n_5(t)m_6(t)+\alpha_1\frac{\mathrm{d}}{\mathrm{d}t}q_9(t)+\alpha_2\frac{\mathrm{d}}{\mathrm{d}t}q_6(t)+\alpha_3\frac{\mathrm{d}}{\mathrm{d}t}q_3(t)\\
&-n_5(t)\frac{\mathrm{d}}{\mathrm{d}t}m_6(t)-n_6(t)\frac{\mathrm{d}}{\mathrm{d}t}m_5(t)-n_2(t)\frac{\mathrm{d}}{\mathrm{d}t}m_9(t)-n_8(t)\frac{\mathrm{d}}{\mathrm{d}t}m_3(t)-n_3(t)\frac{\mathrm{d}}{\mathrm{d}t}m_8(t)-n_9(t)\frac{\mathrm{d}}{\mathrm{d}t}m_2(t)\\
&+m_5(t)\frac{\mathrm{d}}{\mathrm{d}t}n_6(t)+m_6(t)\frac{\mathrm{d}}{\mathrm{d}t}n_5(t)+m_2(t)\frac{\mathrm{d}}{\mathrm{d}t}n_9(t)+m_8(t)\frac{\mathrm{d}}{\mathrm{d}t}n_3(t)+m_3(t)\frac{\mathrm{d}}{\mathrm{d}t}n_8(t)+m_9(t)\frac{\mathrm{d}}{\mathrm{d}t}n_2(t)\\
&+(-n_5(t)m_6(t)+n_3(t)m_8(t)+n_2(t)m_9(t)-n_6(t)m_5(t)-n_8(t)m_3(t)-n_9(t)m_2(t)\\
&-\alpha_1 q_9(t)+\alpha_2 q_6(t)+\alpha_3 q_3(t))\frac{\mathrm{d}}{\mathrm{d}t}B_1(t)+(n_5(t)m_6(t)-n_3(t)m_8(t)-n_2(t)m_9(t)+n_6(t)m_5(t)\\
&-n_8(t)m_3(t)-n_9(t)m_2(t)+\alpha_1 q_9(t)-\alpha_2 q_6(t)+\alpha_3 q_3(t))\frac{\mathrm{d}}{\mathrm{d}t}B_2(t)\\
&+(-n_5(t)m_6(t)-n_3(t)m_8(t)-n_2(t)m_9(t)-n_6(t)m_5(t)+n_8(t)m_3(t)+n_9(t)m_2(t)\\
&+\alpha_1 q_9(t)+\alpha_2 q_6(t)-\alpha_3 q_3(t))\frac{\mathrm{d}}{\mathrm{d}t}B_3(t)+(n_5(t)m_6(t)-n_3(t)m_8(t)+n_2(t)m_9(t)-n_6(t)m_5(t)\\
&+n_8(t)m_3(t)-n_9(t)m_2(t)-\alpha_1 q_9(t)+\alpha_2 q_6(t)+\alpha_3 q_3(t))\frac{\mathrm{d}}{\mathrm{d}t}B_2^*(t)\\
&+(-n_5(t)m_6(t)+n_3(t)m_8(t)-n_2(t)m_9(t)+n_6(t)m_5(t)-n_8(t)m_3(t)+n_9(t)m_2(t)\\
&+\alpha_1 q_9(t)+\alpha_2 q_6(t)+\alpha_3 q_3(t))\frac{\mathrm{d}}{\mathrm{d}t}B_3^*(t)\Big\}\mathrm{e}^{R_3}\,\mathrm{d}t,
\end{aligned}
$$

$$
r_1(t) \;=\; p_1(t)|_{m_v(t)=k_v(t),\alpha_z=\beta_z},\quad r_2(t)=p_2(t)|_{m_v(t)=k_v(t),\alpha_z=\beta_z},\quad r_3(t)=p_3(t)|_{m_v(t)=k_v(t),\alpha_z=\beta_z},
$$

