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Thermodynamics of charged AdS black hole surrounded by quintessence in restricted phase space

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We study thermodynamics of charged AdS black hole surrounded by quintessence in a new formalism which is called the restricted phase space thermodynamics. This context is based on Visser's holographic thermodynamics with a fixed anti-de Sitter radius and a variable Newton constant. The conjugate variables, central charge C and the chemical potential μ, are introduced as a new pair of thermodynamic variables. We find that the iso-e-charge T–S curve becomes non-monotonic when $\hat{Q} < \hat{Q}_c$. Correspondingly, the F–T curve exhibits a swallow tail structure. This behavior is considered as a van der Waals-like phase transition. As the value of $\hat{b}$ related to the energy density of Kiselev's fluid becomes larger, the critical temperature T_c will decrease. Thus, the van der Waals-like phase transition will occur at lower temperature. There is always a non-quilibrium transition from a small unstable black hole to a large stable black hole state in the isocoltage T–S process. There exist a maximum and a Hawking–Page phase transition points in the μ–C plane. As the value of $\hat{b}$ related to Kiselev's fluid becomes larger, the Hawking–Page phase transition will occur at lower temperature in the isovoltage μ–T process. For other values of the state parameter ω, there also exists van der Waals-like phase transition.

Keywords: Reissner–Nordström qAdS black hole, restricted phase space, variable Newton constant

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1. Introduction

Over the last half century, thermodynamics of black hole has been one of the most hot and important topics, especially, after Bekenstein,[1,2] Bardeen, Cater and Hawking[3,4] defined entropy and temperature of black hole. With the development of history and technology, the study of black hole can be classified into two major theories. The traditional formalism focused on the initial establishment of basic laws of black hole thermodynamics and thermodynamics quantities. In traditional theory, the mass of black hole can be interpreted as internal energy. The Smarr relation[5] and the Wald method[6] play important roles in this formalism. Compared to classical thermodynamics, there is no usual P–V term in the first law of black hole thermodynamics. In order to restore the P–V term, the black hole phase space must be extended, so as to accommodate effective thermodynamic pressure and volume. Thus a relatively modern theory is the extended phase space (EPS) theory appearing about fifty years ago.[7–11] This theory is also called black hole chemistry.[12,13] There have been a number of works published about this regard.[11,14–18] In this theory, one of the most recently interesting aspects consists in the possibility of assuming a negative cosmological constant Λ as a thermodynamic variable, which plays the role of a pressure term. The surface gravity and the event horizon area are supposed to be proportional to temperature and entropy. The mass is interpreted as an enthalpy. Simultaneously, the AdS/CFT correspondence proposed by Maldacena[19] has played an important role to understand the behavior of black hole, and this framework brought out a new thermodynamic approach. Under AdS/CFT correspondence theory, the AdS black hole is equivalent to a thermal state in the conformal field theory (CFT). Recently, under the framework of the AdS/CFT correspondence and on the basis of Visser's thermodynamical theory,[20] an alternative restricted version of Visser's formalism was proposed by Zeyuan and Zhao in Ref. [21]. They have established an extensive black hole thermodynamics based on the definition of C ($C = l^{d-2}/G$), where the AdS radius l is fixed, but G is still variable, which is called restricted phase space thermodynamics (RPS).[22,23] Zeyuan and Zhao[21] emphasized the importance of the Euler homogeneity in the black hole thermodynamic system. They have studied Reissner–Nordström (RN) AdS black hole, in detail, in four-dimensional Einstein–Maxwell theory in the RPS. Black hole mass in RPS is reinterpreted as internal energy and RPS introduced a new pair of conjugate thermodynamic variables, which are the chemical potential μ and the central charge C. The central charge C represents the number degrees of freedom in CFT,[20] which is related to the number of colors in the dual gauge theory. For this reason, its conjugate potential is referred as a chemical potential. The central charge is related to the Newton constant G and the AdS curvature radius l via $C = l^{d-2}/G$, where d is the dimension of the AdS spacetime.[21,24] The thermodynamic processes and phase transition in RN-AdS, Kerr-AdS, BTZ, Taub-NUT and 4D-EGB black holes in the context of RPS formalism[22,25,26] are studied re-

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cently.

In the last two decades, many scientists have observed the accelerated expansion of our universe, which has been typically regarded to the existence of dark energy. The quintessence field is one of the most promising dark energy dynamical candidates, which is essentially a slowly rolling scalar field. According to relevant discussions on black hole surrounded by quintessence, the state equation for quintessence reads $p = \omega\rho$, in which p, ω, and ρ are indicate pressure, the state parameter, and energy density, respectively. The state parameter ω must satisfy $-1 < \omega < -1/3$ so as to guarantee the cosmic acceleration and to satisfy the energy condition. The thermodynamic properties of black hole surrounded by quintessence are discussed in Refs. [27–42]. To study numerically a typical case of the black hole surrounded by the quintessential fluid with the state parameter $\omega = -2/3$, Yan *et al.*[43] investigated the Hawking–Page phase transition[44] temperature $T_{\rm HP}$ and the Gibbs free energy G for the special case of $\omega = -2/3$. In this study, we aim to investigate the RPS thermodynamics of charged AdS black hole surrounded by quintessence dark energy (RN-qAdS black hole).

The paper is organized as follows. In Section 2, we briefly review of the RN-qAdS black hole solution. In Section 3, we examine the thermodynamics properties of the RN-qAdS black hole when state parameter $\omega = -2/3$ in the restricted phase space formalism by T–S processes. We investigate the Hawking–Page phase transition in the μ–C process. In Section 4, we simply investigate thermodynamics properties of the RN-qAdS black hole when state parameter ω takes some other values. The results and discussions are presented in Section 5. In this study, we adopt the units $h = c = k_{\rm B} = 1$.

