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On fractional discrete financial system: Bifurcation, chaos, and control

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The dynamic analysis of financial systems is a developing field that combines mathematics and economics to understand and explain fluctuations in financial markets. This paper introduces a new three-dimensional (3D) fractional financial map and we dissect its nonlinear dynamics system under commensurate and incommensurate orders. As such, we evaluate when the equilibrium points are stable or unstable at various fractional orders. We use many numerical methods, phase plots in 2D and 3D projections, bifurcation diagrams and the maximum Lyapunov exponent. These techniques reveal that financial maps exhibit chaotic attractor behavior. This study is grounded on the Caputo-like discrete operator, which is specifically influenced by the variance of the commensurate and incommensurate orders. Furthermore, we confirm the presence and measure the complexity of chaos in financial maps by the 0–1 test and the approximate entropy algorithm. Additionally, we offer nonlinear-type controllers to stabilize the fractional financial map. The numerical results of this study are obtained using MATLAB.

Keywords: financial model, stability, chaos, commensurate and incommensurate orders, complexity

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1. Introduction

The discipline of economics has witnessed the increasing use of dynamic systems.[1] Financial models are complex systems characterized by nonlinear interactions between various factors such as banking, investment and consumption, leading to chaotic behavior in financial phenomena.[2] In the context of systems theory, a system's behavior is divided into various dynamic states, such as stability and multistability. It can also be unstable, with different properties, including chaos. Evidently, chaos highlights how even tiny changes can amplify and decrease stability, ultimately influencing the system's behavior profoundly. The fact that there are unstable equilibrium points can be a contributing factor to the financial system's overall chaotic behavior.[3]

In the past few years, researchers have increasingly concentrated on study of chaos theories, especially the presence of chaos in finance and economics.[4] Moreover, rather than continuous time, chaotic systems are present in a number of nonlinear dynamical models. In Ref. [5] the authors discuss a continuous dynamical system that explains how the government interacts with small enterprises and banks. Even so, discrete-time systems represented by difference equations are often a more suitable tool for modeling, producing specific dynamic properties that can be used in diverse sectors of economics.[6]

Recently, research in this area has gradually increased to include the dynamical analysis of fractional economic models.[7] Additionally, many mathematicians have become interested in discrete fractional calculus, where the operator is derived by discretizing a continuous-time fractional operator, and are intrigued by its implementations in extensive fields, for instance, image encryption,[8] secure communications,[9] economics and finance.[10,11] The behaviors of fractional financial systems have been studied in multiple situations, for example, analysis of fixed points, stability and chaotic motion. As a result, fractional economic models are necessary and even advantageous, for instance, Chu *et al.*[12] introduced a discrete fractional artificial macroeconomic model, while Pourrahimian *et al.*[13] examined chaos theory in project management and its application. However, in the field of discrete fractional calculus, there is a paucity of literature addressing dynamics and control issues in fractional-order systems.[14–19]

The majority of recent works have utilized the fractional Caputo-like operator; it has been widely applied in various economic models and provides a good explanation of the memory characteristics of financial systems. This is because the initial value in the case of fractional orders has the same conditions as for integer orders. Khennaoui *et al.*[20] proposed a novel triopoly game with bounded rationality via a Caputo-like operator. Abbes *et al.*[21] presented a new incommensurate fractional macroeconomic map. Wu and Xia[22]

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introduced a novel sliding mode control method and examined the stability of financial systems using fractional order. In Ref. [23] we delved into the chaotic and bifurcation of a fractional population map. On the other hand, Refs. [24–26] studied various commensurate and incommensurate discrete chaotic systems.

Therefore, it is logical to focus on the dynamical analysis of financial maps with commensurate and incommensurate order. This study aims to examine a new financial model with commensurate and incommensurate orders using certain theoretical and numerical techniques. This article is structured as follows: in Section 2 we define certain basic preliminaries of discrete fractional derivatives and lemmas of stability theory for equilibrium points of nonlinear commensurate and incommensurate financial maps. In Sections 3 and 4 we describe the system and analyze the stability of its equilibrium points in various fractional orders. Phase attractors, bifurcation and the maximum Lyapunov exponent are also analyzed in detail. In Section 5 we validate the chaotic attractor within the proposed financial map and measure complexity, utilizing the approximate entropy and 0–1 test results. In Section 6 we propose nonlinear controllers to ensure the stability of the fractional financial map. In conclusion, we close the paper with a summary of the most important results and perspectives for future works.