$$
\begin{aligned}
R_1 \;=\; &(A_1+A_2+A_3+A_1^*+A_2^*)^3\int\varepsilon(t)\,\mathrm{d}t+B_1(t)+B_2(t)+B_3(t)+B_1^*(t)+B_2^*(t)-[\mathrm{i}A_2^2\\
&+(2\mathrm{i}A_1+2\mathrm{i}A_3+2\mathrm{i}A_1^*+2\mathrm{i}A_2^*)A_2+\mathrm{i}A_3^2+(2\mathrm{i}A_1+2\mathrm{i}A_1^*+2\mathrm{i}A_2^*)A_3+\mathrm{i}A_1^2+(2\mathrm{i}A_1^*+2\mathrm{i}A_2^*)A_1+\mathrm{i}(A_1^*+A_2^*)^2]t,\\
R_2 \;=\; &(A_1+A_2+A_3+A_1^*+A_3^*)^3\int\varepsilon(t)\,\mathrm{d}t+B_1(t)+B_2(t)+B_3(t)+B_1^*(t)+B_3^*(t)-[\mathrm{i}A_2^2\\
&+(2\mathrm{i}A_1+2\mathrm{i}A_3+2\mathrm{i}A_1^*+2\mathrm{i}A_3^*)A_2+\mathrm{i}A_3^2+(2\mathrm{i}A_1+2\mathrm{i}A_1^*+2\mathrm{i}A_3^*)A_3+\mathrm{i}A_1^2+(2\mathrm{i}A_1^*+2\mathrm{i}A_3^*)A_1+\mathrm{i}(A_1^*+A_3^*)^2]t,\\
R_3 \;=\; &(A_1+A_2+A_3+A_2^*+A_3^*)^3\int\varepsilon(t)\,\mathrm{d}t+B_1(t)+B_2(t)+B_3(t)+B_2^*(t)+B_3^*(t)-[\mathrm{i}A_2^2
\end{aligned}
$$

$$
\begin{aligned}
&+(2\mathrm{i}A_1+2\mathrm{i}A_3+2\mathrm{i}A_2^*+2\mathrm{i}A_3^*)A_2+\mathrm{i}A_3^2+(2\mathrm{i}A_1+2\mathrm{i}A_2^*+2\mathrm{i}A_3^*)A_3+\mathrm{i}A_1^2+(2\mathrm{i}A_2^*+2\mathrm{i}A_3^*)A_1+\mathrm{i}(A_2^*+A_3^*)^2]t,\\
l_1(t) \;=\; &(A_1+A_2+A_3+A_1^*+A_2^*+A_3^*)^{-2}\Big\{-n_1(t)q_9(t)(A_1-A_2-A_3+A_1^*-A_2^*-A_3^*)^2\\
&-n_2(t)q_8(t)(A_1-A_2-A_3-A_1^*+A_2^*-A_3^*)^2-n_3(t)q_7(t)(A_1-A_2-A_3-A_1^*-A_2^*+A_3^*)^2\\
&-n_4(t)q_6(t)(A_1-A_2+A_3-A_1^*+A_2^*+A_3^*)^2-n_5(t)q_5(t)(A_1-A_2+A_3+A_1^*-A_2^*+A_3^*)^2\\
&-n_6(t)q_4(t)(A_1-A_2+A_3+A_1^*-A_2^*+A_3^*)^2-n_7(t)q_3(t)(A_1+A_2-A_3-A_1^*+A_2^*+A_3^*)^2\\
&-n_8(t)q_2(t)(A_1+A_2-A_3+A_1^*-A_2^*+A_3^*)^2-n_9(t)q_1(t)(A_1+A_2-A_3+A_1^*+A_2^*-A_3^*)^2\\
&+\sigma[p_1(t)\alpha_3^*+p_2(t)\alpha_2^*+p_3(t)\alpha_1^*+p_1^*(t)\alpha_3+p_2^*(t)\alpha_2+p_3^*(t)\alpha_1\\
&+r_1(t)\beta_3^*+r_2(t)\beta_2^*+r_3(t)\beta_1^*+r_1^*(t)\beta_3+r_2^*(t)\beta_2+r_3^*(t)\beta_1\\
&+m_1(t)m_9^*(t)+m_1^*(t)m_9(t)+m_2(t)m_6^*(t)+m_2^*(t)m_6(t)+m_3(t)m_3^*(t)+m_4(t)m_8^*(t)+m_4^*(t)m_8(t)\\
&+m_5(t)m_5^*(t)+m_7(t)m_7^*(t)+k_1(t)k_9^*(t)+k_1^*(t)k_9(t)+k_2(t)k_6^*(t)+k_6^*(t)k_2(t)+k_3^*(t)k_3(t)\\
&+k_4(t)k_8^*(t)+k_8^*(t)k_4(t)+k_5(t)k_5^*(t)+k_7(t)k_7^*(t)]\Big\},
\end{aligned}
$$

where $j=1,2,3$, $v=1,2,\ldots,9$, $z=1,2,3$.