2. Four-dimensional RN-qAds black hole

The action of RN-qAdS black hole with a negative cosmological constant reads[28,43]

$$S = \int {\rm d}^4x\sqrt{-g}\Big[\frac{1}{16\pi G}(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}) + \mathcal{L}_q\Big], \tag{1}$$

where g and R are the determinant of the metric tensor $g_{\mu\nu}$ and the Ricci scalar. G is the Newton constant and $F^{\mu\nu}$ is the electromagnetic tensor. $\mathcal{L}_q$ is the Lagrangian of the quintessence dark energy, which is considered as a barotropic perfect fluid. $\mathcal{L}_q = -\rho[1 + \omega\ln(\rho/\rho_0)]$ with an integral constant ρ_0. The cosmological constant is given by

$$\Lambda = -\frac{(d-1)(d-2)}{2l^2}, \tag{2}$$

where l denotes the anti-de Sitter (AdS) curvature radius. Consequently, the metric of the RN-qAdS black hole in four dimensions adopts the limit $d \to 4$ and is given by[27,45]

$${\rm d}s^2 = f(r){\rm d}t^2 - \frac{{\rm d}r^2}{f(r)} - r^2{\rm d}\theta^2 - r^2\sin^2\theta{\rm d}\phi^2, \tag{3}$$

where

$$f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} + \frac{r^2}{l^2} - \frac{Gb}{r^{3\omega+1}}. \tag{4}$$

The parameters M and Q are the black hole mass and electric charge; b is a parameter related to the energy density of Kiselev's fluid and it is a positive normalization factor. Solving the equation $f(r) = 0$, $f(r)$ has two distinct real roots $r = r_\pm$, and the larger one r_+ corresponds to the event horizon. Using Eq. (4) we can obtain the black hole mass

$$M = \frac{1}{2}\Big(-br_+^{-3\omega} + \frac{r_+^3}{Gl^2} + \frac{r_+}{G} + \frac{Q^2}{r_+}\Big). \tag{5}$$

We define the normalization factor b as a fixed parameter,[43] which characterizes the energy density of quintessence field. It is different from Refs. [46–48], where b is interpreted as a thermodynamic variable. The macro states of RN-qAdS black hole in RPS thermodynamics are characterized by the following three pairs of conjugate variables (S, T), (Q, Φ) and (C, μ).

The Hawking temperature at the event horizon is written as

$$T = \frac{f'(r)}{4\pi} = \frac{3bGl^2\omega r_+^{1-3\omega} - Gl^2Q^2 + 3l^2r_+^4 + l^2r_+^2}{4\pi l^2 r_+^3}. \tag{6}$$

The entropy S at the event horizon is given as

$$S = \frac{A}{4G} = \frac{\pi r_+^2}{G}. \tag{7}$$

At the event horizon, the electrical potential is written as

$$\Phi = \frac{Q}{r_+}. \tag{8}$$

3. RPS thermodynamics in RN-qAds black hole

Firstly we briefly review the general EPS formalism of the thermodynamics for the black hole. In the context of extended phase space, the thermodynamics of the black hole is carried out with a variable cosmological constant, which leads to the variation of corresponding dynamical equations. In order to avoid this problem, we introduce the thermodynamics of restricted phase space where Newton's constant is assumed to be a thermodynamics variable instead of the cosmological constant, and consequently the field equations remain unchanged. In the RPS formalism, considering the AdS/CFT duality, the gravity model should be fixed. Thus the meanings of the variables pressure P and volume V in the holographic description have changed.[20,21,25] We should forbid the appearance of the V and the P in the first law of thermodynamics and the Euler-like relation. On the other hand, the other two thermodynamic quantities, i.e., central charge C and chemical potential μ, make great sense in the CFT. Since the central charge C determines microscopic freedom of the degree in CFT, its

conjugate μ is considered as a chemical potential.[21] Also, the relationship between the central charge and Newton's constant is expressed as

$$C=\frac{l^2}{G}. \tag{9}$$

The chemical potential μ is given by

$$\mu=\frac{M-TS-\hat{\Phi}\hat{Q}}{C}, \tag{10}$$

where $l^2=-3/\Lambda$.[20–22] In the RPS formalism, the AdS radius l is fixed as a constant, which is different from Visser's formalism. Using the above quantities (see Refs. [21,22,25]), the first law of thermodynamics can also be expressed as

$$\mathrm{d}M=T\,\mathrm{d}S+\hat{\Phi}\,\mathrm{d}\hat{Q}+\mu\,\mathrm{d}C. \tag{11}$$

The Euler-like relation can be written as

$$M=TS+\hat{\Phi}\hat{Q}+\mu C, \tag{12}$$

where M is mass interpreted as an internal energy in RPS, T is temperature, and S is entropy of black hole. Respectively, $\hat{Q}$ is the rescaled electric charge and $\hat{\Phi}$ is the corresponding rescaled Coulomb electric potential. In four-dimensional space, they are defined by the dual CFT quantities as[21,49,50]

$$\hat{Q}=\frac{Ql}{\sqrt{G}},\quad \hat{\Phi}=\frac{\Phi\sqrt{G}}{l}, \tag{13}$$

the free energy can be rewritten as

$$W=\mu C=M-TS-\hat{\Phi}\hat{Q}. \tag{14}$$

With the corresponding conjugate quantities

$$T=\left(\frac{\partial M}{\partial S}\right)_{\hat{Q},C},\quad \hat{\Phi}=\left(\frac{\partial M}{\partial \hat{Q}}\right)_{S,C,},\quad \mu=\left(\frac{\partial M}{\partial C}\right)_{S,\hat{Q}}, \tag{15}$$

