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Dynamics of fundamental and double-pole breathers and solitons for a nonlinear Schrödinger equation with sextic operator under non-zero boundary conditions

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We study the dynamics of fundamental and double-pole breathers and solitons for the focusing and defocusing nonlinear Schrödinger equation with the sextic operator under non-zero boundary conditions. Our analysis mainly focuses on the dynamical properties of simple- and double-pole solutions. Firstly, through verification, we find that solutions under non-zero boundary conditions can be transformed into solutions under zero boundary conditions, whether in simple-pole or double-pole cases. For the focusing case, in the investigation of simple-pole solutions, temporal periodic breather and the spatial-temporal periodic breather are obtained by modulating parameters. Additionally, in the case of multi-pole solitons, we analyze parallel-state solitons, bound-state solitons, and intersecting solitons, providing a brief analysis of their interactions. In the double-pole case, we observe that the two solitons undergo two interactions, resulting in a distinctive "triangle" crest. Furthermore, for the defocusing case, we briefly consider two situations of simple-pole solutions, obtaining one and two dark solitons.

Keywords: double-pole solitons, double-pole breathers, Riemann–Hilbert problem, non-zero boundary conditions, nonlinear Schrödinger equation with sextic operator

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1. Introduction

Nonlinear Schrödinger (NLS) equation is a basic physical model, which can describe the evolution of the complex envelope of the weakly nonlinear dispersive wave trains.[1,2] The NLS equation has been proved to model solitons and breathers.[3–5] Solitons, described by the NLS equation, arise as a result of the balance between group velocity dispersion and self-phase modulation.[6] Breathers are often referred to as "multi-soliton bound states" under the zero background, which propagate with periodic distribution structure.[7,8] So far, certain approaches have been developed to study solitons and breathers, such as Darboux transformation (DT),[9,10] generalized DT,[11–13] binary DT,[14,15] Riemann–Hilbert (RH) approaches,[16–18] and Fokas unified transformation approach.[19,20]

The NLS equation has been widely used in many fields, such as fluid mechanics,[21] nonlinear optics,[22] and plasma physics.[23] However, as the exploration into nonlinear phenomena deepens, it becomes evident that basic NLS-type equations are no longer sufficient for accurately describing these phenomena. For instance, the shorter duration propagating pulse along the fiber, with the amplitude increasing, need to be governed by the NLS-type equations with higher-order operators in nonlinear optics.[24–27] In order to describe the nonlinear phenomena more comprehensively in practical applications, some researchers have investigated the NLS equation with high-order operators as follows:[28]

$$\begin{aligned}
&\mathrm{i}q_\tau+\frac{1}{2}(q_{\varsigma\varsigma}+2\sigma q|q|^2)-\mathrm{i}\beta(q_{\varsigma\varsigma\varsigma}+6\sigma|q|^2q_\varsigma)\\
&+\delta\{q_{\varsigma\varsigma\varsigma\varsigma\varsigma\varsigma}+10q^3[(q^*_\varsigma)^2+2q^*q^*_{\varsigma\varsigma}]\\
&+[60q^*|q_\varsigma|^2+50(q^*)^2q_{\varsigma\varsigma}+2\sigma q^*_{\varsigma\varsigma\varsigma\varsigma}]q^2\\
&+q(12\sigma q^*q_{\varsigma\varsigma\varsigma\varsigma}+8\sigma q_\varsigma q^*_{\varsigma\varsigma\varsigma}+22\sigma|q_{\varsigma\varsigma}|^2)+20\sigma q^*q^2_{\varsigma\varsigma}\\
&+q[18\sigma q_{\varsigma\varsigma\varsigma}q^*_\varsigma+70(q^*)^2q^2_\varsigma]+20\sigma q^2_\varsigma q^*_{\varsigma\varsigma}\\
&+10q_\varsigma(5\sigma q_{\varsigma\varsigma}q^*_\varsigma+3\sigma q^*q_{\varsigma\varsigma\varsigma})+20\sigma q|q|^6\}=0, \qquad (1)
\end{aligned}$$

where $q=q(\varsigma,\tau)$ denotes the envelope of the waves, τ and ς respectively indicate the scaled time and propagation distance, the superscript "$*$" represents the complex conjugation, $\sigma=1$ ($\sigma=-1$) expresses focusing (defocusing) case, and the real parameters β and δ are the coefficients of the third- and sextic-order dispersions, respectively. In nonlinear optics and oceanography, when the parameter β and δ are taken as the specific values, special cases of Eq. (1) have been seen, including:

(i) When $\beta=\delta=0$, equation (1) has been degenerated to the basic NLS equation. In this case, scalar and vector superregular breathers and MI have been reported in Refs. [29,30].

(ii) When $\beta\neq0,\delta=0$, equation (1) has been simplified into the Hirota equation, which can depict the propagation of the subpicosecond or femtosecond optical pulse propagation in the optical fiber.[31,32]

(iii) When $\beta=0,\delta\neq0$, equation (1) has been simplified into a sixth-order NLS equation. For this special form

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of Eq. (1), certain results have been reported. Under zero boundary conditions (ZBCs), breathers have been studied via the DT,[33] and multi-soliton solutions and multi-pole soliton solutions have been investigated via the RH problem.[34] Under non-zero boundary conditions (NZBCs), in Ref. [35] the authors has derived multi-soliton solutions through the inverse scattering transform. Based on the RH problem, the modified Darboux transformation method has been used to derive the rogue wave solutions.[36] In addition, in Ref. [37] the author has investigated the breather-like solitons with imaginary eigenvalues, and analyzed the interactions between two solitons.

To our knowledge, the fundamental and double-pole breathers and solitons for Eq. (1) with NZBCs have not been derived via the RH method. Thus, we will consider the RH problem for Eq. (1) with the following NZBCs

$$\begin{aligned}&\lim_{\varsigma\to\pm\infty} q(\varsigma,\tau)=q_{\pm}\mathrm{e}^{\mathrm{i}b\tau},\quad |q_{\pm}|=q_0\neq 0,\\&b=\frac{1}{2}q_0^2\sigma+20q_0^6\sigma^3\delta,\end{aligned}\tag{2}$$

where q_0 is a constant. To simplify the boundary condition, we take the transformation

$$q\to q\mathrm{e}^{\mathrm{i}b\tau}.\tag{3}$$

Then, equation (1) can be transformed into

$$\begin{aligned}&\mathrm{i}q_\tau+\frac{1}{2}(q_{\varsigma\varsigma}+2\sigma q^*q^2-2q_0^2\sigma q)-\mathrm{i}\beta(q_{\varsigma\varsigma\varsigma}+6\sigma|q|^2q_\varsigma)\\&+\delta[q_{\varsigma\varsigma\varsigma\varsigma\varsigma}+12\sigma|q|^2q_{\varsigma\varsigma\varsigma}+(30\sigma q_\varsigma q^*+18\sigma qq_\varsigma^*)q_{\varsigma\varsigma}\\&+(50|q|^4+50\sigma|q_\varsigma|^2+20\sigma q^*q_{\varsigma\varsigma}+22\sigma qq_{\varsigma\varsigma}^*)q_\varsigma\\&+(70|q|^2q^*q_\varsigma+60q|q|^2q_\varsigma^*)q_\varsigma+2\sigma q^2q_{\varsigma\varsigma\varsigma}^*+8\sigma qq_\varsigma q_{\varsigma\varsigma}^*\\&+(20\sigma q_\varsigma^2+20q^2|q|^2)q_{\varsigma\varsigma}^*\\&+10q^3(q_\varsigma^*)^2+20\sigma q|q|^6-20q_0^6\sigma q]=0,\end{aligned}\tag{4}$$

under the following boundary condition:

$$\lim_{\varsigma\to\pm\infty} q(\varsigma,\tau)=q_{\pm},\quad |q_{\pm}|=q_0\neq 0.\tag{5}$$

Subsequently, the corresponding Lax pair of Eq. (4) can be obtained as

$$\begin{aligned}&\Phi_\varsigma=U\Phi,\quad U=\mathrm{i}k\sigma_3+Q,\\&\Phi_\tau=V\Phi,\\&V=V_0+kV_1+k^2V_2+k^3V_3+k^4V_4+k^5V_5+k^6V_6,\end{aligned}\tag{6}$$

where $\Phi=\Phi(\varsigma,\tau;k)$ is a 2×2 matrix-valued eigenfunction of Lax pair (6), $k\in\mathbb{C}$ is a spectral parameter, $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $Q(\varsigma,\tau)=\begin{pmatrix}0&q(\varsigma,\tau)\\-\sigma q^*(\varsigma,\tau)&0\end{pmatrix}$, and the expressions for V_j's $(j=0,1,\ldots,6)$ are given in Appendix A. Zero curvature equation $U_\varsigma-V_\tau+UV-VU=0$ and equation (1) are equal. On the basis of Lax pair (6), in Ref. [28] the authors has obtained the multi-soliton solutions, breathers and rogue waves for Eq. (1) with variable coefficients via the DT and generalized DT.

In this paper, we will investigate soliton and breather solutions under conditions of reflectionless potential. Additionally, through the parameter modulation, we will explore the dynamical properties of soliton solutions. The structure of this paper will be organized as follows. In Section 2, we will establish the regular RH problem to derive the simple-pole and double-pole solutions for the focusing NLS with the sextic operator under NZBCs. In Section 3, we will deduce the inverse scattering transform of the defocusing NLS with the sextic operator under NZBCs to obtain its general soliton solutions. In Section 4, we will analyze the obtained solutions and study their dynamical properties. Finally, our conclusions and discussions will be given in Section 5.

2. The focusing case of Eq. (4)

2.1. Direct scattering with simple poles under NZBCs

Using the boundary condition (5) and considering the limit as $\varsigma\to\pm\infty$, we can derive the asymptotic spectral problem of the focusing $(\sigma=1)$ case as

$$\begin{aligned}&\Phi_\varsigma=U_\pm\Phi,\quad U_\pm=\lim_{\varsigma\to\pm\infty}U=\mathrm{i}k\sigma_3+Q_\pm,\\&\Phi_\tau=V_\pm\Phi,\quad V_\pm=\lim_{\varsigma\to\pm\infty}V=\Xi^{\mathrm{f}}U_\pm,\end{aligned}\tag{7}$$

with $Q_\pm=\lim_{\varsigma\to\pm\infty}Q(\varsigma,\tau)=\begin{pmatrix}0&q_\pm\\-q_\pm^*&0\end{pmatrix}$. The solutions of spectral problem of Eq. (7) can be written as the following fundamental matrix functions:

$$\begin{aligned}&\Phi_\pm^{\mathrm{f}}(\varsigma,\tau;k)\\&=\begin{cases}Y_\pm^{\mathrm{f}}\mathrm{e}^{\mathrm{i}\theta(\varsigma,\tau;k)\sigma_3}, & k\neq\pm\mathrm{i}q_0,\\I+\left[\varsigma-\left(k+6k^2\beta+60k^5\delta\right)\tau\right]U_\pm(k), & k=\pm\mathrm{i}q_0,\end{cases}\end{aligned}\tag{8}$$

where

$$Y_\pm^{\mathrm{f}}=\begin{pmatrix}1&\dfrac{\mathrm{i}q_\pm}{k+\lambda(k)}\\\dfrac{\mathrm{i}q_\pm^*}{k+\lambda(k)}&1\end{pmatrix},$$

$\theta(\varsigma,\tau;k)=\lambda(k)(\varsigma+\Xi^{\mathrm{f}}\tau)$, $\Xi^{\mathrm{f}}=-[k+(4k^2-2q_0^2)\beta+4(8k^5-4k^3q_0^2+3kq_0^4)\delta]$, and $\lambda^2(k)=k^2+q_0^2$.

In order to avoid multiplicity and complexity of the Riemann surface, we introduce a uniformization variable ρ, and discuss the RH problem on a standard complex ρ-plane. The Riemann surface is divided into two complex k-plane (S_1,S_2), alone the branch line $[-\mathrm{i}q_0,\mathrm{i}q_0]$, of which the segment are $k=\pm\mathrm{i}q_0$. Hence, λ represents a simple-valued function of k, where on the sheet S_1: it is given by $k(\rho)=\frac{1}{2}\left(\rho-q_0^2/\rho\right)$, and on the Sheet S_2: it is given by $\lambda(\rho)=\frac{1}{2}\left(\rho+q_0^2/\rho\right)$. The mappings between the two-sheeted Riemann k-surface, complex λ-plane, and complex ρ-plane are presented in Fig. 1.