3. Discussion

In Eq. (3), the transmission properties of q_1 and q_2 are similar, with only differences in amplitude due to high symmetry of optical fibers. To avoid repeated discussion, we will only discuss the interactions between the three optical solitons in q_1. At first, we assume that $\varepsilon(t)=\sin(t)+\cos(2t)$, $\sigma=1$, $C_1=C_2=C_3=0$, $\alpha_1^i=2$, $\alpha_1^r=1$, $\alpha_2^i=1/3$, $\alpha_2^r=2$, $\alpha_3^i=2/3$, $\alpha_3^r=2$ and $\beta_1^i=1$. In Fig. 1, we see the interaction situation of three optical solitons. We can change the relative amplitudes between the optical solitons, and achieve efficient routing of optical solitons during transmission by adjusting the values of A_1, A_2, and A_3. In Figs. 1(b) and 1(c), one of the optical solitons has a much smaller amplitude. In Fig. 1(d), two of the optical solitons form bound-state solitons that interact with a third optical soliton. They are almost unaffected by the interaction and are transmitted in their original state.

Continuing to change the values of A_1, A_2, and A_3, as illustrated in Fig. 2, parallel transmission of optical solitons is achievable. In Fig. 2(a), two of the optical solitons exhibit sinusoidal oscillations when they are transmitted. And the third optical soliton has little interaction with them due to its small amplitude. In Fig. 2(b), because of the small distance between two optical solitons, the interaction between them is more pronounced and they both exhibit sinusoidal oscillations, but the third optical soliton with smaller oscillations has almost no effect on them. In Fig. 2(c), the distance between two optical solitons increases, so they do not affect each other during transmission. The third optical soliton interacts with them more significantly due to its larger amplitude, but it does not affect the stable transmission of the bound-state soliton after the interaction. In Fig. 2(d), due to the different incidence angles of the two optical solitons, there is a certain angle between them, and parallel transmission cannot be achieved. During the transmission processes, both optical solitons show sinusoidal function-type oscillations.

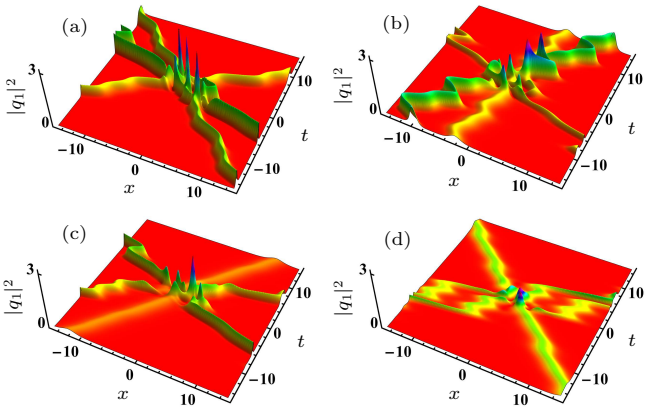


Fig. 1. Three-soliton controllable interaction transmissions. Parameters in solution (3) are $\varepsilon(t)=\sin(t)+\cos(2t)$, $\sigma=1$, $C_1=C_2=C_3=0$, $\alpha_1^i=2$, $\alpha_1^r=1$, $\alpha_2^i=1/3$, $\alpha_2^r=2$, $\alpha_3^i=2/3$, $\alpha_3^r=2$, and $\beta_1^i=1$ with (a) $A_1=-1.3-0.56\mathrm{i}$, $A_2=-1.5-1.3\mathrm{i}$, $A_3=1.1+0.84\mathrm{i}$; (b) $A_1=-1.4+0.34\mathrm{i}$, $A_2=-0.78+0.13\mathrm{i}$, $A_3=0.84-1.1\mathrm{i}$; (c) $A_1=-1.2-1.1\mathrm{i}$, $A_2=-0.63+0.41\mathrm{i}$, $A_3=-1+\mathrm{i}$; (d) $A_1=0.75+\mathrm{i}$, $A_2=0.59+1.1\mathrm{i}$, $A_3=0.69-0.53\mathrm{i}$.