the event horizon radius r_+ in terms of the RN-qAdS black hole entropy and the AdS/CFT quantities reads

$$r_+=l\frac{\sqrt{S}}{\sqrt{\pi C}}. \tag{16}$$

Using Eqs. (5) and (16), we can rewrite the mass of the RN-qAdS black hole to study thermodynamic properties in RPS thermodynamics. For convenience, we set the $b=\hat{b}l^{1-3\omega}C$ under the condition of $T>0$. In this case, we study the case of $\omega=-2/3$ in four-dimensional space. Firstly the mass reads

$$\begin{aligned} M&=M(S,\hat{Q},C)\\ &=\frac{\sqrt{l^2S}(S(\pi C+S)+\pi^2\hat{Q}^2)-\hat{b}\sqrt{\pi C}lS^2}{2\pi^{3/2}\sqrt{C}l^2S}. \end{aligned} \tag{17}$$

Using Eq. (17) and the thermodynamics first law (11), we can easily obtain the equations of state (EOS) as follows:

$$\begin{aligned} T&=\left(\frac{\partial M}{\partial S}\right)_{\hat{Q},C}\\ &=\frac{l(-2\sqrt{\pi}\hat{b}\sqrt{CS}\sqrt{l^2S}+lS(\pi C+3S)+\pi^2(-l)\hat{Q}^2)}{4\pi^{3/2}\sqrt{C}(l^2S)^{3/2}}, \end{aligned} \tag{18}$$

$$\hat{\Phi}=\left(\frac{\partial M}{\partial \hat{Q}}\right)_{C,S}=\frac{\sqrt{\pi}\hat{Q}}{\sqrt{C}\sqrt{l^2S}}, \tag{19}$$

$$\mu=\left(\frac{\partial M}{\partial C}\right)_{\hat{Q},S}=-\frac{S(S-\pi C)+\pi^2\hat{Q}^2}{4\pi^{3/2}C^{3/2}\sqrt{l^2S}}. \tag{20}$$

Rescaling S, $\hat{Q}$, C as $S\to\lambda S$, $\hat{Q}\to\lambda\hat{Q}$, $C\to\lambda C$, M will be rescaled as $M\to\lambda M$. At the same time, Eqs. (18)–(20) indicate that T, $\hat{\Phi}$, μ will not be changed, which tells us that M is a first-order homogeneity, and T, $\hat{\Phi}$ and μ are zeroth-order homogeneity in S, C, and μ. Using Eqs. (10) and (11), we can express the Gibbs–Duheim relation for the chemical potential as follows:

$$\mathrm{d}\mu=-\hat{\mathcal{Q}}\mathrm{d}\hat{\Phi}-\mathcal{S}\mathrm{d}T, \tag{21}$$

where $\mathcal{S}=S/C$ and $\hat{\mathcal{Q}}=\hat{Q}/C$ are the entropy per unit C-charge and the rescaled electric charge per unit C-charge, respectively.

From Eqs. (17)–(20), we can also see that the M is a function of extensive variables, and T, $\hat{\Phi}$, $\hat{b}$ and μ are intensive variables which are written as functions of the extensive variables S, $\hat{Q}$, C. According to the first law of thermodynamics, i.e., Eq. (11), the role of C-charge likes the particle number, so C-charge is obviously extensive. Other extensive variables S and $\hat{Q}$ are proportional to C, and their coefficients of proportionality only depend on the intensive variables. Because of the area dependence of the black hole entropy S, which is often miss-interpreted as nonextensivity. Thus it is regarded as a necessary step to identify the variable S to be extensive.

Let us identify the extensivity of variables S and $\hat{Q}$. Solving $\hat{Q}$ from Eq. (19) yields

$$\hat{Q}=\frac{l\hat{\Phi}\sqrt{CS}}{\sqrt{\pi}}. \tag{22}$$

Substituting Eq. (22) into Eq. (18), we have

$$T=\frac{\pi C(1-l^2\phi^2)-2\hat{b}\sqrt{\pi CS}+3S}{4\pi^{3/2}\sqrt{C}l\sqrt{S}}. \tag{23}$$

Because T is nonnegative, it is necessary to constrain the value of $\hat{\Phi}$. Under the condition of $S\geq 0$, $\hat{\Phi}$ should follow the boundary

$$\hat{\Phi}\leq\left|\frac{\sqrt{-2\sqrt{\pi CS}\hat{b}+\pi C+3S}}{\sqrt{\pi C}l}\right|. \tag{24}$$

Assuming that the above boundary is unsaturated, we can get S from Eq. (23) as follows:

$$S=\mathcal{S}C, \tag{25}$$

$$\mathcal{S}=\frac{1}{18}(\pi^3l^2)\Big\{-\Big[6T_0^2-\Big(\frac{2\hat{b}}{\pi l}+4\tau T_0\Big)^2\Big]$$

$$\pm\sqrt{\left[6T_0^2-\left(\frac{2\hat{b}}{\pi l}+4\tau T_0\right)^2\right]^2+36T_0^4}\Big\}, \tag{26}$$

where

$$\tau=\frac{T}{T_0},\ \ T_0=\frac{(1-l^2\hat{\Phi}^2)^{1/2}}{\pi l}. \tag{27}$$

We can obtain that the entropy S is obviously proportional to C, and the coefficient $\mathcal{S}$ only depends on the intensive variables T, $\hat{\Phi}$ and $\hat{b}$. This identifies the extensivity of S. Substituting Eq. (25) into Eq. (22), we can identify the extensive $\hat{Q}$ as follows:

$$\hat{Q}=\hat{\mathcal{Q}}C,\ \ \hat{\mathcal{Q}}=l\hat{\Phi}\left(\frac{\mathcal{S}}{\pi}\right)^{\frac{1}{2}}. \tag{28}$$

