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Entropy variances of pure coherent states in the diffusion channel

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Using the operator correspondence of the real and fictious modes in the thermo entangled state representation, we solve the quantum master equation describing the diffusion channel and obtain the Kraus operator-sum representation of its analytical solution. we find that the pure coherent states evolve into the new mixed thermal superposed states in the diffusion channel. Also, we investigate the statistical properties of the initial coherent states and their entropy evolutions in the diffusion channel, and find that the entropy evolutions are only related to the decay time and without the amplitudes of the initial coherent states.

Keywords: entangled state representation, diffusion channel, coherent state, entropy evolution

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1. Introduction

In quantum physics, for any open quantum systems that interact with the environment, solving quantum master equations about them has become a very important research topic. The reason is that quantum noise in open systems can cause the deterioration of nonclassicality of light fields and thus affects the transmission quality of light signal through media.

In the previous work, one usually use the Fokker–Planck equation or Langevin equation to treat the quantum master equations of open systems by virtue of rewriting the density operators as their corresponding P-representations or Wigner representations.[1–3] In Refs. [4–6], Fan *et al.* have proposed the thermal field dynamics method and use it to solve a series of master equations describing open systems, such as amplitude decay, phase damping, thermal reservoir, and laser process, *etc*., and obtain the density operators' infinite operator-sum representations as their analytical operator solutions. Based on the new approach and some main results, the evolutions of a class of quantum states in different decoherence channels are investigated via their density-operator evolutions or the function-distribution evolutions in the phase space.[7–13] For examples, the distribution function evolution in phase-sensitive reservoir, the evolution of two-mode squeezed states in the single-mode or two-mode amplitude decay, the evolution of characteristic function of the density operator, etc. In this paper, we use the entangled state approach to solve the diffusion master equation[14,15]

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = -\kappa(a^\dagger a\rho_t - a^\dagger\rho_t a - a\rho_t a^\dagger + \rho_t aa^\dagger), \tag{1}$$

where κ is diffusion coefficient, and especially focus on investigating the time evolution of the pure coherent states including the density operator, statistical properties, and entropy in the diffusion channel.

The paper is arranged as follows. In Section 2, we present the thermo entangled state representation and its well-behaved properties. In Section 3, we derive the Kraus operator-sum representation of the time-dependent density operator as the solution of the diffusion equation. In Sections 4 and 5, we obtain the evolution laws of the pure coherent states in the diffusion channel and analytically investigate their statistical properties. In Section 6, we obtain the entropy evolution of the pure coherent states in the diffusion channel.

2. A useful thermo-entangled state representation

To investigate the evolutions of pure coherent states in a diffusion channel, we need to introduce the thermo-entangled state representation and present its main properties. In terms of Ref. [16], in two-mode Fock space, the thermo-entangled state is defined as

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta a^\dagger - \eta^*\tilde{a}^\dagger + a^\dagger\tilde{a}^\dagger\right)|0\tilde{0}\rangle, \tag{2}$$

where $\tilde{a}^\dagger$ is a fictitious mode associated with the real mode $a^\dagger$, $[\tilde{a},\tilde{a}^\dagger] = 1$ and $[a,\tilde{a}^\dagger] = 0$, and the state $|\eta\rangle$ obeys the following eigenvalue equations

$$(a-\tilde{a}^\dagger)|\eta\rangle = \eta|\eta\rangle, \quad (a^\dagger-\tilde{a})|\eta\rangle = \eta^*|\eta\rangle \tag{3}$$

or

$$\langle\eta|(a^\dagger-\tilde{a}) = \eta^*\langle\eta|, \quad \langle\eta|(a-\tilde{a}^\dagger) = \eta\langle\eta|. \tag{4}$$

Using the normal ordering form of two-mode thermo vacuum $|0\tilde{0}\rangle\langle 0\tilde{0}|$, i.e., $|0\tilde{0}\rangle\langle 0\tilde{0}| =: \mathrm{e}^{-a^\dagger a-\tilde{a}^\dagger\tilde{a}}:$ (: : representing normal ordering) and the integration method within an ordered product of operators (IWOP),[17,18] we can obtain the