2. Basic concepts and model description

Firstly, we present certain definitions and the lemma of discrete fractional calculus, as well as lemmas to assure the stability standards for the equilibrium points of discrete commensurate and incommensurate fractional orders systems.

2.1. Basic concepts

Definition 1[27] Let $F:\mathbb{N}_t\to\mathbb{R}$, on a time scale $\mathbb{N}_t=\{t,t+1,t+2,t+3\ldots\}$, for $t\in\mathbb{R}$. We define $\Delta_t^{-\gamma}$ the γ-th discrete fractional sum for $F(s)$ by

$$\Delta_t^{-\gamma}F(s)=\frac{1}{\Gamma(\gamma)}\sum_{\sigma=t}^{s-\gamma}(s-1-\sigma)^{(\gamma-1)}F(\sigma),\quad s\in\mathbb{N}_{t+\gamma},\ \gamma>0. \tag{1}$$

Definition 2[28] For $\alpha-1<\gamma\le\alpha$ and $\alpha=\lceil\gamma\rceil+1$, the γ-th Caputo discrete difference operator for $F(s)$ is given as

$$\begin{aligned}{}^{C}\Delta_t^{\gamma}F(s)&=\Delta_t^{-(\alpha-\gamma)}\Delta^{\alpha}F(s)\\&=\frac{1}{\Gamma(\alpha-\gamma)}\sum_{\sigma=t}^{s-(\alpha-\gamma)}(s-1-\sigma)^{(\alpha-\gamma-1)}\Delta^{\alpha}F(\sigma),\end{aligned} \tag{2}$$

where $s\in\mathbb{N}_{t+\alpha-\gamma}$. The $(s-1-\sigma)^{(\alpha-1-\gamma)}$ is written as

$$(s-1-\sigma)^{(\alpha-1-\gamma)}=\frac{\Gamma(s-\sigma)}{\Gamma(1+s-\sigma-\alpha-\gamma)}, \tag{3}$$

and

$$\begin{aligned}\Delta^{\alpha}F(s)&=\Delta\Delta^{\alpha-1}F(s)\\&=\sum_{k=0}^{\alpha}\binom{\alpha}{k}(-1)^{\alpha-k}F(s+k).\end{aligned} \tag{4}$$

Now, we offer the lemma below, allowing us to obtain the numerical formula for the identified fractional discrete system.

Lemma 1[29] The solution to the initial state problem

$$\begin{cases}{}^{C}\Delta_t^{\gamma}w(s)=f(s+\gamma-1,w(s+\gamma-1)),\\ \Delta^{k}w(t)=w_k,\quad \alpha=\lceil\gamma\rceil+1,\quad k=0,1,\ldots,\alpha-1\end{cases} \tag{5}$$

is indicated as

$$\begin{aligned}w(s)&=w_0(t)+\frac{1}{\Gamma(\gamma)}\sum_{\sigma=t+\alpha-\gamma}^{s-\gamma}(s+1-\sigma)^{(\gamma-1)}\\&\quad\times F(\sigma+\gamma-1,w(\sigma+\gamma-1)),\end{aligned} \tag{6}$$

where

$$w_0(t)=\sum_{k=0}^{\alpha-1}\frac{(s-t)^k}{\Gamma(k+1)}\Delta^k w(t).$$

Next, we need the following lemma in order to determine the stability standards for the equilibrium points of the fractional map via commensurate order.

Lemma 2[30] The discrete commensurate fractional system is as follows:

$${}^{c}\Delta_t^{\gamma}W(s)=Jw(s-1+\gamma). \tag{7}$$

Let $W(s)=(w_1(s),\ldots,w_n(s))^{\mathrm{T}}$, $0<\gamma<1$ and $\forall s\in\mathbb{N}_{t+1-\gamma}$. $J\in R^{n\times n}$ is a matrix. The system (7) has an asymptotically stable zero equilibrium if

$$\lambda_j\in\left\{\rho\in\mathbb{C}:|\rho|\le\left(2\cos\frac{|\arg\rho|-\pi}{2-\gamma}\right)^{\gamma}\text{ and }|\arg\rho|\ge\frac{\gamma\pi}{2}\right\}, \tag{8}$$

where λ_j are the eigenvalues of J.