3.1. Thermodynamic processes and phase transition

We devote to study the thermodynamic processes and phase transition of RN-qAdS black hole by fixing the electric charge $\hat{Q}$ and the central charge C. The conditions of the critical point in the T–S curve by fixing $\hat{Q}$ and C are

$$\left(\frac{\partial T}{\partial S}\right)_{\hat{Q},C}=0,\ \ \left(\frac{\partial^2 T}{\partial^2 S}\right)_{\hat{Q},C}=0. \tag{29}$$

Combining Eqs. (18) and (29), we can obtain the critical parameters

$$S_{\rm c}=\frac{\pi C}{6},\ \ \hat{Q}_{\rm c}=\frac{C}{6}. \tag{30}$$

Substituting the critical parameters (30) into Eqs. (17) and (18), we have

$$T_{\rm c}=\frac{4-\sqrt{6}\hat{b}}{2\sqrt{6}\pi l},\ \ M_{\rm c}=\frac{4\sqrt{6}C-3C\hat{b}}{36l}. \tag{31}$$

Introducing the relative parameters, we have following three normalized parameters:

$$t=\frac{T}{T_{\rm c}},\ \ s=\frac{S}{S_{\rm c}},\ \ q=\frac{Q}{Q_{\rm c}}. \tag{32}$$

Using Eqs. (18) and (31), the equation of state in the restricted phase space thermodynamics (RPST) formalism can be written as

$$t=\frac{q^2+2\hat{b}s\sqrt{6s}-3s(s+2)}{2s^{3/2}(\sqrt{6}\hat{b}-4)}. \tag{33}$$

Now, we can introduce the Helmholtz free energy as

$$F(S,\hat{Q},C)=M(S,\hat{Q},C)-TS. \tag{34}$$

From Eqs. (31) and (32), the critical point of Helmholtz energy takes the following form:

$$F_{\rm c}=\frac{C}{3\sqrt{6}l}. \tag{35}$$

Using Eqs. (18) and (31), we can rewrite Eq. (34) as

$$f=\frac{F}{F_{\rm c}}=\frac{3q^2-(s-6)s}{8\sqrt{s}}. \tag{36}$$

From Eqs. (33) and (36), we can see that t and f are independent of the central charge C, which is similar to "the law of corresponding states" in standard thermodynamics for ordinary matters.

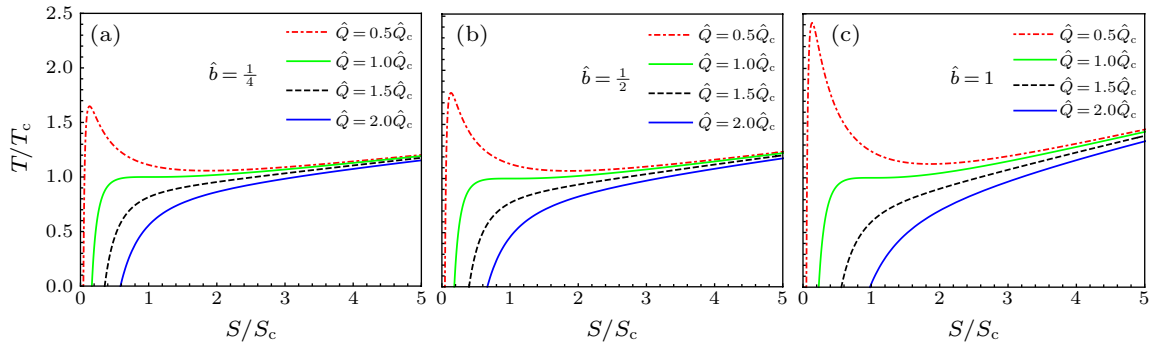


Fig. 1. The T–S curves for different electric charges $\hat{Q}=0.5\hat{Q}_{\rm c}$, $\hat{Q}_{\rm c}$, $1.5\hat{Q}_{\rm c}$, $2\hat{Q}_{\rm c}$ by fixing the energy density of Kiselev's fluid at $\hat{b}=1/4$, $1/2$, 1, respectively.

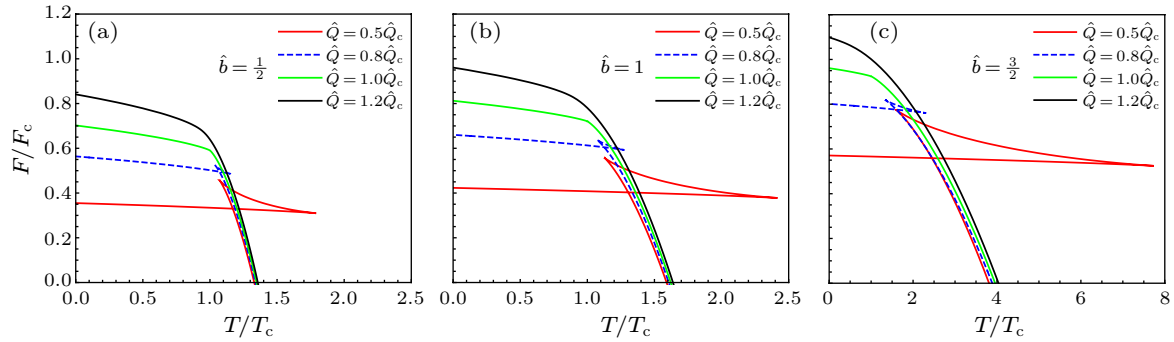


Fig. 2. The F–T curves for different electric charges $\hat{Q}=0.5\hat{Q}_{\rm c}$, $0.8\hat{Q}_{\rm c}$, $1\hat{Q}_{\rm c}$, $1.2\hat{Q}_{\rm c}$ by fixing the energy density of Kiselev's fluid at $\hat{b}=1/2$, 1, $3/2$, respectively.