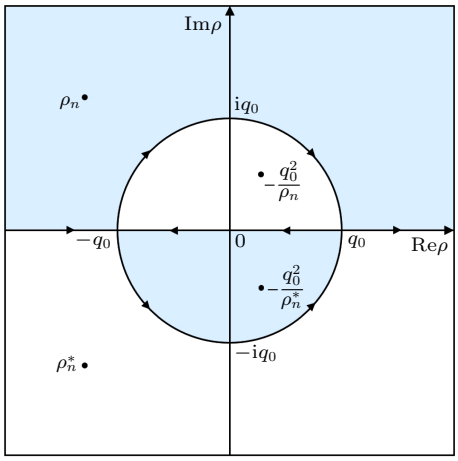


Fig. 1. The discrete spectrum and jump curve distribution of the RH problem on the complex ρ-plane, the wathet area stands for D^+, the white area stands for D^-.

Setting $D^+ = \{\rho \in \mathbb{C} : (|\rho|^2 - q_0^2)\,\mathrm{Im}\,\rho > 0\}$, $D^- = \{\rho \in \mathbb{C} : (|\rho|^2 - q_0^2)\,\mathrm{Im}\,\rho < 0\}$, and the continuous spectrum $\mathbb{R}\cup\mathbb{C}_0, \mathbb{C}_0 = \{|\rho| = q_0, \rho \in \mathbb{C}\}$ on ρ-plane, we can define the Jost solutions $\Phi_\pm(\varsigma,\tau;k)$ as

$$\Phi_\pm(\varsigma,\tau;k) = Y_\pm^{\mathrm{f}}\,\mathrm{e}^{\mathrm{i}\theta(\varsigma,\tau;\rho)\sigma_3} + O(1), \quad \varsigma \to \pm\infty. \tag{9}$$

By the variable transformation

$$\mu_\pm(\varsigma,\tau;\rho) = \Phi_\pm(\varsigma,\tau;\rho)\,\mathrm{e}^{-\mathrm{i}\theta(\varsigma,\tau;\rho)\sigma_3}, \tag{10}$$

one has the modified Jost solutions that $\mu_\pm(\varsigma,\tau;\rho) \to Y_\pm^{\mathrm{f}}(\rho)$, as $\varsigma \to \pm\infty$. The Volterra integral equations for formula (10) can be written as

$$\mu_\pm(\varsigma,\tau;\rho) = \begin{cases} Y_\pm^{\mathrm{f}} + \displaystyle\int_{\pm\infty}^{\varsigma} Y_\pm^{\mathrm{f}}\,\mathrm{e}^{\mathrm{i}\lambda(\varsigma-\varsigma')\hat{\sigma}_3}\left[\left(Y_\pm^{\mathrm{f}}\right)^{-1}\Delta U_\pm(\varsigma',\tau)\mu_\pm(\varsigma',\tau;\rho)\right]\mathrm{d}\varsigma', & \rho \neq \pm\mathrm{i}q_0, \\ Y_\pm^{\mathrm{f}} + \displaystyle\int_{\pm\infty}^{\varsigma}\left[I + (\varsigma-\varsigma')(U_\pm \mp q_0\sigma_3)\right]\Delta U_\pm(\varsigma',\tau)\mu_\pm(\varsigma',\tau;\rho)\mathrm{d}\varsigma', & \rho = \pm\mathrm{i}q_0, \end{cases} \tag{11}$$

where $\Delta U_\pm = U - U_\pm$.

Proposition 1 Setting $q - q_\pm \in L^1(\mathbb{R}^\pm)$, and $\mu_{\pm j}(\varsigma,\tau;\rho)$ stands for the j-th column of $\mu_\pm(\varsigma,\tau;\rho)$ $(j = 1,2)$. In terms of transform (10) and formula (11), we give the following asymptotic and analytical properties:

P1-1 $\mu_{+1}(\varsigma,\tau;\rho)$ and $\mu_{-2}(\varsigma,\tau;\rho)$ are analytical in D^+ and continuous in $D^+\bigcup\Sigma_0$.

P1-2 $\mu_{-1}(\varsigma,\tau;\rho)$ and $\mu_{+2}(\varsigma,\tau;\rho)$ are analytical in D^- and continuous in $D^-\bigcup\Sigma_0$.

P1-3 $\mu_\pm(\varsigma,\tau;\rho) \to I$, as $\rho \to \infty$.

P1-4 $\mu_\pm(\varsigma,\tau;\rho) \to \frac{\mathrm{i}}{\rho}\sigma_3 Q_\pm$, as $\rho \to 0$.

P1-5 $\det\mu_\pm(\varsigma,\tau;\rho) = \det Y_\pm = \gamma = 1 + q_0^2/\rho^2$.

Since the Jost functions $\Phi_\pm(\varsigma,\tau;\rho)$ satisfy the Lax pair (6), there exists a constant matrix $S(\rho) = (s_{ij}(\rho))_{2\times2}$ $(i,j = 1,2)$, that is independent of the time and space variables, and satisfy the following linear relationship:

$$\Phi_+(\varsigma,\tau;\rho) = \Phi_-(\varsigma,\tau;\rho)S(\rho), \quad \rho \in \Sigma. \tag{12}$$

The scattering coefficients $S(\rho) = (s_{ij}(\rho))_{2\times2}$ $(i,j = 1,2)$ can be expressed as

$$\begin{pmatrix} s_{11}(\rho) & s_{12}(\rho) \\ s_{21}(\rho) & s_{22}(\rho) \end{pmatrix} = \begin{pmatrix} \dfrac{\det(\Phi_{+,1}(\varsigma,\tau;\rho),\Phi_{-,2}(\varsigma,\tau;\rho))}{\gamma(\rho)} & \dfrac{\det(\Phi_{+,2}(\varsigma,\tau;\rho),\Phi_{-,2}(\varsigma,\tau;\rho))}{\gamma(\rho)} \\ \dfrac{\det(\Phi_{-,1}(\varsigma,\tau;\rho),\Phi_{+,1}(\varsigma,\tau;\rho))}{\gamma(\rho)} & \dfrac{\det(\Phi_{-,1}(\varsigma,\tau;\rho),\Phi_{+,2}(\varsigma,\tau;\rho))}{\gamma(\rho)} \end{pmatrix}, \tag{13}$$

and the reflection coefficients are defined by $\eta(\rho) = s_{21}(\rho)/s_{11}(\rho)$, $\widehat{\eta}(\rho) = s_{12}(\rho)/s_{22}(\rho)$.

Proposition 2 Supposing $q - q_\pm \in \mathrm{L}^1(\mathbb{R}^\pm)$, as for scattering matrix $S(\rho)$, we have

P2-1 $s_{11}(\rho)$ is analytical in D^- and continuous in $D^-\cup\Sigma_0$.

P2-2 $s_{22}(\rho)$ is analytical in D^+ and continuous in $D^+\cup\Sigma_0$.

P2-3 $S(\rho) \to I$, $\varsigma \to \infty$.

P2-4 $S(\rho) \to \dfrac{q_+}{q_-}I$, $\varsigma \to 0$.

P2-5 $\det S(\rho) = 1$, as $\rho \in \Sigma_0$.

2.2. The inverse scattering and simple poles under NZBCs

In what follows, we mainly analyze the Jost function $\Phi(\varsigma,\tau;\rho)$ and scattering matrix $S(\rho)$ with NZBCs. We can derive the symmetries of $U(\varsigma,\tau;\rho)$, $V(\varsigma,\tau;\rho)$, and $\theta(\varsigma,\tau;\rho)$ as

$$\begin{aligned} &U(\varsigma,\tau;\rho) = -\sigma_1 U^*(\varsigma,\tau;\rho^*)\sigma_1, \\ &V(\varsigma,\tau;\rho) = -\sigma_1 V^*(\varsigma,\tau;\rho^*)\sigma_1, \\ &\theta(\varsigma,\tau;\rho) = \theta^*(\varsigma,\tau;\rho^*), \\ &U(\varsigma,\tau;\rho) = U\left(\varsigma,\tau;-\frac{q_0^2}{\rho}\right), \end{aligned}$$

$$V(\varsigma,\tau;\rho)=V\left(\varsigma,\tau;-\frac{q_0^2}{\rho}\right),$$
$$\theta(\varsigma,\tau;\rho)=-\theta^*\left(\varsigma,\tau;-\frac{q_0^2}{\rho}\right), \tag{14}$$

where $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Moreover, through Lax pair (6) and formula (10), we can get the following symmetries for $\Phi_\pm$ as

$$\begin{aligned}
&\Phi_\pm(\varsigma,\tau;\rho)=-\sigma_1\Phi_\pm^*(\varsigma,\tau;\rho^*)\sigma_1,\\
&\mu_\pm(\varsigma,\tau;\rho)=-\sigma_1\mu_\pm^*(\varsigma,\tau;\rho^*)\sigma_1,\\
&\Phi_\pm(\varsigma,\tau;\rho)=\frac{\mathrm{i}}{\rho}\Phi_\pm\left(\varsigma,\tau;-\frac{q_0^2}{\rho}\right)\sigma_3Q_\pm,\\
&\mu_\pm(\varsigma,\tau;\rho)=\frac{\mathrm{i}}{\rho}\Phi_\pm\left(\varsigma,\tau;-\frac{q_0^2}{\rho}\right)\sigma_3Q_\pm,
\end{aligned} \tag{15}$$

and then

$$\begin{aligned}
&S(\rho)=-\sigma_1S^*(\rho^*)\sigma_1,\\
&S(\rho)=(\sigma_3Q_-)^{-1}S\left(-\frac{q_0^2}{\rho}\right)\sigma_3Q_+.
\end{aligned} \tag{16}$$

Furthermore, we can deduce the symmetries between $\eta(\rho)$ and $\widehat{\eta}(\rho)$ as

$$\eta(\rho)=-\hat{\eta}^*(\rho^*),\quad \eta(\rho)=\frac{q_-^*}{q_-}\widehat{\eta}\left(-\frac{q_0^2}{\rho}\right). \tag{17}$$

The set of discrete spectrum is composed of all ρ that make the eigenfunction exist in $L^2(\mathbb{R}^\pm)$. Assuming that $s_{11}(\rho)$ has K simple zeros ρ_n's $(n=1,2,\ldots,K)$ in $D^+\cap\{\rho\in\mathbb{C}:\mathrm{Im}\rho>0\}$, then we have $s_n(\rho_n)=0$, $s_{11}'(\rho_n)\neq0$ and $s_{22}(\rho_n^*)=s_{22}(-q_0^2/\rho_n)=s_{11}(-q_0^2/\rho_n)=0$. From the symmetry of the scattering matrix, we derive the corresponding discrete spectrum as

$$Z=\left\{\rho_n,-\frac{q_0^2}{\rho_n^*},\rho_n^*,-\frac{q_0^2}{\rho_n}\right\}_{n=1}^K, \tag{18}$$

where ρ_n satisfies that $|\rho_n|=q_0$.

Considering $s_{11}(\rho_0)=0$ and $s_{11}'(\rho_0)=0$ $(\rho\in Z\cap D^+)$, we can verify that there is a norming constant $b_+[\rho_0]$ to satisfy

$$\Phi_{+,1}(\varsigma,\tau;\rho_0)=b_+[\rho_0]\,\Phi_{-,2}(\varsigma,\tau;\rho_0). \tag{19}$$

Therefore, we attain that

$$\begin{aligned}
&\operatorname*{Res}_{\rho=\rho_0}\left[\frac{\Phi_{+,1}(\varsigma,\tau;\rho)}{s_{11}(\rho)}\right]=A_+[\rho_0]\,\Phi_{-,2}(\varsigma,\tau;\rho_0),\\
&\operatorname*{Res}_{\rho=\rho_0^*}\left[\frac{\Phi_{+,2}(\varsigma,\tau;\rho)}{s_{22}(\rho)}\right]=A_-[\rho_0^*]\,\Phi_{-,1}(\varsigma,\tau;\rho_0),
\end{aligned} \tag{20}$$

with

$$A_\pm[\rho_0]=\begin{cases}\dfrac{b_+[\rho_0]}{s_{11}'(\rho_0)}, & \rho_0\in Z\cap D^+,\\[2mm] \dfrac{b_-[\rho_0^*]}{s_{22}'(\rho_0^*)}, & \rho_0^*\in Z\cap D^-,\end{cases}$$

which correspond to the symmetries

$$A_+[\rho_0]=-A_-^*[\rho_0^*],\quad A_+[\rho_0]=\frac{\rho_0^2}{q_-^2}A_-\left[-\frac{q_0^2}{\rho_0}\right]. \tag{21}$$

The symmetries of $A_\pm[\rho_n]$ as $\rho_n\in Z\cap D^+$ are

$$A_+[\rho_n]=-A_-^*[\rho_n^*]=\frac{\rho_n^2}{q_-^2}A_-\left[-\frac{q_0^2}{\rho_n}\right]=\frac{\rho_n^2}{q_-^2}A_+\left[-\frac{q_0^2}{\rho_n^*}\right]. \tag{22}$$

Form Proposition 1, we define the sectionally meromorphic matrix $M(\varsigma,\tau;\rho)$ as

$$\begin{aligned}
&M^+(\varsigma,\tau;\rho)=\left(\frac{\mu_{+1}(\varsigma,\tau;\rho)}{s_{11}(\rho)},\mu_{-2}(\varsigma,\tau;\rho)\right),\quad \rho\in D_+,\\
&M^-(\varsigma,\tau;\rho)=\left(\mu_{-1}(\varsigma,\tau;\rho),\frac{\mu_{+2}(\varsigma,\tau;\rho)}{s_{22}(\rho)}\right),\quad \rho\in D_-.
\end{aligned} \tag{23}$$

Therefore, we can construct the RH problem:

Proposition 3 The matrix $M(\varsigma,\tau;\rho)$ satisfies the following RH problem:

P3-1 $M(\varsigma,\tau;\rho)$ is analytical in $(D^-\cup D^+)\backslash Z$.