The following lemma is for the stability conditions of the incommensurate discrete fractional system.

Lemma 3[31] Consider the system

$$\begin{cases}{}^{c}\Delta_t^{\gamma_1}x_1(s)=w_1(x(s-1+\gamma_1)),\\ {}^{c}\Delta_t^{\gamma_2}x_2(s)=w_2(x(s-1+\gamma_2)),\qquad s=0,1,\ldots.\\ \quad\vdots\\ {}^{c}\Delta_t^{\gamma_n}x_n(s)=w_n(x(s-1+\gamma_n)),\end{cases} \tag{9}$$

Let $x(s)=(x_1(s),\ldots,x_n(s))^{\mathrm{T}}\in\mathbb{R}^n$, $W=(w_1,\ldots,w_n):\mathbb{R}^n\to\mathbb{R}^n$, consider $\gamma_i\in(0,1]$, $i=\overline{1..n}$ and L (LCM) is the lowest common multiple of the denominators $\bar{u}_i$ of γ_i. Added to $\gamma_i=\bar{v}_i\bar{u}_i$, $(\bar{u}_i,\bar{v}_i)=1$, $\bar{v}_i,\bar{u}_i\in\mathbb{Z}_+$, $\forall i=\overline{1,n}$. If each root of Eq. (9)

$$\det(\mathrm{diag}(\lambda^{L\gamma_1},\ldots,\lambda^{L\gamma_n})-(1-\lambda^{L})J)=0 \tag{10}$$

is included in $\mathbb{C}/K^{\mu}$, where J is the Jacobian matrix of system (9). Therefore the trivial solution of system (9) is asymptotically stable, where

$$K^{\mu}=\left\{\rho\in\mathbb{C}:|\rho|\leq\left(2\cos\frac{|\arg\rho|}{\mu}\right)^{\mu}\text{ and }|\arg\rho|\leq\frac{\rho\pi}{2}\right\},\tag{11}$$

and $\mu=1/L$.

2.2. The financial map

Financial non-linear systems refer to the complex interaction of variables that influence financial behavior. Thus, the continuous financial model is given in Ref. [32]. Xin *et al.*[33] describe a discrete-time system, which can be presented as follows:

$$\begin{cases}x(s+1)=x(s)+\delta(z(s)+(y(s)-a)x(s)),\\ y(s+1)=y(s)+\delta(1-by(s)-x^2(s)),\\ z(s+1)=z(s)+\delta(-x(s)-cz(s)),\end{cases}\tag{12}$$

where $\delta\in]0,1[$ is the step size, and a, b and c are positive constant parameters representing the amount of savings, the cost per investment and the demand elasticity of commercial markets. x, y, z are the changes over time in the interest rate, the investment demand and the price index, respectively. When the real parameter $a\in[1.7,2]$ is varied, the financial system (12) is capable of carrying out a range of dynamic cases. For the parameters $c=0.2$, $b=0.6$, $\delta=0.9$ and the initial condition $x(0)=0.4$, $y(0)=0.6$, $z(0)=0.8$, Figs. 1(b) and 1(c) show the bifurcation diagram and the MLE that corresponds to it. In particular, the financial model (12) has chaotic behavior when $a\in[1.7,1.82]$, so that the MLE is positive and periodic in the rest of the field but MLE it eventually drops to almost zero. Furthermore, model (12) could create chaotic behavior as illustrated in Fig. 1(a).

In this work we aim to derive a fractional financial map using the Caputo-like operator. First, we formulate the first-order difference of the financial system as

$$\begin{cases}\Delta x(s)=\delta(z(s)+(y(s)-a)x(s)),\\ \Delta y(s)=\delta(1-by(s)-x^2(s)),\\ \Delta z(s)=\delta(-x(s)-cz(s)),\end{cases}\tag{13}$$

with $\Delta z(s)=z(s+1)-z(s)$ the standard difference operator.