In Figs. 1 and 2, we can see that the T–S and F–T curves of RN-qAdS black hole are similar to the RN-AdS black hole in RPS,[21] so the quintessence does not affect the thermal structure. It is easy to find that the curves of T–S are not monotonous when $0 < \hat{Q} < \hat{Q}_c$. The F–T curves have a swallow tail where the root is located at some supercritical temperature $T > T_c$. From the swallow tail structure on the F–T curves, there is a first-order phase transition from a small black hole to a large black hole through the medium sized black hole for $0 < \hat{Q} < \hat{Q}_c$. This behavior is also considered as a van der Waals-like-first-order phase equilibrium. When the phase transition occurs, there are three black hole states with the same temperature, which means that the states are stable with the smallest entropy and largest entropy at event horizon, while the medium sized black hole state is unstable. When $T_{\rm trans} = T_c$, the swallow tail will disappear in the picture, which tells us that the first-order phase transition will become the second-order phase transition. For $\hat{Q} > \hat{Q}_c$, there are no phase transition. In addition, we notice that the value of $\hat{b}$ will influence the strengths of T–S and F–T curves. As the value of $\hat{b}$ becomes larger, the critical temperature T_c will decreases. Thus, the van der Waals-like phase transition will occur at lower temperature. We can find that the root of swallow tail becomes larger.

Assuming that the bound Eq. (24) is unsaturated, let us consider the isovoltage processes by stimulating the T–S curves at fixed $\hat{\Phi}$, C, $\hat{b}$.

$$S = S_{\rm min} = \frac{1}{3}\pi C(1 - l^2\hat{\Phi}^2), \quad T = T_{\rm min} = \frac{\sqrt{3}\sqrt{1 - l^2\hat{\Phi}^2} - \hat{b}}{2\pi l}. \tag{37}$$

Define

$$\tilde{t} = \frac{T}{T_{\rm min}}, \quad \tilde{s} = \frac{S}{S_{\rm min}}. \tag{38}$$

Using Eq. (18), the $\tilde{t}$ can be expressed in the form

$$\tilde{t} = \frac{3(\tilde{s}+1)\sqrt{\tilde{s}(1 - l^2\hat{\Phi}^2)} - 2\sqrt{3}\hat{b}\tilde{s}}{\tilde{s}(6\sqrt{1 - l^2\hat{\Phi}^2} - 2\sqrt{3}\hat{b})}. \tag{39}$$

Unlike the isovoltage T–S curves in Ref. [21] showing that the isovoltage T–S curves cannot discriminate different values of $\hat{\Phi}$, in the present study the isovoltage T–S curves can discriminate different values of $\hat{\Phi}$, l, and $\hat{b}$. From Fig. 3, as the values of $\hat{b}$ and $l^2\hat{\Phi}^2$ become larger, the curve open mouth will be smaller. At the same $T/T_{\rm min}$, the S monotonously decreases for $S < S_{\rm min}$, and the S monotonously increases for $S > S_{\rm min}$. Although the values of $l^2\hat{\Phi}^2$ and $\hat{b}$ are changed, the point of $T/T_{\rm min}$ does not shift. The temperature of isovoltage process is not allowed to be below $T_{\rm min}$. When $T > T_{\rm min}$, there are two black hole states at the same T, $\hat{\Phi}$, C and $\hat{b}$ with different entropies, which is also expressed by Eq. (39). According to the maximum entropy rule, the state at larger S should be more thermodynamically stable. In other words, the black hole with smaller radius will jump to the black hole with larger radius under any small disturbance. Though there does exist the specific transition temperature, this phenomenon is also considered to be a phase transition. From other aspects, when $S < S_{\rm min}$, the T decreases with S, the heat capacity $C_{\hat{\Phi},C} < 0$ for $S < S_{\rm min}$, so the states of $S < S_{\rm min}$ are unstable. However, when $S > S_{\rm min}$, the $C_{\hat{\Phi},C} > 0$, the states of RN-qAdS black hole are stable.

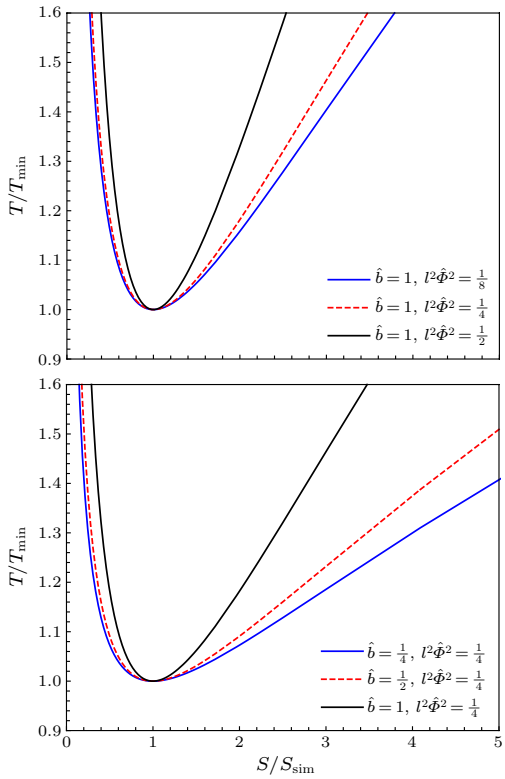


Fig. 3. (a) The T–S curves of the isovoltage processes for different values of $l^2\Phi^2$. (b) T–S curves of the isovoltage processes for different values of $\hat{b}$.