P3-2 $M^-(\varsigma,\tau;\rho)=M^+(\varsigma,\tau;\rho)(I-G(\varsigma,\tau;\rho))$, $\rho\in\Sigma$.

P3-3 $M(\varsigma,\tau;\rho)\to I$, as $\varsigma\to\infty$.

P3-4 $M(\varsigma,\tau;\rho)\to\frac{\mathrm{i}}{\rho}\sigma_3Q_-$, as $\varsigma\to0$,

with the jump matrix as

$$G(\varsigma,\tau;\rho)=\begin{pmatrix}0&-\widehat{\eta}(\rho)\,\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\rho)}\\ \eta(\rho)\,\mathrm{e}^{-2\mathrm{i}\theta(\varsigma,\tau;\rho)}&\eta(\rho)\widehat{\eta}(\rho)\end{pmatrix}.$$

For convenience, we take $\widehat{\xi}_n=-q_0^2/\xi_n$ and define

$$\xi_n=\begin{cases}\rho_n, & n=1,2,\ldots,K\\ -\dfrac{q_0^2}{(\rho_{n-K}^*)}, & n=K+1,K+2,\ldots,2K.\end{cases} \tag{24}$$

The residue of $M(\varsigma,\tau;\rho)$ is

$$\begin{aligned}
&\operatorname*{Res}_{\rho=\xi_n}M^+(\varsigma,\tau;\rho)=\left(A_+[\xi_n]\,\mathrm{e}^{-2\mathrm{i}\theta(\varsigma,\tau;\xi_n)}\mu_{-2}(\varsigma,\tau;\xi_n),0\right),\\
&\operatorname*{Res}_{\rho=\widehat{\xi}_n}M^-(\varsigma,\tau;\rho)=\left(0,A_-\left[\widehat{\xi}_n\right]\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau,\widehat{\xi}_n)}\mu_{-1}(\varsigma,\tau;\widehat{\xi}_n)\right).
\end{aligned} \tag{25}$$

It follows that

$$\begin{aligned}
&\frac{\operatorname*{Res}_{\rho=\xi_n}M^+(\varsigma,\tau;\rho)}{\rho-\xi_n}+\frac{\operatorname*{Res}_{\rho=\widehat{\xi}_n}M^-(\varsigma,\tau;\rho)}{\rho-\widehat{\xi}_n}\\
&=\left(D[\rho]\mu_{-2}(\varsigma,\tau;\xi_n),\widehat{D}[\rho]\mu_{-1}(\varsigma,\tau;\widehat{\xi}_n)\right),
\end{aligned} \tag{26}$$

with

$$\begin{aligned}
&D[\rho]=\frac{A_+[\xi_n]\,\mathrm{e}^{-2\mathrm{i}\theta(\varsigma,\tau;\xi_n)}}{\rho-\xi_n},\\
&\widehat{D}[\rho]=\frac{A_-\left[\widehat{\xi}_n\right]\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\widehat{\xi}_n)}}{\rho-\widehat{\xi}_n}.
\end{aligned}$$

Subtracting out the asymptotic values as $\rho\to\infty,\rho\to0$ and the residue, the regular RH problem can be obtained as

$$M^-(\varsigma,\tau;\rho)-I-\frac{\mathrm{i}}{\rho}\sigma_3Q_-$$

$$
\begin{aligned}
&-\sum_{n=1}^{2K}\left[\frac{\underset{\rho=\xi_n}{\operatorname{Res}} M^{+}(\varsigma,\tau;\rho)}{\rho-\xi_n}+\frac{\underset{\rho=\widehat{\xi}_n}{\operatorname{Res}} M^{-}(\varsigma,\tau;\rho)}{\rho-\widehat{\xi}_n}\right]\\
&=M^{+}(\varsigma,\tau;\rho)-I-\frac{\mathrm{i}}{\rho}\sigma_3 Q_{-}\\
&\quad-\sum_{n=1}^{2K}\left[\frac{\underset{\rho=\xi_n}{\operatorname{Res}} M^{+}(\varsigma,\tau;\rho)}{\rho-\xi_n}+\frac{\underset{\rho=\widehat{\xi}_n}{\operatorname{Res}} M^{-}(\varsigma,\tau;\rho)}{\rho-\widehat{\xi}_n}\right]\\
&\quad-M^{+}(\varsigma,\tau;\rho)G(\varsigma,\tau;\rho). \qquad (27)
\end{aligned}
$$

Therefore, through the Plemelj's formulae, we have

$$
\begin{aligned}
&M(\varsigma,\tau;\rho)\\
&=I+\frac{\mathrm{i}}{\rho}\sigma_3 Q_{-}+\sum_{n=1}^{2K}\left[\frac{\underset{\rho=\xi_n}{\operatorname{Res}} M^{+}(\varsigma,\tau;\rho)}{\rho-\xi_n}+\frac{\underset{\rho=\widehat{\xi}_n}{\operatorname{Res}} M^{-}(\varsigma,\tau;\rho)}{\rho-\widehat{\xi}_n}\right]\\
&\quad+\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{M^{+}(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta)}{\zeta-\rho}\mathrm{d}\zeta,\quad \rho\in\mathbb{C}\backslash\Sigma. \qquad (28)
\end{aligned}
$$

Therefore, the asymptoptic behavior of $M(\varsigma,\tau;\rho)$ is given as

$$
M(\varsigma,\tau;\rho)=I+\frac{1}{\rho}M^{(1)}(\varsigma,\tau;\rho)+O\left(\frac{1}{\rho^2}\right),\quad \rho\to\infty \qquad (29)
$$

with

$$
\begin{aligned}
M^{(1)}(\varsigma,\tau;\rho)\ =\ &\mathrm{i}\sigma_3 Q_{-}-\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}M^{+}(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta)\mathrm{d}\zeta\\
&-\sum_{n=1}^{2K}\Big[A_{+}[\xi_n]\,\mathrm{e}^{-2\mathrm{i}\theta(\varsigma,\tau,\xi_n)}\mu_{-2}(\varsigma,\tau,\xi_n)\\
&\cdot A_{-}\big[\widehat{\xi}_n\big]\,\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau,\widehat{\xi}_n)}\mu_{-1}(\varsigma,\tau,\widehat{\xi}_n)\Big]. \qquad (30)
\end{aligned}
$$

Furthermore, from the second column in formula (26) and formula (27) we can derive that

$$
\begin{aligned}
\mu_{-2}(\varsigma,\tau,\xi_j)\ =\ &\begin{pmatrix}\dfrac{\mathrm{i}q_{-}}{\xi_j}\\ 1\end{pmatrix}+\sum_{n=1}^{2K}\widehat{D}[\xi_j]\mu_{-1}(\widehat{\xi}_n)\\
&+\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{(M^{+}(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta))_2}{\zeta-\xi_j}\mathrm{d}\zeta, \qquad (31)
\end{aligned}
$$

as $\rho=\xi_j$ $(j=1,2,\ldots,2K)$. According to formula (15), we derive

$$
\mu_{-2}(\varsigma,\tau,\xi_j)=\frac{\mathrm{i}q_{-}}{\xi_j}\mu_{-1}(\varsigma,\tau,\widehat{\xi}_j). \qquad (32)
$$

Using Eqs. (31) and (32), we obtain

$$
\begin{aligned}
&\sum_{n=1}^{2K}\left(\widehat{D}[\xi_j]-\frac{\mathrm{i}q_{-}}{\xi_j}\delta_{n,j}\right)\mu_{-11}(\varsigma,\tau,\widehat{\xi}_j)+\frac{\mathrm{i}q_{-}}{\xi_j}\\
&+\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{(M^{+}(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta))_1 2}{\zeta-\xi_j}\mathrm{d}\zeta=0. \qquad (33)
\end{aligned}
$$

with $\delta_{n,j}=\begin{cases}1, & n=j\\ 0, & n\neq j\end{cases}$. Then, the potential $q(\varsigma,\tau)$ of Eq. (4) with simple poles in the focusing case can be expressed as

$$
\begin{aligned}
q(\varsigma,\tau)\ =\ &q_{-}-\mathrm{i}\sum_{n=1}^{2K}A_{-}\left[\widehat{\xi}_n\right]\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau,\widehat{\xi}_n)}\mu_{-11}(\varsigma,\tau;\widehat{\xi}_n)\\
&+\frac{1}{2\pi}\int_{\Sigma}\left(M^{+}(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta)\right)_{12}\mathrm{d}\zeta. \qquad (34)
\end{aligned}
$$

Since $s_{11}(\rho)$ and $s_{22}(\rho)$ are scattering coefficients related to discrete spectrum and reflection coefficients, where $s_{11}(\rho)$ is analytical in D^{+} and $s_{22}(\rho)$ is analytical in D^{-}. The direct spectrum points ξ_n and $\widehat{\xi}_n$ are simple zeros of $s_{11}(\rho)$ and $s_{22}(\rho)$ respectively. Let

$$
\begin{aligned}
\gamma^{+}(\rho)&=s_{11}(\rho)\prod_{n=1}^{2K}\frac{\rho-\widehat{\xi}_n}{\rho-\xi_n},\\
\gamma^{-}(\rho)&=s_{22}(\rho)\prod_{n=1}^{2K}\frac{\rho-\xi_n}{\rho-\widehat{\xi}_n\xi_n},
\end{aligned}
$$

we find that $\gamma^{\pm}(\rho)$ are analytical and have no zeros in $D^{\pm}$. Through Proposition 2, we can derive

$$
\gamma^{+}(\rho)\gamma^{-}(\rho)=s_{11}(\rho)s_{22}(\rho)=\frac{1}{1-\eta(\rho)\widehat{\eta}(\rho)}, \qquad (35)
$$

which can be expressed as

$$
\begin{aligned}
&-\log\gamma^{+}(\rho)-\log\gamma^{-}(\rho)\\
&=\log[1-\eta(\rho)\widehat{\eta}(\rho)]=\log[1+\eta(\rho)\eta^{*}(\rho^{*})]. \qquad (36)
\end{aligned}
$$

According to Cauchy operator and Plemelj's formulas, we obtain that

$$
\begin{aligned}
\log\gamma^{\pm}(\rho)\ =\ &\mp\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{\log[1+\eta(\zeta)\eta^{*}(\zeta^{*})]}{\zeta-\rho}\mathrm{d}\zeta,\\
&\rho\in D^{\pm}, \qquad (37)
\end{aligned}
$$

Hence, the specific expression forms of trace formulas can be written as

$$
\begin{aligned}
s_{11}(\rho)\ =\ &\exp\left(-\frac{\mathrm{i}}{2\pi}\int_{\Sigma}\frac{\log[1+\eta(\zeta)\eta^{*}(\zeta^{*})]}{\zeta-\rho}\mathrm{d}\zeta\right)\\
&\times\prod_{n=1}^{K}\frac{(\rho-\rho_n)\left(\rho+\dfrac{q_0^2}{\rho_n^{*}}\right)}{(\rho-\rho_n^{*})\left(\rho+\dfrac{q_0^2}{\rho_n}\right)}, \qquad (38a)\\
s_{22}(\rho)\ =\ &\exp\left(\frac{\mathrm{i}}{2\pi}\int_{\Sigma}\frac{\log[1+\eta(\zeta)\eta^{*}(\zeta^{*})]}{\zeta-\rho}\mathrm{d}\zeta\right)\\
&\times\prod_{n=1}^{K}\frac{(\rho-\rho_n)\left(\rho+\dfrac{q_0^2}{\rho_n^{*}}\right)}{(\rho-\rho_n^{*})\left(\rho+\dfrac{q_0^2}{\rho_n}\right)}. \qquad (38b)
\end{aligned}
$$