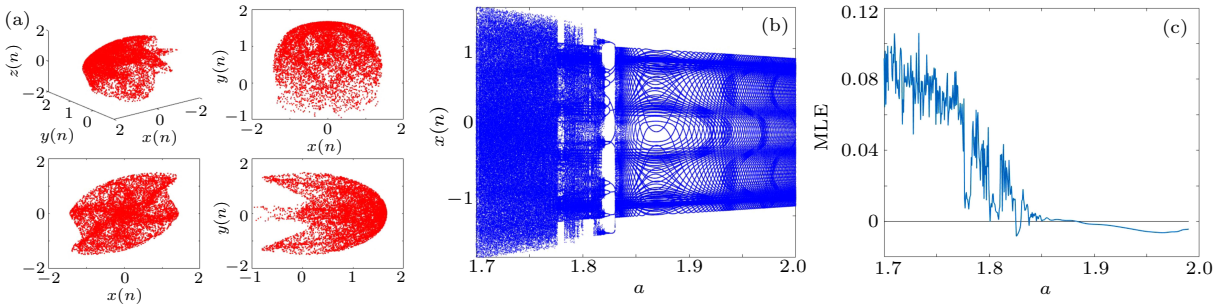


Fig. 1. (a) Chaotic attractor of the financial map (12). (b) Bifurcation diagram from $a\in[1.7,2]$. (c) The corresponding MLE.

3. The commensurate fractional financial map

In this section we discuss into the behavior of the commensurate fractional financial map. We use the difference operator ${}^{c}\Delta_{t}^{\gamma}$. The idea of the commensurate order means that the values of γ are equal for any equation inside the system. The commensurate fractional financial map is represented as follows:

$$\begin{cases}{}^{c}\Delta_{t}^{\gamma}x(s)=\delta(z(s-1+\gamma)\\ \qquad\qquad+(y(s-1+\gamma)-a)x(s-1+\gamma)),\\ {}^{c}\Delta_{t}^{\gamma}y(s)=\delta(1-by(s-1+\gamma)-x^2(s-1+\gamma)),\\ {}^{c}\Delta_{t}^{\gamma}z(s)=\delta(-x(s-1+\gamma)-cz(s-1+\gamma)).\end{cases}\tag{14}$$

3.1. Stability analysis

Here, we evaluate the stability of the equilibrium points of commensurate fractional financial map (14). We solve the following set:

$$\begin{cases}\delta z+\delta(y-a)x=0,\\ \delta-\delta by-\delta x^2=0,\\ -\delta x-\delta cz=0.\end{cases}\tag{15}$$

If $c-b-abc\geq 0$, the map has three equilibrium points $E_1^*=(0,\frac{1}{b},0)$, $E_{2,3}^*=(\pm\sqrt{\frac{c-b-abc}{c}},\frac{1+ac}{c},\mp\frac{1}{c}\sqrt{\frac{c-b-abc}{c}})$, otherwise if $c-b-abc\leq 0$ the map has one equilibrium point $E_1^*=(0,\frac{1}{b},0)$.

Theorem 1 In the event that $b=0.6$ the equilibrium point E_1^* of the map (14) is locally asymptotically stable in $\gamma<0.9$.

Proof We calculate the Jacobian matrix of the system (14)

$$J=\begin{pmatrix}\delta y-\delta a & \delta x & \delta\\ -2\delta x & -\delta b & 0\\ -\delta & 0 & -\delta c\end{pmatrix}.\tag{16}$$

The λ_j, $j=1,2,3$, eigenvalues of the matrix J at the equilibrium point E_1^* for $b=0.6$ are $\lambda_1=-0.105+0.89687\mathrm{i}$,

$\lambda_2 = -0.105 - 0.89687\mathrm{i}$ and $\lambda_3 = -0.54$. Therefore, by substituting the eigenvalues λ_1, λ_2 and λ_3 into the condition stated in Lemma 2 when $a = 1.7$, $c = 0.2$ and $b = 0.6$ the stability of equilibrium point E_1^* of a financial model with commensurate order is given by the next two cases as confirmation of proof.

Case 1 Let $\gamma = 0.69$. We can deduce that $\lambda_j < \left(2\cos\frac{|\arg\lambda_j|-\pi}{2-\gamma}\right)^{\gamma}$ and $|\arg\lambda_j| > \frac{\gamma\pi}{2}$, $\forall j = 1,\ldots,3$. According to Lemma 2, E_1^* for the $b = 0.6$ equilibrium point of the commensurate financial map (14) is asymptotically stable.