From Eq. (20), there exists a maximum μ for the fixed S and $\hat{Q}$,

$$C_{\rm max} = \frac{3(\pi^2\hat{Q}^2 + S^2)}{\pi S}, \quad \mu_{\rm max} = \frac{\sqrt{3}S}{18l\sqrt{\pi^2\hat{Q}^2 + S^2}}. \tag{40}$$

Introducing the dimensionless parameters

$$c = \frac{C}{C_{\rm max}}, \quad m = \frac{\mu}{\mu_{\rm max}}, \tag{41}$$

we can get the rescaled equation states of μ–C, which can be written as

$$m = \frac{3c - 1}{2c\sqrt{c}}. \tag{42}$$

Unlike the iso-e-charge process, we can interestingly see that μ–C curves for different values of $\hat{b}$, $\hat{Q}$ and S are all scaled into a single curve by using relative parameters c and m. At $C = \frac{1}{3}C_{\rm max}$, we obviously have $\mu = 0$, so the Gibbs free energy $G = \mu C$ will equal zero. For $C > \frac{1}{3}C_{\rm max}$, μ is positive and the μ value will not exceed $\mu_{\rm max}$. Thus, at $C > \frac{1}{3}C_{\rm max}$, the RN-qAdS black hole's microscopic degrees of freedom are repulsive, while at $C < \frac{1}{3}C_{\rm max}$, they are attractive.

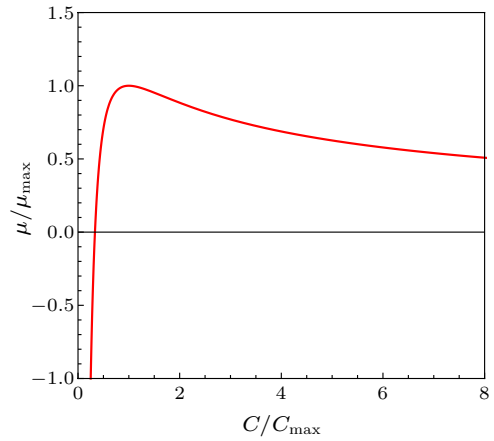


Fig. 4. The μ–C curve in the adiabatic iso-e-charge/isovoltage processes.

In Fig. 4, we can see zero of the chemical potential, which means e-charge $\hat{Q}=0$. This case corresponds to the famous Hawking–Page (HP) transition, which means the AdS black hole state to a pure thermal AdS gas. The HP temperature $T_{\rm HP}$ is an important object related to HP transition, where HP temperature $T_{\rm HP}$ is the temperature when HP transition occurs. In order to obtain the HP temperature $T_{\rm HP}$, we set $\hat{Q}=0$ in Eq. (20), so

$$\mu=\frac{S(\pi C-S)}{4\pi^{3/2}C^{3/2}\sqrt{l^2S}}, \tag{43}$$

where μ is the ratio of the Gibbs free energy to the central charge. Then, substituting Eq. (26) into Eq. (43), we obtain

$$\mu(T)=-\frac{[(8\tau^2+2\hat{b}^2+8\tau\hat{b}-3)\pm2\sqrt{(2\tau+\hat{b})^2(4\tau^2+\hat{b}^2+4\tau\hat{b}-3)}]-9}{108l}\times\frac{\sqrt{(8\tau^2+2\hat{b}^2+8\tau\hat{b}-3)\pm2\sqrt{(2\tau+\hat{b})^2(4\tau^2+\hat{b}^2+4\tau\hat{b}-3)}}}{108l}. \tag{44}$$

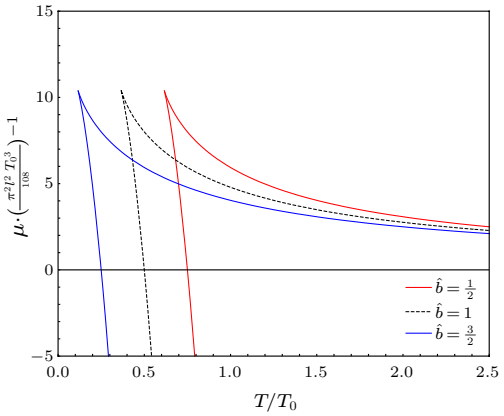


Fig. 5. The μ–T curves of RN-qAdS black hole with different values of parameter $\hat{b}$.

Now, we obtain the μ as an implicit function of black hole temperature T and the parameter $\hat{b}$ related to the density of fluid. In Eq. (44), where τ is defined by Eq. (27) with T_0 replaced by $T_0=\frac{1}{\pi l}$. In Fig. 5, two branches are divided at a cusp in curves, the right branch stands for the small black hole phase, and the other branch behaves the large black hole phase and they meet at the point $(T_{\min},\mu_{\min})$. The Hawking–Page transition occurs at the crossing point with the $\mu(T)=0$ line. We notice that the temperature of the coexistence point of the two branches of $\mu(T)$ is below the temperature of HP phase transition $T_{\rm HP}$. As the parameter $\hat{b}$ related to the density of fluid increases, the point of HP transition $\mu(T)=0$ will move left, which tells us that the HP transition will occur at the lower temperature.