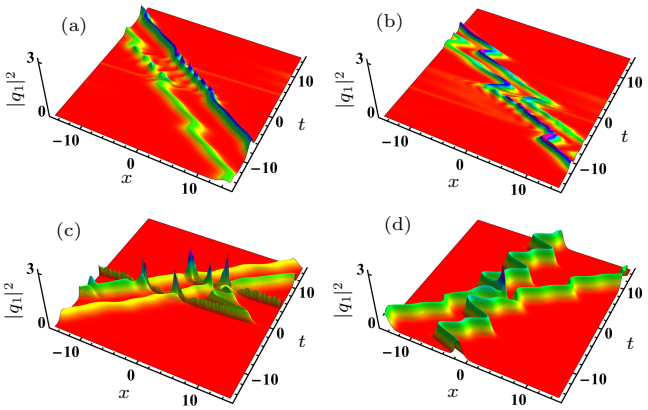


Fig. 2. Three-soliton controllable interaction transmissions. Parameters in solution (3) are $\varepsilon(t)=\sin(t)+\cos(2t)$, $\sigma=1$, $C_1=C_2=C_3=0$, $\alpha_1^i=2$, $\alpha_1^r=1$, $\alpha_2^i=1/3$, $\alpha_2^r=2$, $\alpha_3^i=2/3$, $\alpha_3^r=2$, and $\beta_1^i=1$ with (a) $A_1=-1.1-0.75\mathrm{i}$, $A_2=0.25+1.6\mathrm{i}$, $A_3=0.75-0.69\mathrm{i}$; (b) $A_1=-0.16+1.9\mathrm{i}$, $A_2=-0.81-0.84\mathrm{i}$, $A_3=-0.59-\mathrm{i}$; (c) $A_1=-0.91+0.63\mathrm{i}$, $A_2=-1.1+0.75\mathrm{i}$, $A_3=-1.1+1.9\mathrm{i}$; (d) $A_1=-1.3+0.53\mathrm{i}$, $A_2=0.031$, $A_3=1.4+0.13\mathrm{i}$.

When the hyperbolic secant function is selected, we can control the transmission velocity of the three optical solitons after interaction, as shown in Fig. 3. The transmission velocity of the smaller amplitude optical soliton in Fig. 3(b) is noticeably higher than that of the smaller amplitude optical soliton in Fig. 3(a). In Fig. 3(c), the transmission speed of the optical soliton with smaller amplitude is greater. In Fig. 3(d), due to the larger value of A_3, the speed of the photoresoliton shows a sudden change after the interaction, and it changes directly from a low speed to a high speed for transmission.

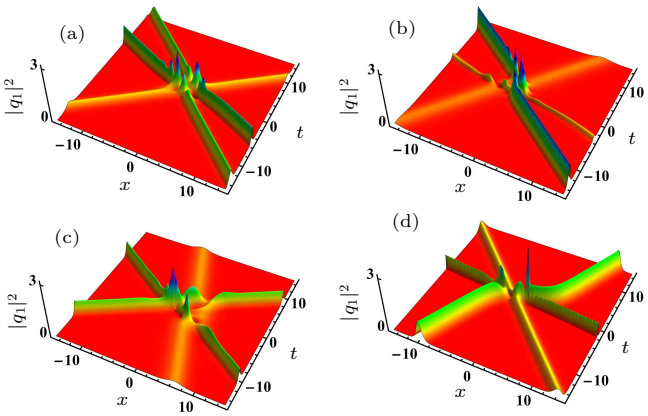


Fig. 3. Three-soliton controllable interaction transmissions. Parameters in solution (3) are $\varepsilon(t) = \mathrm{sech}(t)$, $\sigma = 1$, $C_1 = C_2 = C_3 = 0$, $\alpha_1^i = 2$, $\alpha_1^r = 1$, $\alpha_2^i = 1/3$, $\alpha_2^r = 2$, $\alpha_3^i = 2/3$, $\alpha_3^r = 2$, and $\beta_1^i = 1$ with (a) $A_1 = 0.72 + 0.66\mathrm{i}$, $A_2 = 1.4 - \mathrm{i}$, $A_3 = 1.2 - 0.56\mathrm{i}$; (b) $A_1 = -1.6 - 0.66\mathrm{i}$, $A_2 = 0.53 + 0.38\mathrm{i}$, $A_3 = -0.72 - 1.2\mathrm{i}$; (c) $A_1 = 1.2 - 0.75\mathrm{i}$, $A_2 = -0.63 - 0.19\mathrm{i}$, $A_3 = -1.2 + 0.84\mathrm{i}$; (d) $A_1 = 0.97 - 0.5\mathrm{i}$, $A_2 = -1.5 + 0.22\mathrm{i}$, $A_3 = 1.3 - 1.8\mathrm{i}$.