In this case, we can arrange the limit $\rho\to 0$ in formula (38a). Note that $s_{11}(\rho)\to q_{+}/q_{-}$ from Proposition 2. Thus, we can deduce that

$$
\begin{aligned}
\arg\left(\frac{q_{+}}{q_{-}}\right)\ =\ &4\sum_{n=1}^{K}\arg\rho_n\\
&+\frac{1}{2\pi}\int_{\Sigma}\frac{\log[1+\eta(\zeta)\eta^{*}(\zeta^{*})]}{\zeta}\mathrm{d}\zeta. \qquad (39)
\end{aligned}
$$

Particularly, we consider a special kind of solutions as $\eta(\rho)=\widehat{\eta}(\rho)=0$. Therefore, formula (33) can be simplified to

$$\sum_{n=1}^{2K}\left(\frac{A_-\left[\widehat{\xi}_n\right]\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\widehat{\xi}_n)}}{\xi_j-\widehat{\xi}_n}-\frac{\mathrm{i}q_-}{\xi_j}\delta_{n,j}\right)\times\mu_{-11}(\varsigma,\tau,\widehat{\xi}_j)+\frac{\mathrm{i}q_-}{\xi_j}=0. \tag{40}$$

To this end, we have the following proposition:

Proposition 4 The reflectionless potential with simple poles under NZBCs of Eq. (4) can be obtained as the determinants

$$q(\varsigma,\tau)=q_-+\mathrm{i}\frac{\det\begin{bmatrix}F & g\\ h^{\mathrm{T}} & 0\end{bmatrix}}{\det F}, \tag{41}$$

where $F=(f_{sj})_{2K\times 2K}$, $g=(g_j)_{2K\times 1}$, and $h=(h_j)_{2K\times 1}$ with

$$f_{sj}=\frac{h_j}{\xi_s-\widehat{\xi}_j}+g_s\delta_{s,j},$$
$$g_j=-\frac{\mathrm{i}q_-}{\xi_j},\quad h_j=A_-\left[\widehat{\xi}_j\right]\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau,\widehat{\xi}_j)}.$$

2.3. Direct scattering problem with double poles under NZBCs

The direct scattering and inverse scattering problems, under the conditions of double poles and simple poles, have similarities except for the direct spectrum. Via formula (24), we obtain the scattering coefficient $s_{11}(\rho)$ and $s_{22}(\rho)$ as

$$s_{11}(\rho_0)=s'_{11}(\rho_0),\quad s''_{11}(\rho_0)\neq 0,\quad \rho_0\in Z\cap\mathrm{D}^+, \tag{42a}$$
$$s_{22}(\rho_0)=s'_{22}(\rho_0),\quad s''_{22}(\rho_0)\neq 0,\quad \rho_0\in Z\cap\mathrm{D}^-. \tag{42b}$$

By taking the first-order partial derivative of formula (13) and $s_{11}(\rho_0)=s'_{11}(\rho_0)=0$, we can find that another constant $c[\rho_0]$ satisfies the following expressions:

$$b_+[\rho_0]=\frac{\Phi_{+,1}(\varsigma,\tau;\rho_0)}{\Phi_{-,2}(\varsigma,\tau;\rho_0)},$$
$$c_+[\rho_0]=\frac{\Phi'_{+,1}(\varsigma,\tau;\rho_0)-b_+[\rho_0]\Phi'_{-,2}(\varsigma,\tau;\rho_0)}{\Phi_{-,2}(\varsigma,\tau;\rho_0)},\quad \rho_0\in Z\cap D^+, \tag{43a}$$
$$b_-[\rho_0^*]=\frac{\Phi_{+,2}(\varsigma,\tau;\rho_0^*)}{\Phi_{-,1}(\varsigma,\tau;\rho_0^*)},$$
$$c_-[\rho_0^*]=\frac{\Phi'_{+,2}(\varsigma,\tau;\rho_0^*)-b_-[\rho_0^*]\Phi'_{-,1}(\varsigma,\tau;\rho_0^*)}{\Phi_{-,1}(\varsigma,\tau;\rho_0^*)},\quad \rho_0\in Z\cap D^-. \tag{43b}$$

Replacing $\Phi_\pm$ by the modified eigenfunctions, we have

$$A_+[\rho_0]=\frac{2b_+[\rho_0]}{s''_{11}(\rho_0)},$$
$$B_+[\rho_0]=\frac{c_+[\rho_0]}{b_+[\rho_0]}-\frac{s'''_{11}(\rho_0)}{3s''_{11}(\rho_0)},\quad \rho_0\in Z\cap D^+, \tag{44a}$$
$$A_-[\rho_0^*]=\frac{2b_-[\rho_0^*]}{s''_{22}(\rho_0^*)},$$
$$B_-[\rho_0^*]=\frac{c_-[\rho_0^*]}{b_-[\rho_0^*]}-\frac{s'''_{22}(\rho_0^*)}{3s''_{22}(\rho_0^*)},\quad \rho_0\in Z\cap D^-, \tag{44b}$$

and get the symmetries of $A_\pm[\rho_n]$ and $B_\pm[\rho_n]$ as $\rho_n\in Z\cap D^\pm$:

$$A_+[\rho_n]=-A_-^*[\rho_n^*]=\frac{\rho_n^4q_-^*}{q_0^4q_-}A_-\left[-\frac{q_0^2}{\rho_n}\right], \tag{45a}$$
$$B_+[\rho_n]=B_-^*[\rho_n^*]=\frac{q_0^2}{\rho_n^2}B_-\left[-\frac{q_0^2}{\rho_n}\right]+\frac{2}{\rho_n}. \tag{45b}$$

Then, we attain that

$$\begin{aligned}
\underset{\rho=\rho_0}{P_{-2}}\left[\frac{\Phi_{+,1}(\varsigma,\tau;\rho)}{s_{11}(\rho)}\right]&=\frac{2b_+[\rho_0]}{s''_{11}(\rho_0)}\Phi_{-2}(\varsigma,\tau;\rho_0)=A_+[\rho_0]\Phi_{-,2}(\varsigma,\tau;\rho_0),\quad \rho_0\in Z\cap D^+,\\
\underset{\rho=\rho_0^*}{P_{-2}}\left[\frac{\Phi_{+,2}(\varsigma,\tau;\rho)}{s_{22}(\rho)}\right]&=\frac{2b_-[\rho_0^*]}{s''_{22}(\rho_0^*)}\Phi_{-,1}(\varsigma,\tau;\rho_0^*)=A_-[\rho_0^*]\Phi_{-,1}(\varsigma,\tau;\rho_0^*),\quad \rho_0\in Z\cap D^-,\\
\operatorname*{Res}_{\rho=\rho_0}\left[\frac{\Phi_{+,1}(\varsigma,\tau;\rho)}{s_{11}(\rho)}\right]&=\frac{2b_+[\rho_0]}{s''_{11}(\rho_0)}\left[\Phi'_{-,2}(\varsigma,\tau;\rho_0)+\left(\frac{c_+[\rho_0]}{b_+[\rho_0]}-\frac{s'''_{11}(\rho_0)}{3s''_{11}(\rho_0)}\right)\Phi_{-,2}(\varsigma,\tau;\rho_0)\right]\\
&=A_+[\rho_0]\left[\Phi'_{-,2}(\varsigma,\tau;\rho_0)+B_+[\rho_0]\Phi_{-,2}(\varsigma,\tau;\rho_0)\right],\quad \rho_0\in Z\cap D^+,\\
\operatorname*{Res}_{\rho=\rho_0^*}\left[\frac{\Phi_{+,2}(\varsigma,\tau;\rho)}{s_{22}(\rho)}\right]&=\frac{2b_-[\rho_0^*]}{s''_{22}(\rho_0^*)}\left[\Phi'_{-,1}(\varsigma,\tau;\rho_0^*)+\left(\frac{c_-[\rho_0^*]}{b_-[\rho_0^*]}-\frac{s'''_{22}(\rho_0^*)}{3s''_{22}(\rho_0^*)}\right)\Phi_{-,2}(\varsigma,\tau;\rho_0^*)\right]\\
&=A_-[\rho_0^*]\left[\Phi'_{-,1}(\varsigma,\tau;\rho_0^*)+B_-[\rho_0^*]\Phi_{-,2}(\varsigma,\tau;\rho_0^*)\right],\ \rho_0\in Z\cap D^-.
\end{aligned} \tag{46}$$

According to formulas (10) and (23), we deduce that

$$\underset{\rho=\xi_n}{P_{-2}}M_1^+(\varsigma,\tau;\rho)=\underset{\rho=\xi_n}{P_{-2}}\left[\frac{\mu_{+1}(\varsigma,\tau;\rho)}{s_{11}(\rho)}\right]=A[\xi_n]\mathrm{e}^{-2\mathrm{i}\theta(\varsigma,\tau;\xi_n)}\mu_{-2}(\varsigma,\tau;\xi_n),$$
$$\underset{\rho=\widehat{\xi}_n}{P_{-2}}M_2^-(\varsigma,\tau;\rho)=\underset{\rho=\widehat{\xi}_n}{P_{-2}}\left[\frac{\mu_{+2}(\varsigma,\tau;\rho)}{s_{22}(\rho)}\right]=A\left[\widehat{\xi}_n\right]\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\widehat{\xi}_n)}\mu_{-1}(\varsigma,\tau;\widehat{\xi}_n),$$

$$\begin{aligned}
\operatorname*{Res}_{\rho=\xi_n} M_1^+(\varsigma,\tau;\rho) &= \operatorname*{Res}_{\rho=\xi_n}\left[\frac{\mu_{+1}(\varsigma,\tau;\rho)}{s_{11}(\rho)}\right] = A[\xi_n]\,\mathrm{e}^{-2\mathrm{i}\theta(\varsigma,\tau;\xi_n)}\{\mu'_{-2}(\varsigma,\tau;\xi_n)+[B[\xi_n]-2\mathrm{i}\theta'(\varsigma,\tau;\xi_n)]\,\mu_{-2}(\varsigma,\tau;\xi_n)\},\\
\operatorname*{Res}_{\rho=\widehat{\xi}_n} M_2^-(\varsigma,\tau;\rho) &= \operatorname*{Res}_{\rho=\widehat{\xi}_n}\left[\frac{\mu_{+2}(\varsigma,\tau;\rho)}{s_{22}(\rho)}\right] = A\left[\widehat{\xi}_n\right]\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\widehat{\xi}_n)}\left\{\mu'_{-1}(\varsigma,\tau;\widehat{\xi}_n)+\left[B\left[\widehat{\xi}_n\right]+2\mathrm{i}\theta'(\varsigma,\tau;\widehat{\xi}_n)\right]\mu_{-1}(\varsigma,\tau;\widehat{\xi}_n)\right\}. \quad (47)
\end{aligned}$$

In the case of double poles, the RH problem given by Proposition 3 still holds. However, solving this kind of RH problem requires subtracting the asymptotic values as $\rho\to\infty$ and $\rho\to 0$, and considering the singular contributions. Consequently, we can derive the regular RH problem as follows:

$$M^-(\varsigma,\tau;\rho)-M_{\mathrm{fdp}}(\varsigma,\tau;\rho)=M^+(\varsigma,\tau;\rho)-M_{\mathrm{fdp}}(\varsigma,\tau;\rho)-M^+(\varsigma,\tau;\rho)G, \quad (48)$$

where

$$M_{\mathrm{fdp}}(\varsigma,\tau;\rho)=I+\frac{\mathrm{i}}{\rho}\sigma_3 Q_- +\sum_{n=1}^{2K}\left[\frac{\underset{\rho=\xi_n}{P_{-2}M^+}}{(\rho-\xi_n)^2}+\frac{\underset{\rho=\xi_n}{\operatorname{Res}M^+}}{\rho-\xi_n}+\frac{\underset{\rho=\widehat{\xi}_n}{P_{-2}M^-}}{(\rho-\widehat{\xi}_n)^2}+\frac{\underset{\rho=\widehat{\xi}_n}{\operatorname{Res}M^+}}{\rho-\widehat{\xi}_n}\right].$$