4. State parameter $\omega=-1/2$ and $\omega=-2/5$

When state parameter ω takes some other values, we can obtain

$$M=M(S,\hat{Q},C)=\frac{\left(\frac{\sqrt{l^2S}}{\sqrt{C}}\right)^{1-3\omega}\left(l(S(\pi C+S)+\pi^2\hat{Q}^2)\left(\frac{\sqrt{l^2S}}{\sqrt{C}}\right)^{3\omega}-\hat{b}C^{3/2}\pi^{\frac{3(\omega+1)}{2}}\sqrt{l^2S}l^{3\omega}\right)}{2\pi^{3/2}l^3S}, \tag{45}$$

$$T=\left(\frac{\partial M}{\partial S}\right)_{\hat{Q},C}=\frac{\left(\frac{\sqrt{l^2S}}{\sqrt{C}}\right)^{1-3\omega}\left(3\hat{b}C^{3/2}\pi^{\frac{3(\omega+1)}{2}}\omega\sqrt{l^2S}l^{3\omega}-l(\pi^2\hat{Q}^2-S(\pi C+3S))\left(\frac{\sqrt{l^2S}}{\sqrt{C}}\right)^{3\omega}\right)}{4\pi^{3/2}l^3S^2}, \tag{46}$$

$$\hat{\Phi}=\left(\frac{\partial M}{\partial\hat{Q}}\right)_{C,S}=\frac{\sqrt{\pi}\hat{Q}}{\sqrt{C}\sqrt{l^2S}}, \tag{47}$$

$$\mu=\left(\frac{\partial M}{\partial C}\right)_{\hat{Q},S}=-\frac{\hat{b}\pi^{\frac{3\omega}{2}}(3\omega+2)l^{3\omega+1}\left(\frac{\sqrt{l^2S}}{\sqrt{C}}\right)^{-3\omega}+\frac{\sqrt{l^2S}(S(S-\pi C)+\pi^2\hat{Q}^2)}{\pi^{3/2}C^{3/2}S}}{4l^2}. \tag{48}$$

This is still satisfied that M is a first-order homogeneity, and T, $\hat{\Phi}$ and μ are zeroth-order homogeneity in S, C, and μ.

Using the same method as in the above section, we analyze two other cases at $\omega = -1/2$ and $\omega = -2/5$. We get the critical entropy S and charge $\hat{Q}^2$ when $\omega = -1/2$ and $\omega = -2/5$ in Table 1.

Table 1. Critical points with $\omega = -1/2$ (left) and $\omega = -2/5$ (right).

$\hat{b}$	S_c	$\hat{Q}_c^2$	$\hat{b}$	S_c	$\hat{Q}_c^2$
1/4	0.44818C	$0.02172C^2$	1/4	0.41081C	$0.01795C^2$
1/2	0.37896C	$0.01677C^2$	1/2	0.30467C	$0.01063C^2$
1	0.26025C	$0.00964C^2$	1	0.12354C	$0.00246C^2$

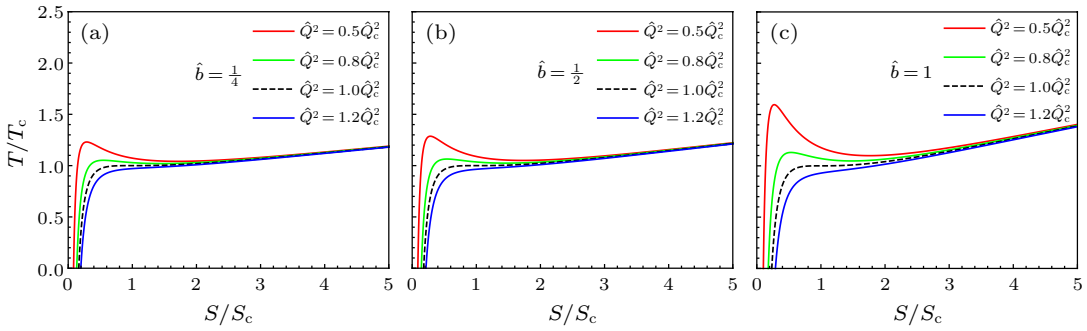


Fig. 6. For the case of $\omega = -1/2$: the T–S curves for different electric charges $\hat{Q}^2 = 0.5\hat{Q}_c^2$, $0.8\hat{Q}_c^2$, $1\hat{Q}_c^2$, $1.2\hat{Q}_c^2$ by fixing the energy density of Kiselev's fluid $\hat{b} = 1/4$, $1/2$, 1, respectively.

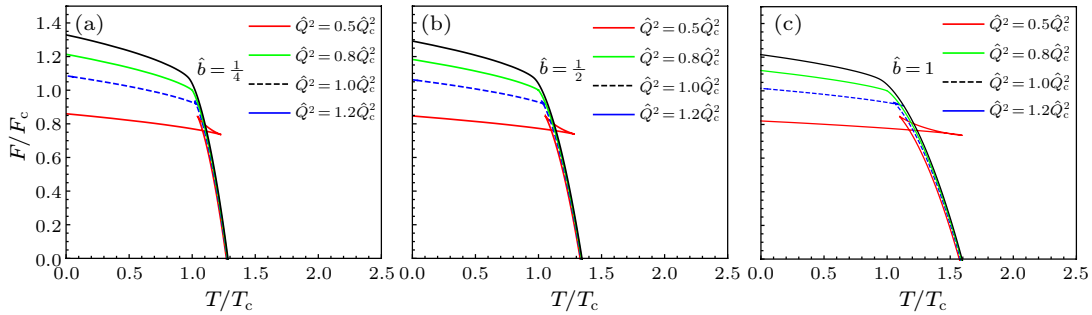


Fig. 7. For the case of $\omega = -1/2$: the F–T curves for different electric charges $\hat{Q}^2 = 0.5\hat{Q}_c^2$, $0.8\hat{Q}_c^2$, $1\hat{Q}_c^2$, $1.2\hat{Q}_c^2$ by fixing the energy density of Kiselev's fluid $\hat{b} = 1/4$, $1/2$, 1, respectively.