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following orthogonal completeness relations, i.e.,

$$\begin{aligned}\left\langle \eta' \right| \left. \eta \right\rangle &= \pi\delta\left(\eta'-\eta\right)\delta\left(\eta'^{*}-\eta^{*}\right) \quad (5)\\ \int \frac{\mathrm{d}^2\eta}{\pi}\left|\eta\right\rangle\left\langle\eta\right| &= \int \frac{\mathrm{d}^2\eta}{\pi}\colon \exp(-\left|\eta\right|^2+\eta a^{\dagger}-\eta^{*}\tilde{a}^{\dagger}\\ &\quad +a^{\dagger}\tilde{a}^{\dagger}+\eta^{*}a-\eta\tilde{a}+a\tilde{a}-a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a})\colon\\ &= 1. \quad (6)\end{aligned}$$

Taking

$$\left|\eta=0\right\rangle = \mathrm{e}^{a^{\dagger}\tilde{a}^{\dagger}}\left|0\tilde{0}\right\rangle \equiv \left|I\right\rangle, \quad (7)$$

and then operating the operators a, $a^{\dagger}$, and $(a^{\dagger}a)^n$ on the entangled state $\left|I\right\rangle$, we find

$$a\left|I\right\rangle = \tilde{a}^{\dagger}\left|I\right\rangle, \quad a^{\dagger}\left|I\right\rangle = \tilde{a}\left|I\right\rangle, \quad (a^{\dagger}a)^n\left|I\right\rangle = (\tilde{a}^{\dagger}\tilde{a})^n\left|I\right\rangle. \quad (8)$$

This means that, in the entangled state $\left|I\right\rangle$ representation, there exist the following operator exchange relationships of the real and fictious modes:

$$a \Leftrightarrow \tilde{a}^{\dagger}, \ \ a^{\dagger} \Leftrightarrow \tilde{a}, \ \ (a^{\dagger}a)^n \Leftrightarrow (\tilde{a}^{\dagger}\tilde{a})^n, \quad (9)$$

which can help to rewrite the master equations of the time-dependent density operators as the equations of the corresponding state vectors, and give the evolution laws of density operators over time.

3. Solving the diffusion equation

Acting both sides of Eq. (1) on the entangled state $\left|I\right\rangle$, noting that the real (denoted as the density operator ρ_t) and fictitious fields are independent of each other, i.e., $\left[\rho_t,\tilde{a}\right] = \left[\rho_t,\tilde{a}^{\dagger}\right] = 0$, $\left[a,\tilde{a}\right] = \left[a,\tilde{a}^{\dagger}\right] = 0$, using Eq. (8) and taking $\rho_t\left|I\right\rangle \equiv \left|\rho_t\right\rangle$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_t\left|I\right\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\left|\rho_t\right\rangle\\ &= -\kappa\left(a^{\dagger}a\rho_t - a\rho_t a^{\dagger} - a^{\dagger}\rho_t a + \rho_t a a^{\dagger}\right)\left|I\right\rangle\\ &= -\kappa\left(a^{\dagger}a\rho_t - a\tilde{a}\rho_t - a^{\dagger}\tilde{a}^{\dagger}\rho_t + \tilde{a}\tilde{a}^{\dagger}\rho_t\right)\left|I\right\rangle\\ &= -\kappa\left(a^{\dagger}-\tilde{a}\right)\left(a-\tilde{a}^{\dagger}\right)\left|\rho_t\right\rangle. \quad (10)\end{aligned}$$

Combining the first and fourth lines of Eq. (10), and letting $\left|\rho_0\right\rangle = \rho_0\left|I\right\rangle$, we can obtain the solution of Eq. (10) as

$$\left|\rho_t\right\rangle = \exp\left[-\kappa t\left(a^{\dagger}-\tilde{a}\right)\left(a-\tilde{a}^{\dagger}\right)\right]\left|\rho_0\right\rangle. \quad (11)$$

Projecting Eq. (11) onto the state $\left\langle\eta\right|$ and using the eigenvalue equation (3), we therefore obtain

$$\begin{aligned}\left\langle\eta\right|\left.\rho_t\right\rangle &= \left\langle\eta\right|\exp\left[-\kappa t\left(a^{\dagger}-\tilde{a}\right)\left(a-\tilde{a}^{\dagger}\right)\right]\left|\rho_0\right\rangle\\ &= \mathrm{e}^{-\kappa t\left|\eta\right|^2}\left\langle\eta\right.\left|\rho_0\right\rangle. \quad (12)\end{aligned}$$