Case 2 Let $\gamma = 0.97$. We can conclude that $\lambda_j \geq \left(2\cos\frac{|\arg\lambda_j|-\pi}{2-\gamma}\right)^{\gamma}$, $\forall j = 1,\ldots,3$. From Lemma 2, the equilibrium E_1^* of the model is then unstable and in this case the fractional commensurate financial system (14) fulfills the prerequisite for the presence of chaos.

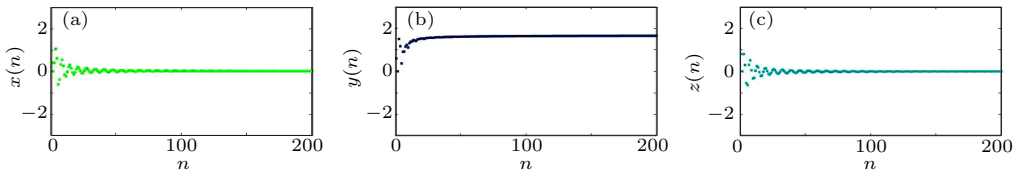


Fig. 2. Time evolution of the commensurate financial map (14) with $\gamma = 0.69$.

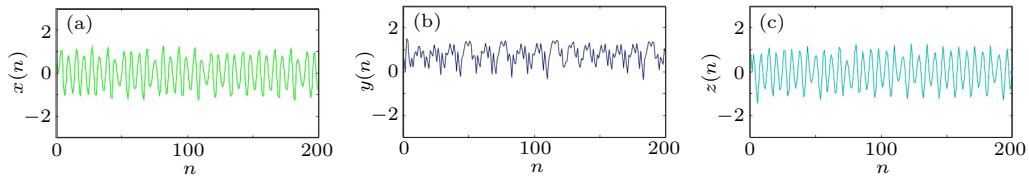


Fig. 3. Time evolution of the commensurate financial map (14) with $\gamma = 0.97$.

3.2. Chaos analysis

Based on Lemma 1, the numerical procedure of the commensurate fractional financial map (14) is as follows:

$$\begin{cases} x(n) = x(0) + \sum_{j=0}^{n-1} \frac{\Gamma(\gamma+n-j-1)}{\Gamma(\gamma)\Gamma(n-j)} \\ \qquad \times \left(\delta(z(j) + y(j)x(j) - ax(j))\right), \\ y(n) = y(0) + \sum_{j=0}^{n-1} \frac{\Gamma(\gamma+n-j-1)}{\Gamma(\gamma)\Gamma(n-j)} \\ \qquad \times \left(\delta(1 - by(j) - x^2(j))\right), \\ z(n) = z(0) + \sum_{j=0}^{n-1} \frac{\Gamma(\gamma+n-j-1)}{\Gamma(\gamma)\Gamma(n-j)} \\ \qquad \times \left(\delta(-cz(j) - x(j))\right). \end{cases} \tag{17}$$

The Lyapunov exponents are crucial tools for demonstrating chaos in fractional systems maps where they correspond to a bifurcation diagram. In Ref. [34], Wu *et al.* considered the prolongation of the Jacobian matrix algorithm to fractional discrete systems that are used to apply the states in the fractional financial map (14). The tangent map holds a discrete memory effect by

$$J_i = \begin{pmatrix} d_1 & d_2 & d_3 \\ h_1 & h_2 & h_3 \\ v_1 & v_2 & v_3 \end{pmatrix}, \tag{18}$$

where

$$\begin{cases} d_i(n) = d_i(0) + \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{n-1} \frac{\Gamma(\gamma+n-j-1)}{\Gamma(n-j)} \\ \qquad \times \Big(\left(\delta(d_i(j)(y(j)-1) + h_i(j)x(j) + v_i(j))\right)\Big), \\ h_i(n) = h_i(0) + \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{n-1} \frac{\Gamma(\gamma+n-j-1)}{\Gamma(n-j)} \\ \qquad \times \Big(\delta(d_i(j)(-2x(j)) - bh_i(j))\Big), \quad i = 1,2,3, \\ v_i(n) = v_i(0) + \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{n-1} \frac{\Gamma(\gamma+n-j-1)}{\Gamma(n-j)} \\ \qquad \times \Big(\delta(-d_i(j) - cv_i(j))\Big), \end{cases} \tag{19}$$

with

$$\begin{pmatrix} d_1(0) & d_2(0) & d_3(0) \\ h_1(0) & h_2(0) & h_3(0) \\ v_1(0) & v_2(0) & v_3(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