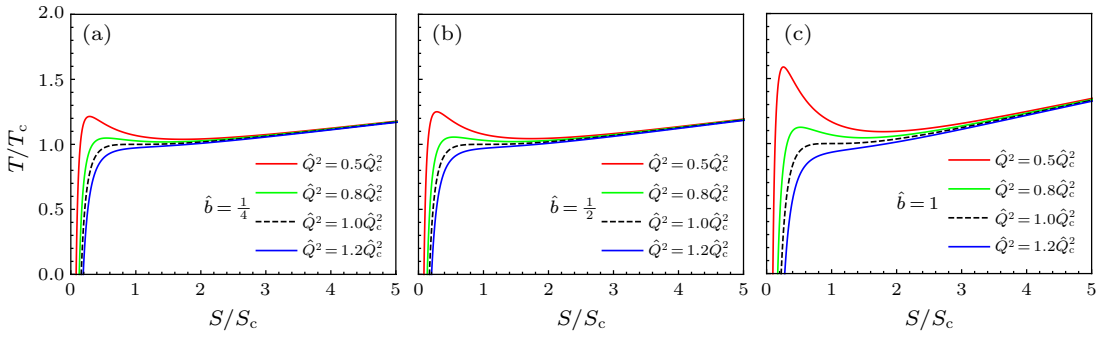


Fig. 8. For the case of $\omega = -2/5$: the T–S curves for different electric charges $\hat{Q}^2 = 0.5\hat{Q}_c^2$, $0.8\hat{Q}_c^2$, $1\hat{Q}_c^2$, $1.2\hat{Q}_c^2$ by fixing the energy density of Kiselev's fluid $\hat{b} = 1/4$, $1/2$, 1, respectively.

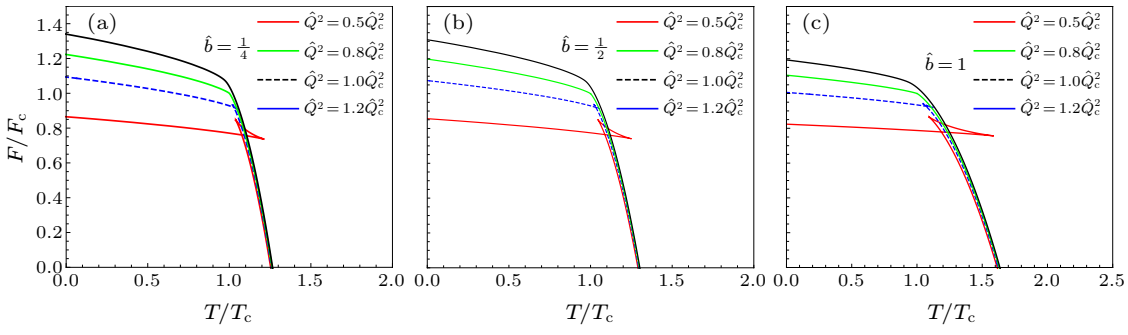


Fig. 9. For the case of $\omega = -2/5$: the F–T curves for different electric charges $\hat{Q}^2 = 0.5\hat{Q}_c^2$, $0.8\hat{Q}_c^2$, $1\hat{Q}_c^2$, $1.2\hat{Q}_c^2$ by fixing the energy density of Kiselev's fluid $\hat{b} = 1/4$, $1/2$, 1, respectively.

We briefly study the T–S and F–T curves of the RN-q black hole at $\omega = -1/2$ and $\omega = -2/5$ by numerical simulation. It is found that the black hole's thermodynamic behavior is similar to Section 4, as shown in Figs. 6–9. In this section, the T–S curve becomes non-monotonic when $\hat{Q}^2 < \hat{Q}_c^2$ in Figs. 6 and 8. Correspondingly, the F–T curve exhibits a swallow tail structure in Figs. 7 and 9. This behavior is considered as a van der Waals-like phase transition. In these two cases, when the value ω is fixed, the critical entropy S_c will decrease and the critical charge will be suppressed as the value of $\hat{b}$ becomes larger.

5. Conclusions

In summary, we have investigated the RN-AdS black hole surrounded by the quintessence in the restricted phase space built on the thermodynamics of Visser's holography. By numerically simulating the T–S curves and F–T curves, we have found that the T–S curves become non-monotonic when $\hat{Q} < \hat{Q}_C = C/6$. Correspondingly, the F–T curves exhibit a swallow tail structure. If the e-charge $\hat{Q}$ takes subcritical value $0 < \hat{Q} < \hat{Q}_c = C/6$, there exists a first-order van der Waals-like phase transition in the iso-e-charge on the T–S plane. When the $\hat{Q}$ approaches the critical value $\hat{Q} = \hat{Q}_c$, the phase transition becomes of second order. In addition, the parameter $\hat{b}$ related to the energy of Kiselev's fluid will influence van der Waals-like phase transition. As the value $\hat{b}$ becomes larger, the range of $T = T_{trans}$ will get larger, then the swallow tail will get larger. However, when considering the isovoltage processes, the T–S curves with different values of $\hat{b}$ have a same single minimum, which indicates a non-equilibrium phase transition. The constants S and $\hat{Q}$ on the μ–C plane contain single maxima and a Hawking–Page phase transition point. The energy of Kiselev's fluid $\hat{b}$ will influence the μ–T curves in the isovoltage processes, which is deserved to study. When the value $\hat{b}$ becomes larger, the HP phase transition will occur at the lower-temperature phase transition.

Finally, we have a simple study for the cases of $\omega = -2/5$ and $\omega = -1/2$. As the value of $\hat{b}$ becomes larger, the critical entropy S_c will decrease and the critical charge is suppressed when the value ω is fixed. There also exists van der Waals-like phase transition in the iso-e-charge on the $(T$–$S)$ plane when $0 < \hat{Q}^2 < \hat{Q}_c^2$. The results further prove the feasibility of this method.

Although many researchers have studied the thermodynamic properties of various black holes in RPS, their works only focus on four-dimensional space. Therefore, exploring the thermodynamic properties of black holes in higher dimension is an interesting topic. We will investigate them in the near future.

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