Multiplying both sides of Eq. (12) by $\int\frac{\mathrm{d}^2\eta}{\pi}\left|\eta\right\rangle$, and then using Eq. (6) and the operator identity[19,20]

$$\exp(\lambda a^{\dagger}a) =\colon \exp\left[\left(\mathrm{e}^{\lambda}-1\right)a^{\dagger}a\right]\colon, \quad (13)$$

where λ is a general parameter, we get

$$\begin{aligned}\left|\rho_t\right\rangle &= \int\frac{\mathrm{d}^2\eta}{\pi}\mathrm{e}^{-\kappa t\left|\eta\right|^2}\left|\eta\right\rangle\left\langle\eta\right.\left|\rho_0\right\rangle\\ &= \int\frac{\mathrm{d}^2\eta}{\pi}\colon \exp[-(1+\kappa t)\left|\eta\right|^2+\eta\left(a^{\dagger}-\tilde{a}\right)\\ &\quad +\eta^{*}\left(a-\tilde{a}^{\dagger}\right)+a^{\dagger}\tilde{a}^{\dagger}+a\tilde{a}-a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a}]\colon\left|\rho_0\right\rangle\\ &= \frac{1}{1+\kappa t}\colon\exp\left[\frac{\kappa t}{1+\kappa t}(a^{\dagger}\tilde{a}^{\dagger}+a\tilde{a}\right.\\ &\quad \left.-a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a})\right]\colon\left|\rho_0\right\rangle. \quad (14)\end{aligned}$$

Further, using the operator identities $\left[\tilde{a},\rho_0\right]=0$ and $\tilde{a}\left|I\right\rangle = a^{\dagger}\left|I\right\rangle$, we thus have

$$\begin{aligned}\exp\left(\frac{\kappa t}{1+\kappa t}a\tilde{a}\right)\rho_0\left|I\right\rangle &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\kappa t}{1+\kappa t}a\right)^n\rho_0\tilde{a}^n\left|I\right\rangle\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\kappa t}{1+\kappa t}\right)^n a^n\rho_0 a^{\dagger n}\left|I\right\rangle, \quad (15)\end{aligned}$$

then noting that the operator $\tilde{a}^{\dagger}\tilde{a}$ is commutable with all real field operators, we naturally obtain

$$\begin{aligned}\left|\rho_t\right\rangle &= \frac{1}{1+\kappa t}\exp\left(\frac{\kappa t}{1+\kappa t}a^{\dagger}\tilde{a}^{\dagger}\right)\left(\frac{1}{1+\kappa t}\right)^{a^{\dagger}a}\\ &\quad\times\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\kappa t}{1+\kappa t}\right)^n a^n\rho_0 a^{\dagger n}\left(\frac{1}{1+\kappa t}\right)^{\tilde{a}^{\dagger}\tilde{a}}\left|I\right\rangle\\ &= \frac{1}{1+\kappa t}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\kappa t}{1+\kappa t}\right)^m a^{\dagger m}\left(\frac{1}{1+\kappa t}\right)^{a^{\dagger}a}\\ &\quad\times\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\kappa t}{1+\kappa t}\right)^n a^n\rho_0 a^{\dagger n}\left(\frac{1}{1+\kappa t}\right)^{a^{\dagger}a}\tilde{a}^{\dagger m}\left|I\right\rangle. \quad (16)\end{aligned}$$

Using the identity $\tilde{a}^{\dagger m}\left|I\right\rangle = a^m\left|I\right\rangle$ and removing the entangled state $\left|I\right\rangle$ from both sides of Eq. (16), we get the analytical form of the time-dependent density operator ρ_t, that is

$$\begin{aligned}\rho_t &= \sum_{m,n=0}^{\infty}\frac{1}{m!n!}\frac{(\kappa t)^{m+n}}{(\kappa t+1)^{m+n+1}}\\ &\quad\times a^{\dagger m}\left(\frac{1}{1+\kappa t}\right)^{a^{\dagger}a}a^n\rho_0 a^{\dagger n}\left(\frac{1}{1+\kappa t}\right)^{a^{\dagger}a}a^m. \quad (17)\end{aligned}$$