So, the Lyapunov exponents of the fractional order system could be given by the following formula:

$$\lambda_r(x_0) = \lim_{i\to\infty} \frac{1}{i} \ln|\lambda_r^{(i)}|, \quad \text{for} \quad r = 1,2,3, \tag{21}$$

with λ_r being the eigenvalues of J_i.

Now, we evaluate the dynamics of the commensurate fractional financial map (14). We set $x(0) = 0.4$, $y(0) = 0.6$, $z(0) = 0.8$ as the initial conditions and $a = 1.7$, $c = 0.2$, $b = 0.6$ are the positive parameters; the step size $\delta = 0.9$. Figure 4 illustrates the phase portraits in the 2D and 3D planes for values γ ($\gamma = 0.95$, $\gamma = 0.97$, $\gamma = 0.99$ and $\gamma = 1$) of the commensurate order. As seen, the trajectories in the suggested commensurate financial map transition between periodic behavior

and chaotic attractor as the γ commensurate values vary. The bifurcation diagram and the MLE display the variations in the behaviors of the commensurate fractional financial map (14). Figure 5 considers the variation of parameter a. We plot bifurcation of map (14) with $\in [1,2]$, which corresponds to $\gamma = 0.97$, $\gamma = 0.8$, $\gamma = 0.99$. It is clear that each parameter a and the commensurate order γ have an impact on the circumstances of the commensurate financial map (14). For instance, when $\gamma = 0.97$, the chaos in the region for a within $[1.57, 1.73]$. The map is stable with period-six orbits occurring when $a \in (1.74, 2)$. For $\gamma = 0.8$, the system indicates chaotic behavior for a within of $[1.12, 1.2]$. Furthermore, when $\gamma = 0.99$, chaos is present in $[1.67, 1.78]$. If the MLE value is positive chaos exists. Otherwise, when MLE is negative the behavior is periodic. Hence, this variation in chaotic regions across different parameter values sheds important light on the intricate behavior of the commensurate financial map.