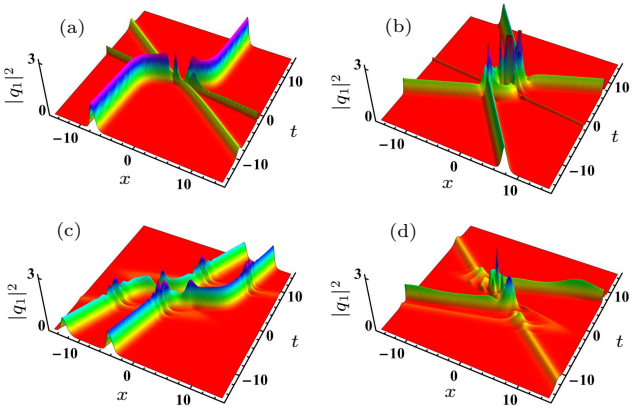


Fig. 4. Three-soliton controllable interaction transmissions. Parameters in solution (3) are $\varepsilon(t) = \mathrm{sech}(t)$, $\sigma = 1$, $C_1 = C_2 = C_3 = 0$, $\alpha_1^i = 2$, $\alpha_1^r = 1$, $\alpha_2^i = 1/3$, $\alpha_2^r = 2$, $\alpha_3^i = 2/3$, $\alpha_3^r = 2$, and $\beta_1^i = 1$ with (a) $A_1 = 1.8 + 0.03\mathrm{i}$, $A_2 = -1 - 1.8\mathrm{i}$, $A_3 = -0.94 - 0.72\mathrm{i}$; (b) $A_1 = -1.3 + 1.3\mathrm{i}$, $A_2 = 0.59 - 1.8\mathrm{i}$, $A_3 = 1.7 - 0.38\mathrm{i}$; (c) $A_1 = -1.6 + 0.13\mathrm{i}$, $A_2 = 0.41 + 1.8\mathrm{i}$, $A_3 = 1.4 + 0.22\mathrm{i}$; (d) $A_1 = -1.3 + 1.7\mathrm{i}$, $A_2 = 0.75 - 0.63\mathrm{i}$, $A_3 = -0.5 + 2\mathrm{i}$.

In Fig. 3(d), we can also modify the amplitude of the optical solitons by varying the values of A_1, A_2, and A_3, and results are illustrated in Fig. 4(a). Due to $\varepsilon(t) = \mathrm{sech}(t)$, the hyperbolic secant function results in a sudden change in the velocity of the soliton at zero. In Fig. 3, all three optical solitons have different incident phases. We change the incident phase of the optical solitons by changing the values of A_1, A_2, and A_3. As shown in Fig. 4(b), the incident phases of the two photoresolitons are just opposite, so that they have certain local bound states during the interaction. In Figs. 4(c) and 4(d), two of the optical solitons have the same incident phase, showing parallel transmission, which enables optical fibers to simultaneously transmit multiple optical solitons without mutual interference. In addition, the third optical soliton has a smaller amplitude, which has almost no effect on their parallel transport.

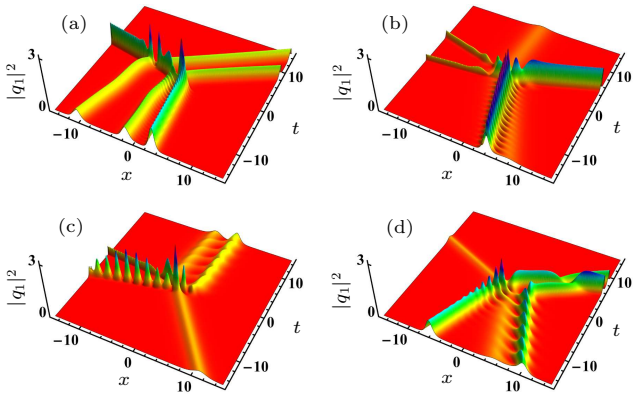


Fig. 5. Three-soliton controllable interaction transmissions. Parameters in solution (3) are $\varepsilon(t) = \tanh(t)$, $\sigma = 1$, $C_1 = C_2 = C_3 = 0$, $\alpha_1^i = 2$, $\alpha_1^r = 1$, $\alpha_2^i = 1/3$, $\alpha_2^r = 2$, $\alpha_3^i = 2/3$, $\alpha_3^r = 2$, and $\beta_1^i = 1$ with (a) $A_1 = -1.2 + 0.44\mathrm{i}$, $A_2 = -1.6 - 1.3\mathrm{i}$, $A_3 = -1.1 + 0.44\mathrm{i}$; (b) $A_1 = 1.6 + 0.6\mathrm{i}$, $A_2 = 0.53 - 0.1\mathrm{i}$, $A_3 = 0.84 + 1.6\mathrm{i}$; (c) $A_1 = -0.97 + 0.97\mathrm{i}$, $A_2 = 0.72 - 0.16\mathrm{i}$, $A_3 = 1 - 1.8\mathrm{i}$; (d) $A_1 = 1.2 + 0.34\mathrm{i}$, $A_2 = 0.56 - 0.66\mathrm{i}$, $A_3 = 1.3 + 0.56\mathrm{i}$.