Or the sum representation of the Kraus operator, i.e.,

$$\rho_t = \sum_{m,n=0}^{\infty}\mathcal{M}_{m,n}\rho_0\mathcal{M}_{m,n}^{\dagger}, \quad (18)$$

where $\mathcal{M}_{m,n}$ is the Kraus operator, i.e.,

$$\mathcal{M}_{m,n} = \sqrt{\frac{(\kappa t)^{m+n}}{m!n!\,(\kappa t+1)^{m+n+1}}}a^{\dagger m}\left(\frac{1}{1+\kappa t}\right)^{a^{\dagger}a}a^n. \quad (19)$$

4. Evolution of pure coherent state

Using Eq. (17), we now plan to study the analytical evolutions of pure coherent states in the diffusion channel. Inserting a pure coherent state with the density operator $\rho_0^{\rm c} \equiv |\alpha\rangle\langle\alpha|$ into Eq. (18) leads to

$$\rho_t^{\rm c} = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \frac{(\kappa t)^{m+n}}{(\kappa t+1)^{m+n+1}} a^{\dagger m} \exp\left(a^\dagger a \ln \frac{1}{1+\kappa t}\right) \times |\alpha|^{2n} |\alpha\rangle\langle\alpha| \exp\left(a^\dagger a \ln \frac{1}{1+\kappa t}\right) a^m. \tag{20}$$

After summing over n and using the identity

$$\exp\left(a^\dagger a \ln \frac{1}{1+\kappa t}\right)|\alpha\rangle = {\rm e}^{-|\alpha|^2/2} \left\| \frac{\alpha}{1+\kappa t} \right\rangle, \tag{21}$$

where $\left\| \frac{\alpha}{1+\kappa t} \right\rangle$ is an unnormalized coherent state. By using operator identity (13), we obtain the final output state as

$$\begin{aligned}
\rho_t^{\rm c} &= \sum_{m}^{\infty} \frac{(\kappa t)^m}{m!(\kappa t+1)^{m+1}} \exp\left(|\alpha|^2 \frac{\kappa t}{\kappa t+1}\right) a^{\dagger m} \\
&\quad \times \exp\left(a^\dagger a \ln \frac{1}{1+\kappa t}\right) |\alpha\rangle\langle\alpha| \\
&\quad \times \exp\left(a^\dagger a \ln \frac{1}{1+\kappa t}\right) a^m \\
&= \sum_{m}^{\infty} \frac{(\kappa t)^m}{m!(\kappa t+1)^{m+1}} \\
&\quad \times \exp\left(\frac{-|\alpha|^2}{\kappa t+1}\right) a^{\dagger m} \left\| \frac{\alpha}{1+\kappa t} \right\rangle \left\langle \frac{\alpha}{1+\kappa t} \right\| a^m \\
&= \sum_{m}^{\infty} \frac{(\kappa t)^m}{m!(\kappa t+1)^{m+1}} \exp\left(\frac{-|\alpha|^2}{\kappa t+1}\right) {:}\, a^{\dagger m} a^m \\
&\quad \times \exp\left(\frac{a^\dagger \alpha}{1+\kappa t} + \frac{a\alpha^*}{1+\kappa t} - a^\dagger a\right) : \\
&= \frac{1}{\kappa t+1} {:}\, \exp\left[\frac{-1}{\kappa t+1}\left(|\alpha|^2 - a^\dagger \alpha - a\alpha^* + a^\dagger a\right)\right] {:}\,.
\end{aligned} \tag{22}$$

On the other hand, using the diffusion equation

$$\frac{\partial P_t}{\partial t} = -\kappa \frac{\partial^2 P_t}{\partial \alpha \partial \alpha^*} \tag{23}$$

and the P-representations of coherent states, we can also verify the correctness of Eq. (22). For the initial coherent state $\rho_0^{\rm c}$, its P-representation reads

$$P_0^{\rm c} = \delta(\alpha^* - z^*)\delta(\alpha - z), \tag{24}$$

thus we can solve the diffusion equation (23) and obtain its solution as

$$P_t^{\rm c} = \frac{1}{\kappa t} \exp\left[-\frac{1}{\kappa t}(\alpha^* - z^*)(\alpha - z)\right], \tag{25}$$