Therefore, through the Plemelj's formulae, we acquire that

$$M(\varsigma,\tau;\rho) = M_{\mathrm{fdp}}+\frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{M^+(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta)}{\zeta-\rho}\mathrm{d}\zeta,\quad \rho\in\mathbb{C}\setminus\Sigma. \quad (49)$$

By means of formulas (48) and (49), we deduce that

$$\begin{aligned}
&\frac{\underset{\rho=\xi_n}{P_{-2}M^+}}{(\rho-\xi_n)^2}+\frac{\underset{\rho=\xi_n}{\operatorname{Res}M^+}}{\rho-\xi_n}+\frac{\underset{\rho=\widehat{\xi}_n}{P_{-2}M^-}}{\left(\rho-\widehat{\xi}_n\right)^2}+\frac{\underset{\rho=\widehat{\xi}_n}{\operatorname{Res}M^+}}{\rho-\widehat{\xi}_n}\\
&= \left(D[\rho]\left[\mu'_{-2}(\xi_n)+\left(C_n+\frac{1}{\rho-\xi_n}\right)\mu_{-2}(\xi_n)\right],\widehat{D}[\rho]\left[\mu'_{-2}(\widehat{\xi}_n)+\left(\widehat{C}_n+\frac{1}{\rho-\widehat{\xi}_n}\right)\mu_{-1}(\widehat{\xi}_n)\right]\right), \quad (50)\\
&C_n=B_+[\xi_n]-2\mathrm{i}\theta'(\xi_n),\quad \widehat{C}_n=B_-\left[\widehat{\xi}_n\right]+2\mathrm{i}\theta'(\widehat{\xi}_n). \quad (51)
\end{aligned}$$

Furthermore, we derive the asymptotic behavior of $M(\varsigma,\tau;\zeta)$:

$$\begin{aligned}
M^{(1)}(\varsigma,\tau,\rho) &= \mathrm{i}\sigma_3 Q_- -\frac{1}{2\pi\mathrm{i}}\int_\Sigma M^+(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta)\mathrm{d}\zeta+\sum_{n=1}^{2K}\left[A_+[\xi_n]\,\mathrm{e}^{-2\mathrm{i}\theta(\varsigma,\tau;\xi_n)}\left(\mu'_{-2}(\varsigma,\tau;\xi_n)+C_n\mu_{-2}(\varsigma,\tau;\widehat{\xi}_n)\right)\right.\\
&\left.\times A_-\left[\widehat{\xi}_n\right]\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\widehat{\xi}_n)}\left(\mu'_{-1}(\varsigma,\tau;\widehat{\xi}_n)+\widehat{C}_n\mu_{-1}(\varsigma,\tau;\widehat{\xi}_n)\right)\right]. \quad (52)
\end{aligned}$$

From formula (49) and the second column in formula (50), we have

$$\mu_{-2}(\varsigma,\tau;\rho) = \begin{pmatrix}\frac{\mathrm{i}q_-}{\rho}\\ 1\end{pmatrix}+\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{(M^+(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta))_2}{\zeta-\rho}\mathrm{d}\zeta+\sum_{n=1}^{2K}\widehat{D}[\rho]\left[\mu'_{-1}(\widehat{\xi}_n)+\left(\widehat{C}_n+\frac{1}{\rho-\widehat{\xi}_n}\right)\mu_{-1}(\varsigma,\tau;\widehat{\xi}_n)\right]. \quad (53)$$

By computing the first-order partial derivative of formula (53), we can obtain as

$$\mu'_{-2}(\varsigma,\tau;\rho)=\begin{pmatrix}\frac{\mathrm{i}q_-}{\rho^2}\\ 0\end{pmatrix}+\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{(M^+(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta))_2}{(\zeta-\rho)^2}\mathrm{d}\zeta+\sum_{n=1}^{2K}\frac{\widehat{D}_n[\rho]}{\rho-\widehat{\xi}_n}\left[\mu'_{-1}(\widehat{\xi}_n)+\left(\widehat{C}_n+\frac{2}{\rho-\widehat{\xi}_n}\right)\mu_{-1}(\varsigma,\tau;\widehat{\xi}_n)\right]. \quad (54)$$

According to the symmetry of μ, we can get

$$\mu'_{-2}(\varsigma,\tau;\rho)=-\frac{\mathrm{i}q_-}{\rho^2}\mu_{-1}\left(-\frac{q_0^2}{\rho}\right)+\frac{\mathrm{i}q_0^2q_-}{\rho^3}\mu'_{-1}\left(-\frac{q_0^2}{\rho}\right). \quad (55)$$

Similar to the case of simple poles, we substitute formulas (32) and (55) into formulas (49) and (54). By setting $\rho=\xi_s$, $s=1,2,\ldots,2K$, we deduce the following $4K$ linear systems:

$$\left[\widehat{D}_n[\xi_s]\left(\widehat{C}_n+\frac{1}{\xi_s-\xi_n}\right)-\frac{\mathrm{i}q_-}{\xi_s}\delta_{n,s}\right]\mu_{-1}(\widehat{\xi}_n)+\sum_{n=1}^{2K}\widehat{D}_n[\xi_s]\mu'_{-1}(\widehat{\xi}_n)=-\begin{pmatrix}\frac{\mathrm{i}q_-}{\xi_s}\\ 1\end{pmatrix}-\int_\Sigma\frac{(M^+(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta))_2}{2\pi\mathrm{i}(\zeta-\xi_s)}\mathrm{d}\zeta, \quad (56a)$$

$$\left[\widehat{D}_n[\xi_s]\left(\widehat{C}_n+\frac{2}{\xi_s-\widehat{\xi}_n}\right)-\frac{\mathrm{i}q_-}{\xi_s^2}\delta_{n,s}\right]\mu_{-1}(\widehat{\xi}_n)+\sum_{n=1}^{2K}\left(\frac{\widehat{D}_n[\xi_s]}{\xi_s-\widehat{\xi}_n}+\frac{\mathrm{i}q_0^2q_-}{\xi_s^3}\delta_{n,s}\right)\mu'_{-1}(\widehat{\xi}_n)=\begin{pmatrix}\frac{\mathrm{i}q_-}{\xi_s^2}\\ 0\end{pmatrix}+\int_\Sigma\frac{(M^+(\varsigma,\tau;\zeta)G(\varsigma,\tau,\zeta))_2}{2\pi\mathrm{i}(\zeta-\xi_s)^2}\mathrm{d}\zeta. \quad (56b)$$

Therefore, the potential $q(\varsigma,\tau)$ with double poles of Eq. (4) is given by

$$\begin{aligned} q(\varsigma,\tau) = q_- - \mathrm{i}\sum_{n=1}^{2K} A_-\left[\widehat{\xi}_n\right] \mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\widehat{\xi}_n)} \\ \times\left(\mu'_{-11}(\varsigma,\tau;\widehat{\xi}_n)+\widehat{C}_n\mu_{-11}(\varsigma,\tau;\widehat{\xi}_n)\right) \\ +\frac{1}{2\pi}\int_\Sigma \left(M^+(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta)\right)_{12}\mathrm{d}\zeta. \end{aligned} \tag{57}$$

Based on the preceding analysis of the simple poles, we can derive the trace formulae to double poles as

$$\begin{aligned} s_{11}(\rho) = \exp\left[\frac{\mathrm{i}}{2\pi}\int_\Sigma \frac{\log\left[1+\eta(\zeta)\eta^*(\zeta^*)\right]}{\zeta-\rho}\mathrm{d}\zeta\right] \\ \times\prod_{n=1}^{2K}\frac{(\rho-\rho_n)^2\left(\rho+\frac{q_0^2}{\rho_n^*}\right)^2}{(\rho-\rho_n^*)^2\left(\rho+\frac{q_0^2}{\rho_n}\right)^2}, \quad \rho\in D_f^+, \end{aligned} \tag{58a}$$

$$\begin{aligned} s_{22}(\rho) = \exp\left[-\frac{\mathrm{i}}{2\pi}\int_\Sigma \frac{\log\left[1+\eta(\zeta)\eta^*(\zeta^*)\right]}{\zeta-\rho}\mathrm{d}\zeta\right] \\ \times\prod_{n=1}^{2K}\frac{(\rho-\rho_n^*)^2\left(\rho+\frac{q_0^2}{\rho_n}\right)^2}{(\rho-\rho_n)^2\left(\rho+\frac{q_0^2}{\rho_n^*}\right)^2}, \quad \rho\in D_f^-. \end{aligned} \tag{58b}$$

Consequently, taking the limit $\rho\to 0$ in $s_{11}(\rho)$, we obtain the theta condition as

$$\begin{aligned} \arg\left(\frac{q_+}{q_-}\right) = 8\sum_{n=1}^{K}\arg(\rho_n) \\ +\frac{1}{2\pi}\int_\Sigma \frac{\log\left[1+\eta(\zeta)\eta^*(\zeta^*)\right]}{\zeta}\mathrm{d}\zeta. \end{aligned} \tag{59}$$

Particularly, when $\eta(\rho)=\widehat{\eta}(\rho)=0$, it is a reflectionless case. Therefore, we have the following proposition:

Proposition 5 The reflectionless potential with double poles under NZBCs of Eq. (4) can be obtained in the form of determinants

$$q(\varsigma,\tau) = q_- + \mathrm{i}\frac{\det\begin{bmatrix} F & g \\ h^{\mathrm{T}} & 0\end{bmatrix}}{\det F}, \tag{60}$$

where

$$\begin{aligned} &F=\begin{pmatrix} F^{11} & F^{12} \\ F^{21} & F^{22}\end{pmatrix}, \\ &F^{(sj)}=\left(f_{mn}^{sj}\right)_{2K\times 2K} \quad (s,j=1,2), \\ &g=\begin{pmatrix} g_n^1 \\ g_n^2\end{pmatrix} \quad (n=1,\ldots,2N), \\ &h=\begin{pmatrix} h_n^1 \\ h_n^2\end{pmatrix} \quad (n=1,\ldots,2N), \end{aligned}$$

with

$$\begin{aligned} &f_{mn}^{11}=\widehat{D}_n(\xi_m)\left(\widehat{C}_n+\frac{1}{\xi_m-\widehat{\xi}_n}\right)-\frac{\mathrm{i}q_-}{\xi_m}\delta_{m,n}, \\ &f_{mn}^{12}=\widehat{D}_n(\xi_m), \\ &f_{mn}^{22}=\frac{\widehat{D}_n(\xi_m)}{\xi_m-\widehat{\xi}_n}+\frac{\mathrm{i}q_-q_0^2}{\xi_m^3}\delta_{m,n}, \\ &f_{mn}^{21}=\frac{\widehat{D}_n(\xi_m)}{\xi_m-\widehat{\xi}_n}\left(\widehat{C}_n+\frac{2}{\xi_m-\widehat{\xi}_n}\right)-\frac{\mathrm{i}q_-}{\xi_m^2}\delta_{m,n}, \\ &g_n^1=-\frac{\mathrm{i}q_-}{\xi_n}, \quad g_n^2=-\frac{\mathrm{i}q_-}{\xi_n^2}, \\ &h_n^1=A_-\left[\widehat{\xi}_n\right]\mathrm{e}^{2\mathrm{i}\theta\left(\varsigma,\tau,\widehat{\xi}_n\right)}\widehat{C}_n, \\ &h_n^2=A_-\left[\widehat{\xi}_n\right]\mathrm{e}^{2\mathrm{i}\theta\left(\varsigma,\tau,\widehat{\xi}_n\right)}. \end{aligned}$$

Moreover, the trace formulae and theta condition are

$$\begin{aligned} &s_{11}(\rho)=\prod_{n=1}^{2K}\frac{(\rho-\rho_n)^2\left(\rho+\frac{q_0^2}{\rho_n^*}\right)^2}{(\rho-\rho_n^*)^2\left(\rho+\frac{q_0^2}{\rho_n}\right)^2}, \quad \rho\in D_f^+, \\ &s_{22}(\rho)=\prod_{n=1}^{2K}\frac{(\rho-\rho_n^*)^2\left(\rho+\frac{q_0^2}{\rho_n}\right)^2}{(\rho-\rho_n)^2\left(\rho+\frac{q_0^2}{\rho_n^*}\right)^2}, \quad \rho\in D_f^-, \\ &\arg\left(\frac{q_+}{q_-}\right)=8\sum_{n=1}^{K}\arg(\rho_n). \end{aligned}$$

3. The defocusing case of Eq. (4)

In this section, we deduce the soliton solutions of Eq. (4) for defocusing ($\sigma=-1$) case under NZBCs.