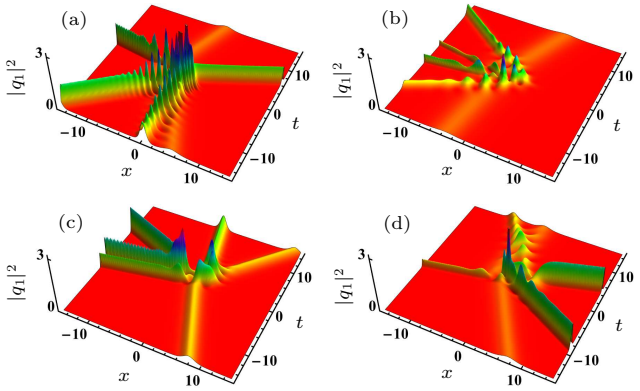


Fig. 6. Three-soliton controllable interaction transmissions. Parameters in solution (3) are $\varepsilon(t) = \tanh(t)$, $\sigma = 1$, $C_1 = C_2 = C_3 = 0$, $\alpha_1^i = 2$, $\alpha_1^r = 1$, $\alpha_2^i = 1/3$, $\alpha_2^r = 2$, $\alpha_3^i = 2/3$, $\alpha_3^r = 2$, and $\beta_1^i = 1$ with (a) $A_1 = -1.4 + 0.53\mathrm{i}$, $A_2 = -1.6 - 1.5\mathrm{i}$, $A_3 = 0.63 - 0.03\mathrm{i}$; (b) $A_1 = -1 + 1.3\mathrm{i}$, $A_2 = 0.75 - 1.3\mathrm{i}$, $A_3 = 0.44 + 1.3\mathrm{i}$; (c) $A_1 = 1.7 + 1.7\mathrm{i}$, $A_2 = -0.81 + 0.1\mathrm{i}$, $A_3 = -1.3 + 1.2\mathrm{i}$; (d) $A_1 = -0.91 + 1.1\mathrm{i}$, $A_2 = 1.7 + 0.34\mathrm{i}$, $A_3 = 0.63 - 0.13\mathrm{i}$.

In Figs. 3 and 4, we choose $\varepsilon(t)$ as a hyperbolic tangent function, at which point we can achieve parallel transmission of the optical solitons being relatively easy, as shown in Fig. 5. In Fig. 5(a), three-optical solitons present parallel transmission. Changing the values of A_1, A_2, and A_3, the spacing between the optical solitons becomes smaller, and at this time they interact violently with each other, as shown in Fig. 5(b). At the same time, we can also increase the spacing between them so that the interaction between the optical solitons becomes weaker, as shown in Figs. 5(c) and 5(d). Thus, during

the interactions of three optical solitons, we achieved parallel transmission of two optical solitons and formed a bound state optical soliton. In Fig. 6, the transmission direction of the optical solitons can be effectively controlled. By choosing different values of A_1, A_2, and A_3, the optical solitons' transmission direction is modifiable so that they transmit in the direction we expected.

4. Conclusion

The study of three optical soliton interactions contributes to the development of all-optical information processing technology, which is of great significance for the construction of all-optical networks. This paper analytically studies Eq. (1), and the three-soliton solutions (3) of them are obtained. Three optical solitons' interactions are investigated when the higher-order dispersion and nonlinear functions are sinusoidal, hyperbolic secant, and hyperbolic tangent function, respectively. The effective control of three optical solitons' interactions is achieved by selecting different parameter values. By changing the values of A_1, A_2, and A_3, the amplitude and velocity of optical solitons, as well as the interaction processes between optical solitons, can be effectively controlled. It is possible to achieve soliton-free interaction transmission when the distance between optical solitons is small. The findings of the paper on three optical solitons' interactions can optimize the design of the optical communication system and improve the transmission capacity and efficiency of the system.

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