which is indeed the coherent state representation of the evolved density operator $\rho_t^{\rm c}$ in the diffusion channel. Further, inserting Eq. (25) into the P-representation formula of the density operator ρ, i.e.,

$$\rho = \int \frac{{\rm d}^2 z}{\pi} P(z) {:}\, \exp[-(z^* - a^\dagger)(z-a)] {:}\,, \tag{26}$$

we can obtain

$$\begin{aligned}
\rho_t^{\rm c} &= \frac{1}{\kappa t} \int \frac{{\rm d}^2 z}{\pi} {:}\, \exp\left[-\frac{1}{\kappa t}(\alpha^* - z^*)(\alpha - z)\right. \\
&\quad \left. -(z^* - a^\dagger)(z-a)\right] {:}\,,
\end{aligned} \tag{27}$$

which exactly coincides with the result in Eq. (22) by using the following integral formula:[21]

$$\int \frac{{\rm d}^2 z}{\pi} {\rm e}^{-h|z|^2 + sz + tz^*} = \frac{1}{h} \exp\left(\frac{st}{h}\right), \quad {\rm Re}\, h > 0. \tag{28}$$

Further, removing the symbol : : from Eq. (22), we have

$$\begin{aligned}
\rho_t^{\rm c} &= \frac{1}{\kappa t+1} \exp\left(-\frac{1}{\kappa t+1}|\alpha|^2\right) \exp\left(\frac{1}{\kappa t+1} a^\dagger \alpha\right) \\
&\quad \times \exp\left(\ln \frac{\kappa t}{\kappa t+1} a^\dagger a\right) \exp\left(\frac{1}{\kappa t+1} \alpha^* a\right),
\end{aligned} \tag{29}$$

which is just the superposition of multi-photon added thermal states. This shows that, after undergoing the diffusion channel, the initial pure coherent state $\rho_0^{\rm c}$ can be evolved into a mixed thermal superposed state $\rho_t^{\rm c}$ as a result of diffusion noise. For the case of $\alpha = 0$, $\rho_t^{\rm c}$ reduces to a thermal state with the density operator $\frac{1}{\kappa t+1} \exp\left(\ln \frac{\kappa t}{\kappa t+1} a^\dagger a\right)$. That is, a vacuum state evolves into a chaotic field in the diffusion channel, a very meaningful result. However, for $t = 0$, $\rho_t^{\rm c}$ becomes $\rho_0^{\rm c} = |\alpha\rangle\langle\alpha|$.

On the flip side, using IWOP method, we can prove the normalization of $\rho_t^{\rm c}$, i.e.,

$$\begin{aligned}
{\rm tr}\rho_t^{\rm c} &= \frac{1}{\kappa t+1} \exp\left(-\frac{1}{\kappa t+1}|\alpha|^2\right) \\
&\quad \times \int \frac{{\rm d}^2 z}{\pi} \langle z| \exp\left(\frac{1}{\kappa t+1} a^\dagger \alpha\right) \\
&\quad \times \exp\left(\ln \frac{\kappa t}{\kappa t+1} a^\dagger a\right) \exp\left(\frac{1}{\kappa t+1} \alpha^* a\right) |z\rangle \\
&= 1,
\end{aligned} \tag{30}$$

where we have used the identity in Eq. (13).

5. Quantum statistical features in the diffusion channel

5.1. Average number of photons

To obtain the evolution of the average number of photons of the initial coherent state $\rho_0^{\rm c}$ in the diffusion channel, we now calculate the anti-normal ordering product of $\rho_t^{\rm c}$ as

$$\rho_t^{\rm c} = \frac{1}{\kappa t} \exp\left(-\frac{1}{\kappa t}|\alpha|^2\right) {:}\, \exp\left[\frac{1}{\kappa t}\left(\alpha a^\dagger + \alpha^* a - a^\dagger a\right)\right] {:}\, \tag{31}$$

by using the formula that can transform the density operator ρ into its anti-normal ordering form ($\vdots\,\vdots$ denoting antinormal ordering), that is[4]

$$\rho=\int\frac{\mathrm{d}^2\alpha}{\pi}\langle-\alpha|\rho|\alpha\rangle\vdots\exp\left(|\alpha|^2+\alpha^*a-\alpha a^\dagger+a^\dagger a\right)\vdots. \quad (32)$$