3.1. Direct scattering under NZBCs

We study the asymptotic problem ($\varsigma\to\pm\infty$) of the Lax pair (6) for the defocusing case:

$$\begin{aligned} &\Phi_\varsigma = U_\pm\Phi, \quad U_\pm=\lim_{\varsigma\to\pm\infty}U=\mathrm{i}k\sigma_3+Q_\pm^{\mathrm{d}}, \\ &\Phi_\tau = V_\pm\Phi, \quad V_\pm=\lim_{\varsigma\to\pm\infty}V=\Xi^{\mathrm{d}}U_\pm, \end{aligned} \tag{61}$$

with $Q_\pm^{\mathrm{d}}=\begin{pmatrix} 0 & q_\pm \\ -q_\pm^* & 0\end{pmatrix}$. In this situation, Lax pairs (61) admit the fundamental matrix solutions

$$\begin{aligned} &\Phi_\pm^{\mathrm{d}}(\varsigma,\tau;k) \\ &=\begin{cases} Y_\pm^{\mathrm{d}}\mathrm{e}^{\mathrm{i}\theta(\varsigma,\tau;k)\sigma_3}, & k\neq\pm q_0, \\ I+\left[\varsigma-\left(k+2k^2\beta+28k^5\delta\right)\tau\right]U_\pm, & k=\pm q_0, \end{cases} \end{aligned} \tag{62}$$

where

$$Y_\pm^{\mathrm{d}}=\begin{pmatrix} 1 & \frac{\mathrm{i}q_\pm}{k+\lambda(k)} \\ -\frac{\mathrm{i}q_\pm^*}{k+\lambda(k)} & 1\end{pmatrix},$$

$\theta(\varsigma,\tau;k)=\lambda(k)(\varsigma+\Xi^{\mathrm{d}}\tau)$, $\Xi^{\mathrm{d}}=-[k+(4k^2+2q_0^2)\beta+4(8k^5+4k^3q_0^2+3kq_0^4)\delta]$, $\lambda^2(k)=k^2-q_0^2$.

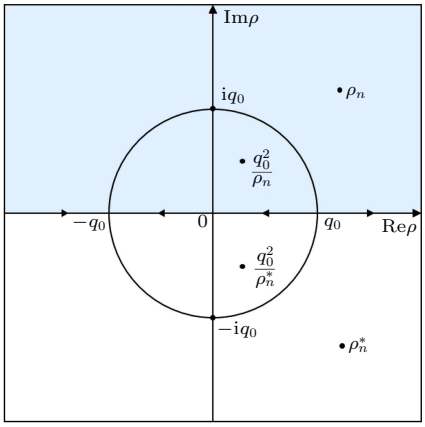


Fig. 2. The discrete spectrum and jump curve distribution of the RH problem on the complex ρ-plane, the wathet area stands for D^+, the white area stands for D^-.

Unlike the focusing case, $\lambda(k)$ satisfies $\lambda^2(k)=k^2-q_0^2$, and the branch cut being the segment $q_0[-1,1]$. The mappings between the two-sheeted Riemann k-surface, complex λ-plane and complex ρ-plane are presented in Fig. 2.

Under boundary conditions (5), we derive the Jost solution

$$\Phi_{\pm}^{\mathrm{d}}(\varsigma,\tau;k)=Y_{\pm}^{\mathrm{d}}\,\mathrm{e}^{\mathrm{i}\theta(\varsigma,\tau;\rho)\sigma_3}+O(1),\quad \varsigma\to\pm\infty. \tag{63}$$

Through the modified Jost solution

$$\mu_{\pm}(\varsigma,\tau;\rho)=\Phi_{\pm}^{\mathrm{d}}(\varsigma,\tau;\rho)\,\mathrm{e}^{-\mathrm{i}\theta(\varsigma,\tau;\rho)\sigma_3}, \tag{64}$$

we obtain

$$\mu_{\pm}(\varsigma,\tau;\rho)=\begin{cases}Y_{\pm}^{\mathrm{d}}+\displaystyle\int_{\pm\infty}^{\varsigma}Y_{\pm}^{\mathrm{d}}\mathrm{e}^{\mathrm{i}\lambda(\varsigma-\varsigma')\hat{\sigma}_3}\left[(Y_{\pm}^{\mathrm{d}})^{-1}\Delta U_{\pm}(\varsigma',\tau)\mu_{\pm}(\varsigma',\tau;\rho)\right]\mathrm{d}\varsigma', & \rho\neq\pm q_0,\\ Y_{\pm}^{\mathrm{d}}+\displaystyle\int_{\pm\infty}^{\varsigma}\left[I+(\varsigma-\varsigma')(Q_{\pm}\pm\mathrm{i}q_0\sigma_3)\right]\Delta U_{\pm}(\varsigma',\tau)\mu_{\pm}(\varsigma',\tau;\rho)\mathrm{d}\varsigma', & \rho=\pm q_0.\end{cases} \tag{65}$$

Proposition 6 Set $q-q_{\pm}\in L^1(\mathbb{R}^+)$, the modified forms $\mu_{\pm}(\varsigma,\tau;\rho)$ have unique solutions in $\Sigma_0^{\mathrm{d}}=\Sigma\setminus\{\pm q_0\}$, and $\mu_{\pm}(\Sigma,\tau;\rho)$ yields:

P6-1 $\mu_{-1}(\varsigma,\tau;\rho)$ and $\mu_{+2}(\varsigma,\tau;\rho)$ are analytical in $\mathbb{C}^-$ and continuous in $\mathbb{C}^-\bigcup\Sigma_0^{\mathrm{d}}$.

P6-2 $\mu_{+1}(\varsigma,\tau;\rho)$ and $\mu_{-2}(\varsigma,\tau;\rho)$ are analytical in $\mathbb{C}^+$ and continuous in $\mathbb{C}^+\bigcup\Sigma_0^{\mathrm{d}}$.

P6-3 $\mu_{\pm}(\varsigma,\tau;\rho)\to I$, as $\rho\to\pm\infty$.

P6-4 $\mu_{\pm}(\varsigma,\tau;\rho)\to\frac{\mathrm{i}}{\rho}\sigma_3Q_{\pm}$, as $\rho\to 0$.

P6-5 $\det\mu_{\pm}(\varsigma,\tau;\rho)=\det Y_{\pm}=\gamma=1-q_0^2/\rho^2$.

In this case, there remains a constant matrix $S(\rho)$ that satisfies the relationship $\phi_+(\varsigma,\tau;\rho)=\phi_-(\varsigma,\tau;\rho)S(\rho)$, $\rho\in\Sigma_0^{\mathrm{d}}$. We can derive the symmetries of $U(\varsigma,\tau;\rho)$, $V(\varsigma,\tau;\rho)$ and $\theta(\varsigma,\tau;\rho)$ as

$$\begin{aligned}
&U(\varsigma,\tau;\rho)=-\sigma_1U^*(\varsigma,\tau;\rho^*)\sigma_1,\\
&V(\varsigma,\tau;\rho)=-\sigma_1V^*(\varsigma,\tau;\rho^*)\sigma_1,\\
&\theta(\varsigma,\tau;\rho)=\theta^*(\varsigma,\tau;\rho^*),\\
&U(\varsigma,\tau;\rho)=U\left(\varsigma,\tau;\frac{q_0^2}{\rho}\right),\\
&V(\varsigma,\tau;\rho)=V\left(\varsigma,\tau;\frac{q_0^2}{\rho}\right),\\
&\theta(\varsigma,\tau;\rho)=-\theta^*\left(\varsigma,\tau;\frac{q_0^2}{\rho}\right).
\end{aligned} \tag{66}$$

Therefore, we have

$$\begin{aligned}
&\Phi_{\pm}(\varsigma,\tau;\rho)=\sigma_1\Phi_{\pm}^*(\varsigma,\tau;\rho^*)\sigma_1,\\
&\mu_{\pm}(\varsigma,\tau;\rho)=\sigma_1\mu_{\pm}^*(\varsigma,\tau;\rho^*)\sigma_1,\\
&\Phi_{\pm}(\varsigma,\tau;\rho)=\frac{\mathrm{i}}{2}\Phi_{\pm}\left(\varsigma,\tau;\frac{q_0^2}{\rho}\right),\\
&\mu_{\pm}(\varsigma,\tau;\rho)=\frac{\mathrm{i}}{2}\Phi_{\pm}\left(\varsigma,\tau;\frac{q_0^2}{\rho}\right))\sigma_3Q_{\pm},
\end{aligned} \tag{67}$$

and hence

$$\begin{aligned}
&S(\rho)=\sigma_1S^*(\varsigma,\tau;\rho^*)\sigma_1,\\
&S(\rho)=(\sigma_3Q_-)^{-1}S\left(\varsigma,\tau;\frac{q_0^2}{\rho}\right)\sigma_3Q_+.
\end{aligned} \tag{68}$$

Furthermore, we can deduce the symmetries between $\eta(\rho)$ and $\widehat{\eta}(\rho)$ as

$$\eta(\rho)=\widehat{\eta}^*(\rho^*),\quad \eta(\rho)=-\frac{q_-^*}{q_-}\widehat{\eta}\left(\frac{q_0^2}{\rho}\right). \tag{69}$$

3.2. The inverse scattering and simple poles

Because of $(1+|\varsigma|)^4(q-q_{\pm}\in L^1(\mathbb{R}^+))$, there are limited discrete spectral points belonging to $\{|\rho|=q_0\}$. In this case, we have the discrete spectrum as $Z^{\mathrm{d}}=\left\{\rho_n,\widehat{\rho_n^*}\right\}_{n=1}^{K}$, $\rho_n^*=q_0^2/\rho_n$ with $\mathrm{Im}(\rho_n)>0$. Next, we attain the symmetry of $A[\rho_0]$ as

$$A_+[\rho_0]=A^*[\rho_0^*]=\frac{\rho_0^2}{q_-^2}A_-\left[\frac{q_0^2}{\rho_0}\right],\quad \rho_0\in\rho^{\mathrm{d}}\cap\mathbb{C}^+, \tag{70}$$

which yields

$$\mathrm{Im}A_+[\rho_n]=\left(\mathrm{Re}\left(\frac{q_-^2}{\rho_n^2}\right)-1\right)\frac{\mathrm{Re}A_+(\rho_n)}{\mathrm{Im}\left(q_-^2/\rho_n\right)}, \tag{71}$$

with $\mathrm{Im}\left(q_-^2/\rho_n^2\right)\neq 0$, $A_+[\rho_n]\in\mathbb{R}$, as $\mathrm{Re}\left(q_-^2/\rho_n^2\right)=1$ and $A_+[\rho_n]\in\mathrm{i}\mathbb{R}$, as $\mathrm{Re}\left(q_-^2/\rho_n^2\right)=-1$. By comparing and analyzing the asymptotic behaviors for modified Jost solution and scattering data of focusing and defocusing cases, we conclude that the RH problem in these two cases is the same.