So, using the completeness relation of coherent states and Eq. (31), we finally obtain the average number of photons of ρ_t^{c}, that is

$$\begin{aligned}
&\mathrm{tr}\left(\rho_t^{\mathrm{c}}a^\dagger a\right)\\
&=\frac{1}{\kappa t}\exp\left(\frac{-|\alpha|^2}{\kappa t}\right)\mathrm{tr}\left[\vdots\exp\left(\frac{1}{\kappa t}\left(a^\dagger\alpha+a\alpha^*-a^\dagger a\right)\right)\vdots a^\dagger a\right]\\
&=\frac{1}{\kappa t}\exp\left(\frac{-|\alpha|^2}{\kappa t}\right)\\
&\quad\times\mathrm{tr}\left[\int\frac{\mathrm{d}^2z}{\pi}\exp\left(\frac{1}{\kappa t}\left(z^*\alpha+\alpha^*z-|z|^2\right)\right)|z\rangle\langle z|a^\dagger a\right]\\
&=\frac{1}{\kappa t}\exp\left(\frac{-|\alpha|^2}{\kappa t}\right)\\
&\quad\times\int\frac{\mathrm{d}^2z}{\pi}\exp\left(\frac{1}{\kappa t}\left(z^*\alpha+\alpha^*z-|z|^2\right)\right)|z|^2\\
&=\kappa t+|\alpha|^2, \qquad (33)
\end{aligned}$$

where we have used the integration formula[22]

$$\begin{aligned}
&\int\frac{\mathrm{d}^2z}{\pi}z^nz^{*m}\exp(h|z|^2+sz+\eta z^*)\\
&=\exp\left(-\frac{s\eta}{h}\right)\sum_{l=0}^{\min(m,n)}\frac{m!n!s^{m-l}\eta^{n-l}}{l!(m-l)!(n-l)!(-h)^{m+n-l+1}}. \quad (34)
\end{aligned}$$

Obviously, equation (33) refers to the average number of photons of ρ_t^{c}, which increases linearly with time t, in the diffusion channel. This result is reasonable because coherent states always exchange with particles in the surrounding environment as a result of diffusion. Specially, at the initial time, i.e., $t=0$, $\mathrm{tr}\left(\rho_0^{\mathrm{c}}a^\dagger a\right)=|\alpha|^2$, corresponding to the average number of photons of the coherent state ρ_0^{c}. However, for $\alpha=0$, $\mathrm{tr}\left(\rho_t^{\mathrm{c}}a^\dagger a\right)=\kappa t$, it is the evolution of the average number of photons of vacuum in the diffusion channel, which means that the average number of photons of vacuum monotonically increases over time t in the diffusion channel.

5.2. Photon number distribution

This subsection is devoted to the calculation of the probability of detecting m photoelectrons at time t. It is given by the formula

$$\begin{aligned}
p(m,t)&=\mathrm{tr}\left(|m\rangle\langle m|\rho_t^{\mathrm{c}}\right)\\
&=\frac{\exp\left(-\frac{1}{\kappa t+1}|\alpha|^2\right)}{\kappa t+1}\\
&\quad\times\langle m|:\exp\left(\frac{1}{\kappa t+1}\left(a^\dagger\alpha+a\alpha^*-a^\dagger a\right)\right):|m\rangle\\
&=\frac{\exp\left(-\frac{1}{\kappa t+1}|\alpha|^2\right)}{(\kappa t+1)m!}d_z^md_{\beta^*}^m\\
&\quad\times\langle\beta\|\exp\left(\frac{1}{\kappa t+1}\left(\beta^*\alpha+z\alpha^*-\beta^*z\right)\right)\|z\rangle\Big|_{z=\beta^*=0}\\
&=\frac{\exp\left(-\frac{1}{\kappa t+1}|\alpha|^2\right)}{(\kappa t+1)m!}d_z^md_{\beta^*}^m\\
&\quad\times\exp\left(\frac{1}{\kappa t+1}\left(\alpha\beta^*+\alpha^*z+\kappa t\beta^*z\right)\right)\Big|_{z=\beta^*=0}, \quad (35)
\end{aligned}$$

where we have used the identity $|m\rangle=(m!)^{-1/2}d_z^m\,\|z\rangle|_{z=0}$ and the inner product $\langle\beta\|\,z\rangle=\mathrm{e}^{\beta^*z}$, $\|z\rangle$ and $\langle\beta\|$ are unnormalized coherent states. Further, using the generating function for two-variable Hermite polynomial, we obtain

$$p(m,t)=\frac{(-\kappa t)^m\exp\left(-\frac{1}{\kappa t+1}|\alpha|^2\right)}{(\kappa t+1)^{m+1}m!}\mathrm{H}_{m,m}(\rho,\rho^*), \quad (36)$$

where $\rho=\left[-(\kappa t+1)\kappa t\right]^{-1/2}\alpha$. Equation (36) shows that the evolution of the photon number distribution is exactly related to a two-variable Hermite polynomial in the diffusion channel. For the special case, $t=0$, $p(m,0)=\frac{1}{m!}|\alpha|^{2m}\mathrm{e}^{-|\alpha|^2}$, it is indeed the photon number distribution of the state ρ_0^{c}. For $\alpha=0$, $p(m,t)=(\kappa t)^m(\kappa t+1)^{-(m+1)}$, corresponding to the photon number distribution evolution of vacuum in the diffusion channel.

5.3. Wigner distribution

Now we want to investigate the Wigner distribution evolution of the initial state ρ_0^{c} in the diffusion channel. In the coherent state $|z\rangle$ representation, the Wigner operator can be represented as[23,24]

$$\Delta(z)=\mathrm{e}^{2|z|^2}\int\frac{\mathrm{d}^2z'}{\pi^2}|z'\rangle\langle-z'|\exp[2(zz'^*-z^*z')], \quad (37)$$

so the Wigner distribution of ρ_t^{c} can be obtained as

$$\begin{aligned}
w(z,t)&=\mathrm{tr}\left(\rho_t^{\mathrm{c}}\Delta(z)\right)\\
&=\frac{\mathrm{e}^{2|z|^2}}{\kappa t+1}\int\frac{\mathrm{d}^2z'}{\pi^2}\langle-z'|:\exp\left[-\frac{1}{\kappa t+1}\right.\\
&\quad\left.\times\left(|\alpha|^2-a^\dagger\alpha-a\alpha^*+a^\dagger a\right)\right]:|z'\rangle\,\mathrm{e}^{2(zz'^*-z^*z')}\\
&=\frac{1}{\pi(2\kappa t+1)}\exp\left(-\frac{2|z-\alpha|^2}{2\kappa t+1}\right). \quad (38)
\end{aligned}$$

Obviously, the Wigner distribution $w(z,t)$ always keeps Gaussian in the diffusion channel. In particular, for $t=0$, $w(z,0)$ refers to the Wigner distribution of the state ρ_0^{c}, that is

$$w(z,0)=\frac{1}{\pi}\exp\left(-2|z-\alpha|^2\right). \quad (39)$$

For the case of $\alpha=0$, $w(z,t)$ becomes the Wigner distribution evolution for vacuum in the diffusion channel, that is

$$w(z,t)=\frac{1}{\pi(2\kappa t+1)}\exp\left(-\frac{2|z|^2}{2\kappa t+1}\right). \quad (40)$$

6. Entropy evolution

In this section, we shall calculate the entropy evolution of the initial state $\rho_0^{\rm c}$ in the diffusion channel. For any quantum state ρ, the von Neumann entropy is given as $\mathcal{S}(\rho) = -\mathrm{tr}(\rho\ln\rho)$. In terms of the definition of $\mathcal{S}(\rho)$, we need to obtain the natural exponential form of the density operator ρ. Using Eq. (29) and the operator formula[25]

$$\mathrm{e}^{\mathcal{A}}\mathrm{e}^{\mathcal{B}} = \exp\left(\mathcal{A} + \frac{\mu\mathcal{B}+f}{1-\mathrm{e}^{-\mu}} - \frac{f}{\mu}\right), \quad [\mathcal{A},\mathcal{B}] = \mu\mathcal{B} + f, \tag{41}$$

we can rewrite $\rho_t^{\rm c}$ in Eq. (29) as a single exponential, i.e.,

$$\rho_t^{\rm c} = \frac{1}{\kappa t+1}\exp\left[\left(|\alpha|^2 - \alpha a^\dagger - \alpha^* a + a^\dagger a\right)\ln\frac{\kappa t}{\kappa t+1}\right]. \tag{42}$$