As mentioned above, the first (second) column of formula (23) exhibits a simple pole at $\rho=\rho_n(\rho=\rho_n^*)$. Thus, the residue of $M(\varsigma,\tau;\rho)$ is

$$\operatorname*{Res}_{\rho=\rho_n}M^+(\varsigma,\tau;\rho)$$

$$= \left(A_+[\rho_n]\,\mathrm{e}^{-2\mathrm{i}\theta(\varsigma,\tau;\rho_n)}\mu_{-2}(\varsigma,\tau;\rho_n),0\right), \tag{72a}$$

$$\operatorname*{Res}_{\rho=\widehat{\rho}_n} M^-(\varsigma,\tau;\rho)$$

$$= \left(0,A_-[\widehat{\rho}_n]\,\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\widehat{\rho}_n)}\mu_{-1}(\varsigma,\tau;\widehat{\rho}_n)\right), \tag{72b}$$

which leads the following formula:

$$\begin{aligned}&\frac{\operatorname*{Res}_{\rho=\rho_n} M^+(\varsigma,\tau;\rho)}{\rho-\rho_n}+\frac{\operatorname*{Res}_{\rho=\widehat{\rho}_n} M^-(\varsigma,\tau;\rho)}{\rho-\widehat{\rho}_n}\\ &=\left(\frac{A_+[\rho_n]\,\mathrm{e}^{-2\mathrm{i}\theta(\varsigma,\tau;\rho_n)}}{\rho-\rho_n}\mu_{-2}(\varsigma,\tau;\rho_n),\right.\\ &\left.\frac{A_-[\widehat{\rho}_n]\,\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\widehat{\rho}_n)}}{\rho-\widehat{\rho}_n}\mu_{-1}(\varsigma,\tau;\widehat{\rho}_n)\right).\end{aligned} \tag{73}$$

Substracying out the asymptotic value as $\rho\to\infty$, $\rho\to 0$ and the residue, we derive that

$$\begin{aligned}&M^-(\varsigma,\tau;\rho)-I-\frac{\mathrm{i}}{\rho}\sigma_3 Q_-\\ &-\sum_{n=1}^{K}\left[\frac{\operatorname*{Res}_{\rho=\rho_n} M^+(\varsigma,\tau;\rho)}{\rho-\rho_n}+\frac{\operatorname*{Res}_{\rho=\widehat{\rho}_n} M^-(\varsigma,\tau;\rho)}{\rho-\widehat{\rho}_n}\right]\\ &=M^+(\varsigma,\tau;\rho)-I-\frac{\mathrm{i}}{\rho}\sigma_3 Q_-\\ &-\sum_{n=1}^{K}\left[\frac{\operatorname*{Res}_{\rho=\rho_n} M^+(\varsigma,\tau;\rho)}{\rho-\rho_n}+\frac{\operatorname*{Res}_{\rho=\widehat{\rho}_n} M^-(\varsigma,\tau;\rho)}{\rho-\widehat{\rho}_n}\right]\\ &-M^+(\varsigma,\tau;\rho)G(\varsigma,\tau;\rho).\end{aligned} \tag{74}$$

Therefore, through the Plemelj's formulae, we have

$$\begin{aligned}M(\varsigma,\tau;\rho) &= I+\frac{\mathrm{i}}{\rho}\sigma_3 Q_-\\ &+\sum_{n=1}^{K}\left[\frac{\operatorname*{Res}_{\rho=\rho_n} M^+(\varsigma,\tau;\rho)}{\rho-\rho_n}+\frac{\operatorname*{Res}_{\rho=\widehat{\rho}_n} M^-(\varsigma,\tau;\rho)}{\rho-\widehat{\rho}_n}\right]\\ &+\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{M^+(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta)}{\zeta-\rho}\mathrm{d}\zeta,\\ &\rho\in\mathbb{C}\backslash\Sigma^{\mathrm{d}}.\end{aligned} \tag{75}$$

In this end, let

$$M(\varsigma,\tau;\rho)=I+\frac{1}{\rho}M^{(1)}(\varsigma,\tau;\rho)+O\left(\frac{1}{\rho^2}\right),\quad \rho\to\infty,$$

with

$$\begin{aligned}M^{(1)}(\varsigma,\tau;\rho) &= \mathrm{i}\sigma_3 Q_- -\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}M^+(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta)\mathrm{d}\zeta\\ &+\sum_{n=1}^{K}\left[A_+[\rho_n]\,\mathrm{e}^{-2\mathrm{i}\theta(\varsigma,\tau;\rho_n)}\mu_{-2}(\varsigma,\tau;\rho_n),\right.\\ &\left.A_-[\widehat{\rho}_n]\,\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau,\widehat{\rho}_n)}\mu_{-1}(\varsigma,\tau,\widehat{\rho}_n)\right].\end{aligned}$$

Similarly, considering the second column of formulas (73) and (75) at $\rho=\rho_j$, $(j=1,2,\ldots,K)$, we derive that

$$\begin{aligned}\mu_{-2}(\varsigma,\tau;\rho_j) &= \begin{pmatrix}\frac{\mathrm{i}q_-}{\rho_j}\\ 1\end{pmatrix}+\sum_{n=1}^{K}\frac{A_-[\widehat{\rho}_n]\,\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\widehat{\rho}_n)}}{\rho_j-\widehat{\rho}_n}\mu_{-1}(\widehat{\rho}_n)\\ &+\frac{1}{2\pi\mathrm{i}}\int_{\Sigma^{\mathrm{d}}}\frac{(M^+G)_2(\varsigma,\tau;\zeta)}{\zeta-\rho_j}\mathrm{d}\zeta.\end{aligned} \tag{76}$$

Furthermore, according to formula (15), we get

$$\begin{aligned}\mu_{-2}(\varsigma,\tau;\rho_j) &= \frac{\mathrm{i}q_-}{\rho_j}\mu_{-1}(\varsigma,\tau;\widehat{\rho}_j)\\ &= \frac{\mathrm{i}q_-}{\rho_j}\mu_{-1}\left(\varsigma,\tau;\frac{q_0^2}{\rho_j}\right).\end{aligned} \tag{77}$$

By employing formulas (76) and (77), we obtain

$$\begin{aligned}&\sum_{n=1}^{K}\left(\frac{A_-[\widehat{\rho}_n]\,\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\widehat{\rho}_n)}}{\rho_j-\widehat{\rho}_n}-\frac{\mathrm{i}q_-}{\rho_j}\delta_{n,j}\right)\\ &\times\mu_{-1}(\varsigma,\tau;\widehat{\rho}_n)+\begin{pmatrix}\frac{\mathrm{i}q_-}{\rho_j}\\ 1\end{pmatrix}\\ &=-\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{(M^+G)_2(\varsigma,\tau;\zeta)}{\zeta-\rho_j}\mathrm{d}\zeta.\end{aligned} \tag{78}$$

Then, the potential $q(\varsigma,\tau)$ of Eq. (4) for defocusing $(\sigma=-1)$ case can be expressed as

$$\begin{aligned}q(\varsigma,\tau) &= q_- -\mathrm{i}\sum_{n=1}^{K}A_-[\widehat{\rho}_n]\,\mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau;\widehat{\rho}_n)}\mu_{-11}(\varsigma,\tau;\widehat{\rho}_n)\\ &+\frac{1}{2\pi}\int_{\Sigma^{\mathrm{d}}}\left(M^+(\varsigma,\tau;\zeta)G(\varsigma,\tau;\zeta)\right)_{12}\mathrm{d}\zeta.\end{aligned} \tag{79}$$

Similar to the focusing case, the trace formulae and theta condition can be obtained as

$$\begin{aligned}s_{11}(\rho) &= \exp\left(\frac{\mathrm{i}}{2\pi}\int_{\Sigma^{\mathrm{d}}}\frac{\log[1+\eta(\zeta)\eta^*(\zeta^*)]}{\zeta-\rho}\mathrm{d}\zeta\right)\\ &\times\prod_{n=1}^{K}\frac{(\rho-\rho_n)}{(\rho-\rho_n^*)},\end{aligned} \tag{80a}$$

$$\begin{aligned}s_{22}(\rho) &= \exp\left(-\frac{\mathrm{i}}{2\pi}\int_{\Sigma^{\mathrm{d}}}\frac{\log[1+\eta(\zeta)\eta^*(\zeta^*)]}{\zeta-\rho}\mathrm{d}\zeta\right)\\ &\times\prod_{n=1}^{K}\frac{(\rho-\rho_n^*)}{(\rho-\rho_n)},\end{aligned} \tag{80b}$$

$$\begin{aligned}\arg\left(\frac{q_+}{q_-}\right) &= 2\sum_{n=1}^{K}\arg\rho_n\\ &+\frac{1}{2\pi}\int_{\Sigma}\frac{\log[1+\eta(\zeta)\eta^*(\zeta^*)]}{\zeta}\mathrm{d}\zeta.\end{aligned} \tag{81}$$

Particularly, considering a special kind of solutions as $\eta(\rho)=\widehat{\eta}(\rho)=0$, we can get the following proposition:

Proposition 7 The reflection potential with simple poles under NZBCs of Eq. (4) can be derived as the determinants

$$q(\varsigma,\tau)=q_-+\mathrm{i}\frac{\det\begin{bmatrix}F & g\\ h^{\mathrm{T}} & 0\end{bmatrix}}{\det F}, \tag{82}$$

where $F=(f_{sj})_{K\times K}$, $g=(g_s)_{K\times 1}$, and $h=(h_j)_{K\times 1}$ with

$$f_{sj}=\frac{h_s}{\rho-\widehat{\rho}_s}+g_s\delta_{s,j},\quad g_s=-\frac{\mathrm{i}q_-}{\rho_s},$$

$$h_j = A_-[\widehat{\rho}_j]\, \mathrm{e}^{2\mathrm{i}\theta(\varsigma,\tau,\widehat{\rho}_j)}.$$

Moreover, the trace formulae and theta condition are

$$s_{11}(\rho) = \prod_{n=1}^{K} \frac{(\rho-\rho_n)}{(\rho-\rho_n^*)},\ \rho \in \mathbb{C}^+,$$

$$s_{22}(\rho) = \prod_{n=1}^{K} \frac{(\rho-\rho_n^*)}{(\rho-\rho_n)}\ \rho \in \mathbb{C}^-,$$

$$\arg\left(\frac{q_+}{q_-}\right) = 2\sum_{n=1}^{K} \arg(\rho_n).$$

4. Discussion on fundamental and double-pole breathers and solitons for Eq. (4)

In what follows, based on simple-pole solutions (41) and (82), and double-pole solution (60), we will perform an analysis of soliton propagation and interaction characteristics through the adjustment of relevant parameters, and investigate the dynamical behaviors of both breathers and solitons.

Firstly, we consider the focusing case ($\sigma = 1$). When $K = 1$, $q_- = 1$, $\rho = \frac{3}{2}\mathrm{i}$, $\beta = \delta = 0.1$, we present fundamental breathers with the periodicity in time, which exhibit strong interactions between solitons, as shown in Fig. 3(a). Keeping β invariant and δ increasing, we find that the period of the breather is shortened, as illustrated in Fig. 3(b). On the other hand, when β changes, we observe that the temporal periodic breathers transform into the spatial-temporal periodic breathers, as shown in Fig. 3(c).

Comparing Fig. 4(a) with Fig. 4(b), we notice that by keeping the other parameters invariant and only varying q_-, the fundamental breather gradually transforms into a bright soliton. In addition, we analyze the influence of the eigenvalue ρ on this solution. Comparing Fig. 4(c) with Fig. 4(a), we find that by changing the eigenvalue ρ, the temporal periodic breather can be converted into the spatial-temporal periodic breather.

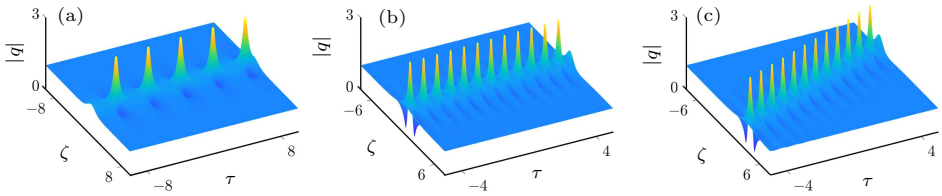


Fig. 3. Fundamental breathers via simple-pole solutions (41) with $q_- = 1$, $\rho = 1.5\mathrm{i}$: (a) $\beta = 0.01$, $\delta = 0.01$; (b) $\beta = 0.01$, $\delta = 0.1$; (c) $\beta = 0.1$, $\delta = 0.1$.

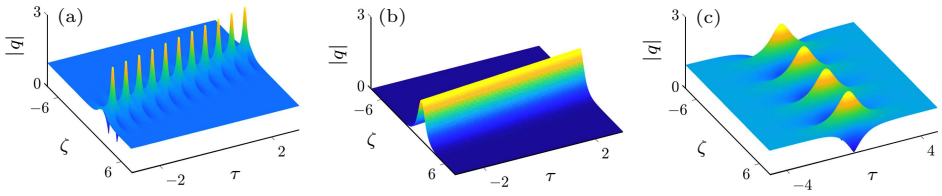


Fig. 4. (a) Temporal periodic breather via simple-pole solutions (41) with NZBC $q_- = 1$, $\beta = \delta = 0.1$, $\rho = \frac{5}{3}\mathrm{i}$; (b) Bright soliton via simple-pole solutions (41) with ZBC $q_- \to 0$, $\beta = \delta = 0.1$, $\rho = \frac{5}{3}\mathrm{i}$; (c) Spatio periodic breather via the breather simple-pole solutions (41) with NZBC $q_- = 1$, $\beta = \delta = 0.1$, $\rho = \frac{3}{4} + \frac{2}{3}\mathrm{i}$.

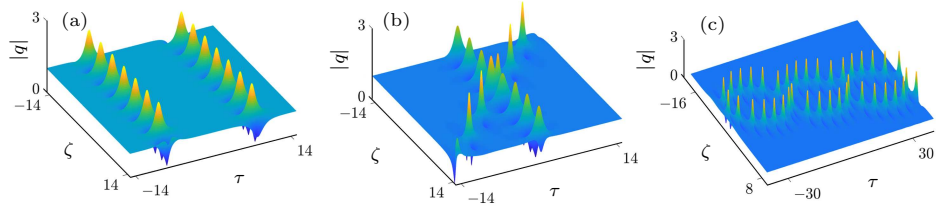


Fig. 5. Simple-pole two breathers via simple-pole solutions (41) with $q_- = 1$, $\beta = \delta = 0.01$: (a) $\rho_1 = \frac{3}{4} + \frac{2}{3}\mathrm{i}$, $\rho_2 = -\frac{3}{4} + \frac{2}{3}\mathrm{i}$; (b) $\rho_1 = \frac{3}{4} + \frac{2}{3}\mathrm{i}$, $\rho_2 = -\frac{1}{5} + \frac{4}{3}\mathrm{i}$; (c) $\rho_1 = -\frac{1}{60} + \frac{2}{3}\mathrm{i}$, $\rho_2 = -\frac{1}{60} + \frac{21}{30}\mathrm{i}$.

When $K = 2$, through analysis of different discrete eigenvalues, we obtain three distinct kinds of simple-pole two breathers, as shown in Fig. 5. Figure 5(a) depicts a pair of parallel breathers; figure 5(b) illustrates the interaction between a pair of intersecting solitons, where there is no change in the wave peaks before and after interaction, indicating the elastic interaction; figure 5(c) presents a pair of bound-state solitons. Similarly, by utilizing analogous calculations, higher-order breather solutions can be obtained.

Additionally, we consider the case of double-pole breather solutions for the NLS equation with the sextic operator. By selecting different characteristic parameter values, we obtain the interaction between two breathers, which reveals distinct features, as depicted in Figs. 6(a), 6(d) and 6(b), 6(e). When $q_- = 1$, $\beta = \delta = 0.01$ and $\rho = -\frac{3}{2}\mathrm{i}$, figures 6(a) and 6(d) illustrate a single peak in the interaction region. Conversely, when $\rho = -\frac{3}{2}\mathrm{i}$, we observe a distinctive "triangle" crest in the region of the interaction, as shown in Figs. 6(b) and

6(e). Through the density plot, we observe that the wave peaks remain unchanged before and after the intersection interaction of the two breathers. Furthermore, similar to the simple-pole case, when $q_- \to 0$, the boundary conditions become ZBCs, resulting in a smooth breather, as shown in Figs. 6(c) and 6(f).

In the following, we also briefly discuss the defocusing case $(\sigma = -1)$ of the NLS equation with the sextic operator. Figure 7(a) depicts a dark soliton. Setting $K = 2$, we observe the interaction between two dark solitons, as shown in Fig. 7(b). Analyzing the wave plots at different time points along the ς-axis in Fig. 7(c), we identify an elastic collision between the two dark solitons. During this interaction, the wave peaks attain their maximum height.

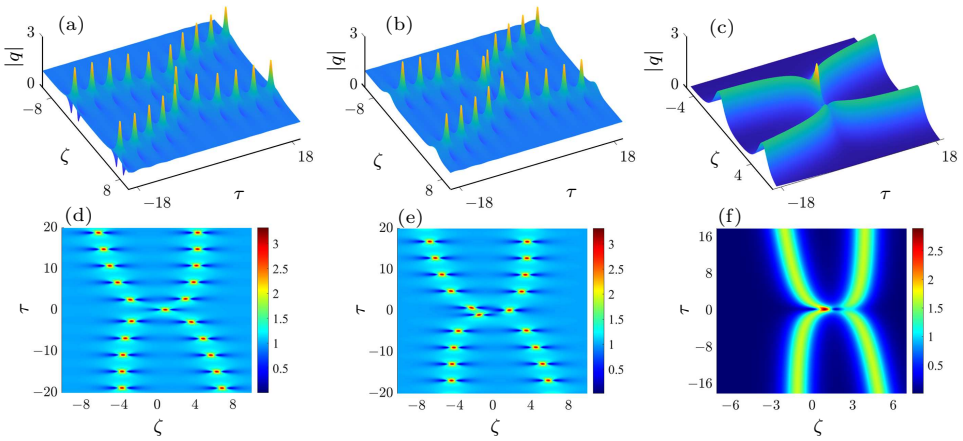


Fig. 6. Double-pole breathers via double-pole solutions (60) with (a) and (d) $q_- = 1$, $\beta = \delta = 0.01$, $\rho = \frac{3}{2}\mathrm{i}$; (b) and (e) $q_- = 1$, $\beta = \delta = 0.01$, $\rho = -\frac{3}{2}\mathrm{i}$; (c) and (f) $q_- \to 0$, $\beta = \delta = 0.01$, $\rho = -\frac{3}{2}\mathrm{i}$.

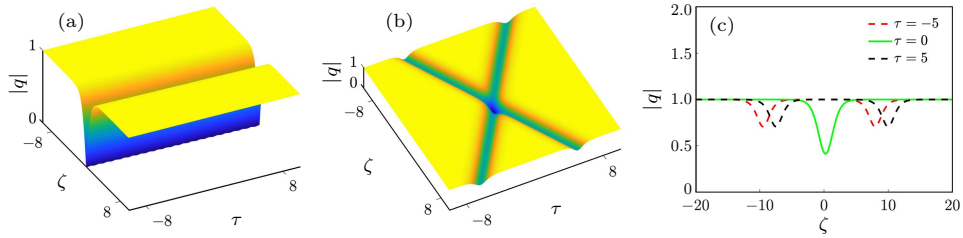


Fig. 7. Simple-pole dark solution via simple-pole solutions (82) with $q_- = 1$, (a) $K = 1$, $\beta = \delta = 0.01$, $\rho = \mathrm{i}$; (b) $K = 2$, $\beta = \delta = 0.05$, $\rho = \exp\left(\frac{\pi}{4}\mathrm{i}\right)$, $\rho = \exp\left(\frac{3\pi}{4}\mathrm{i}\right)$. (c) The wave propagation of simple pole solution along the ς axis at different time, with the same parameters as those in panel (b).

5. Conclusions

In this work, we have obtained simple-pole solutions (41) and (82), and double-pole solution (60) through the RH problem of the NLS with the sextic operator under NZBCs. Starting from Lax pair (6), we have constructed a generalized RH problem using the Jost function and the scattering data and its properties. Finally, the expressions of the solutions for Eq. (4) have been worked out by the Plemelj formula.

By means of simple-pole solutions (41) and double-pole solution (60) for the focusing case, the propagation and interaction characteristics of breathers have been studied, and the dynamic behaviors of solitons and breathers have been discussed. Moreover, we have got some interesting conclusions: Coefficients β and δ have an influence on the propagation direction, velocity and amplitude of solitons. For example, when the coefficient β changes, the temporal periodic breathers could be converted to the spatial-temporal periodic breathers in Fig. 3. We also have obtained different forms of two-breather solutions in Fig. 5. When considering double poles, we observed that the two breathers undergo two interactions, resulting in a distinctive "triangle" crest, as depicted in Fig. 6. When q_- tends to zero, we have found that the NZBCs problem could be transformed into the ZBCs problem, regardless of whether it is the simple pole or the double pole, as depicted in Figs. 4(b) and 6(c). We have also briefly discussed the defocusing case of the NLS with the sextic operator. For $N = 2$, we have observed that the two dark solitons interact with each other, reaching their peak height during the interaction, as presented in Fig. 7.

Some scholars have studied NLS equations with third-order operators, i.e., Hirota equations or NLS equations with sixth-order operators, via DT,[38–40] RH problem,[41,42] and so on.[43,44] NLS equations with both third- and sixth-order operators are still rarely studied. In this paper, we have discussed Eq. (4) under NZBCs through RH problem, and studied the interaction, evolution and propagation of waves under the combined influence of third-order and sixth-order operators. This study not only deepens our understanding of nonlinear wave phenomena, but also provides guidance for specific applications such as transmission media in fiber optic communications and boundary effects in plasma.

Appendix A

$$V = V_0 + kV_1 + k^2V_2 + k^3V_3 + k^4V_4 + k^5V_5 + k^6V_6.$$

The expressions for V_j $(j = 0, 1, \ldots, 6)$ are given as follows:

$$V_0 = \begin{pmatrix} A_0 & B_0 \\ -\sigma B_0^* & -A_0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} A_1 & B_1 \\ -\sigma B_1^* & -A_1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} A_2 & B_2 \\ -\sigma B_2^* & -A_2 \end{pmatrix}, \quad V_3 = \begin{pmatrix} A_3 & B_3 \\ -\sigma B_3^* & -A_3 \end{pmatrix},$$
$$V_4 = \begin{pmatrix} A_4 & B_4 \\ -\sigma B_4^* & -A_4 \end{pmatrix}, \quad V_5 = \begin{pmatrix} A_5 & B_5 \\ -\sigma B_5^* & -A_5 \end{pmatrix}, \quad V_6 = \begin{pmatrix} A_6 & B_6 \\ -\sigma B_6^* & -A_6 \end{pmatrix},$$

where

$$B_6 = 0, \quad A_6 = -32\mathrm{i}\delta, \quad B_5 = -32\delta q, \quad A_5 = 0, \quad B_4 = 16\mathrm{i}\delta q_\varsigma, \quad A_4 = 16\mathrm{i}\sigma\delta|q|^2,$$
$$B_3 = 8\delta\left(q_{\varsigma\varsigma} + 2\sigma|q|^2 q\right), \quad A_3 = -4\mathrm{i}\beta + 8\sigma\delta(q^* q_\varsigma - q q_\varsigma^*),$$
$$B_2 = -4\left(\beta q + 6\mathrm{i}\sigma\delta|q|^2 q_\varsigma + \mathrm{i}\delta q_{\varsigma\varsigma\varsigma}\right), \quad A_2 = -\mathrm{i} - 4\mathrm{i}\delta\sigma\left(3\sigma|q|^4 - |q_\varsigma|^2 + q_{\varsigma\varsigma}^* q + q_{\varsigma\varsigma} q^*\right),$$
$$B_1 = -q + 2\mathrm{i}\beta q_\varsigma + 2\delta\sigma\left(-8|q|^2 q_{\varsigma\varsigma} - 6q^* q_\varsigma^2 - 6\sigma|q|^4 q - 4q|q_\varsigma|^2 - 2q^2 q_{\varsigma\varsigma}^*\right) - 2\delta q_{\varsigma\varsigma\varsigma\varsigma},$$
$$A_1 = 2\mathrm{i}\sigma\beta|q|^2 + 12\sigma^2\delta|q|^2\left(q q_\varsigma^* - q_\varsigma q^*\right) + 2\sigma\delta(-q_\varsigma q_{\varsigma\varsigma}^* + q_\varsigma^* q_{\varsigma\varsigma} - q^* q_{\varsigma\varsigma\varsigma} + q q_{\varsigma\varsigma\varsigma}^*),$$
$$B_0 = \frac{\mathrm{i}}{2} q_\varsigma + \beta\left(q_{\varsigma\varsigma} + 2\sigma|q|^2 q\right) + \mathrm{i}\delta q_{\varsigma\varsigma\varsigma\varsigma\varsigma} + 10\mathrm{i}\sigma\delta\left(q q_\varsigma^* q_{\varsigma\varsigma} + 3\sigma|q|^4 q_\varsigma + |q|^2 q_{\varsigma\varsigma\varsigma} + 2q^* q_\varsigma q_{\varsigma\varsigma} + q q_{\varsigma\varsigma}^* q_\varsigma + q_\varsigma|q_\varsigma|^2\right),$$
$$A_0 = -\frac{\mathrm{i}}{2}\sigma\left(q_0^2 - |q|^2\right) + \beta\sigma\left(-q q_\varsigma^* + q^* q_\varsigma\right) + \mathrm{i}\sigma\delta\left(-q_\varsigma q_{\varsigma\varsigma\varsigma}^* - q_\varsigma^* q_{\varsigma\varsigma\varsigma} + q^* q_{\varsigma\varsigma\varsigma\varsigma} + q q_{\varsigma\varsigma\varsigma\varsigma}^* + |q_{\varsigma\varsigma}|^2\right)$$
$$+ 5\mathrm{i}\sigma^2\delta\left(-2\sigma q_0^6 + 2\sigma|q|^6 + 2|q|^2 q_{\varsigma\varsigma} q^* + (q^*)^2 q_\varsigma^2 + (q_\varsigma^*)^2 q^2 + 2q|q|^2 q_{\varsigma\varsigma}^*\right).$$

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