It then follows that

$$\ln\rho_t^{\rm c} = \ln\frac{1}{\kappa t+1} + \left(|\alpha|^2 - \alpha a^\dagger - \alpha^* a + a^\dagger a\right)\ln\frac{\kappa t}{\kappa t+1}. \tag{43}$$

Further, using Eqs. (22) and (43), thus the von Neumann entropy evolution of $\rho_0^{\rm c}$ in the diffusion channel reads

$$\begin{aligned}\mathcal{S}(\rho_t^{\rm c}) =& -\ln\frac{1}{\kappa t+1} - |\alpha|^2\ln\frac{\kappa t}{\kappa t+1} \\ & - \frac{1}{\kappa t+1}\exp\left(-\frac{1}{\kappa t+1}|\alpha|^2\right) \\ & \times \mathrm{tr}\left\{\exp\left(\frac{1}{\kappa t+1}\alpha a^\dagger\right)\exp\left(a^\dagger a\ln\frac{\kappa t}{\kappa t+1}\right)\right. \\ & \left.\times\exp\left(\frac{1}{\kappa t+1}\alpha^* a\right)[a^\dagger a - (\alpha a^\dagger + \alpha^* a)]\right\} \\ & \times\ln\frac{\kappa t}{\kappa t+1}. \end{aligned} \tag{44}$$

Noting that the operator identities $\mathrm{e}^{\omega a}a^{\dagger i}\mathrm{e}^{-\omega a} = (a^\dagger+\omega)^i$ and $\mathrm{e}^{a^\dagger a\ln\omega}a^{\dagger i}\mathrm{e}^{-a^\dagger a\ln\omega} = \omega^i a^{\dagger i}$, and using the completeness relation of coherent states, we obtain

$$\begin{aligned}\mathcal{S}(\rho_t^{\rm c}) =& -\ln\frac{1}{\kappa t+1} - |\alpha|^2\ln\frac{\kappa t}{\kappa t+1} - \frac{\exp\left(-\frac{1}{\kappa t+1}|\alpha|^2\right)}{\kappa t+1} \\ & \times\mathrm{tr}\left(:\exp\left(\frac{1}{\kappa t+1}\alpha a^\dagger\right)\right. \\ & \times\exp\left(-\frac{1}{\kappa t+1}a^\dagger a\right) \\ & \times\exp\left(\frac{1}{\kappa t+1}\alpha^* a\right)\left\{\frac{\kappa t}{\kappa t+1}[a^\dagger a\right. \\ & \left.\left.\left.-(\alpha a^\dagger+\alpha^* a)] - \frac{1}{\kappa t+1}|\alpha|^2\right\}:\right)\ln\frac{\kappa t}{\kappa t+1} \\ =& -\ln\frac{1}{\kappa t+1} - \kappa t\ln\frac{\kappa t}{\kappa t+1}, \end{aligned} \tag{45}$$

which means that the entropy evolution $\mathcal{S}(\rho_t^{\rm c})$ only depends on the decay time κt in the diffusion channel and is independence of the amplitude α of the initial coherent state $\rho_0^{\rm c}$. That is because the entropy of the initial pure coherent state $\rho_0^{\rm c}$ is always zero, no matter how large the amplitude α of the coherent state is. In other words, the entropy evolution $S(\rho_t^{\rm c})$ is only related to the diffusion coefficient κ of the diffusion channel.

7. Conclusions

In sum, based on the thermo entangled state representation, the correspondence of the operators between the real and fictious modes are obtained and, using this, the diffusion equation is successfully solved and thus its analytical solution with the Kraus operator is presented. Taking the pure coherent states as initial states, we find that they can evolve into the new mixed thermal superposed states, i.e., the superpositions of multi-photon added thermal states, in the diffusion channel.

On the other hand, the statistical properties of the initial coherent states are investigated, and the results show that the average number of photons of $\rho_t^{\rm c}$ increases monotonically over the decay time κt, the photon number distribution evolution is always related to a two-variable Hermite polynomial, and the Wigner distribution always maintains Gaussian. More meaningly, the entropy evolutions of the coherent states in the diffusion channel are treated, and the entropy evolutions only depend upon the time κt and have nothing to do with the amplitudes of the initial coherent states.

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