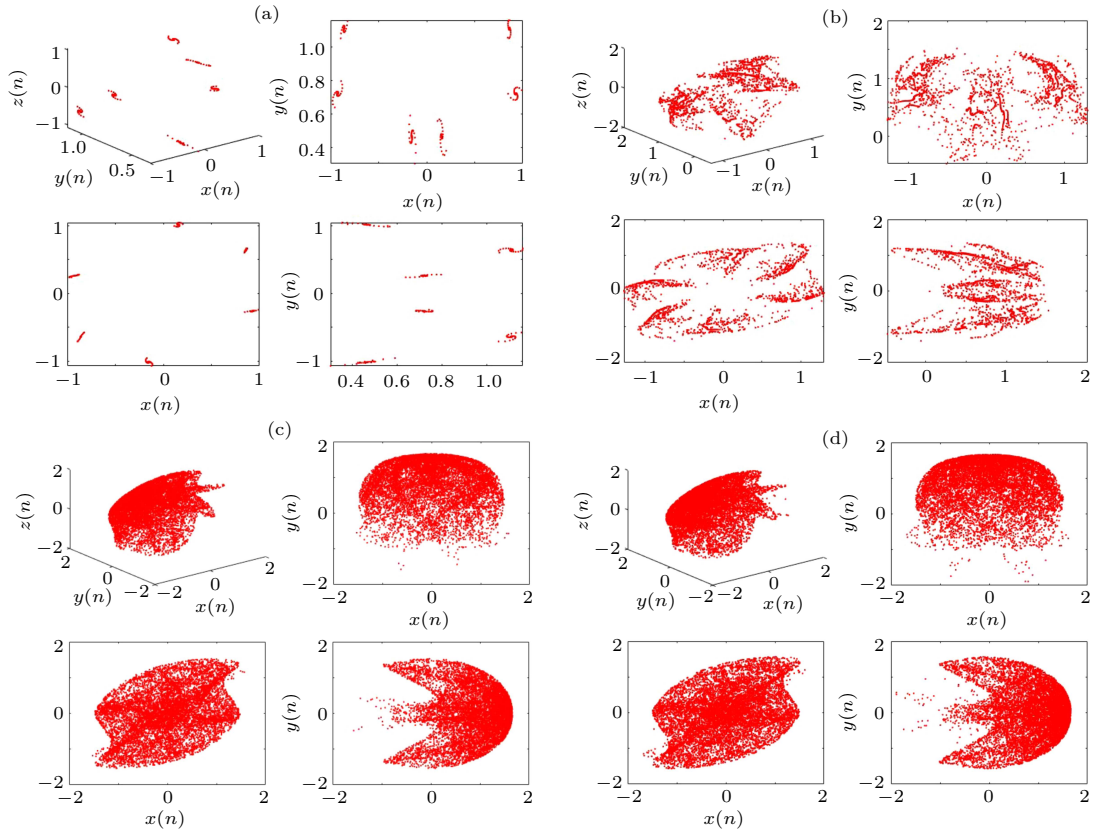


Fig. 4. Portraits of the commensurate financial map (14) for different values of γ: (a) $\gamma = 0.95$, (b) $\gamma = 0.97$, (c) $\gamma = 0.99$, (d) $\gamma = 1$.

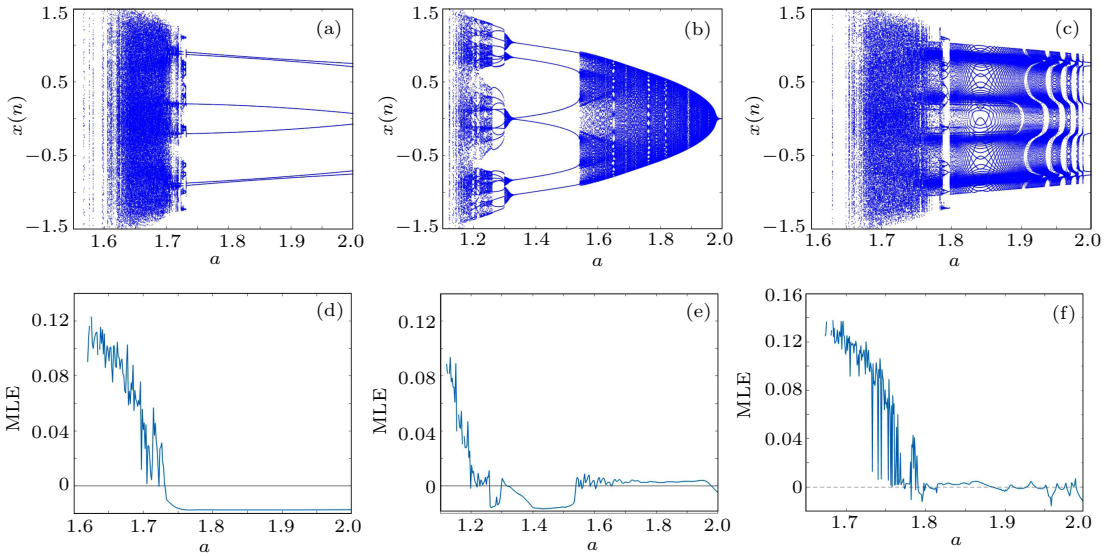


Fig. 5. Bifurcation of the commensurate financial map (14) with $a \in [1,2]$, for (a) $\gamma = 0.97$, (b) $\gamma = 0.8$, (c) $\gamma = 0.99$. Panels (d)–(f) show the corresponding MLE.

Figure 6 shows γ varying with a step size 0.001 from 0 to 1. The system (14) illustrates varied dynamics involving chaos and no chaos in distinct regions. Certainly, we can observe chaotic attraction for $\gamma \in (0.96, 1)$. Furthermore, when $\gamma \in (0.73, 0.82)$ very slight chaos appears. Similarly, periodic trajectories occur for γ within the interval of $[0.1, 0.7]$. In addition, when $\gamma \in (0, 0.09)$, the trajectories exhibit regular patterns. MLE varies from positive to negative. Conversely, the chaos present in the commensurate financial map is inferred when MLE is positive.

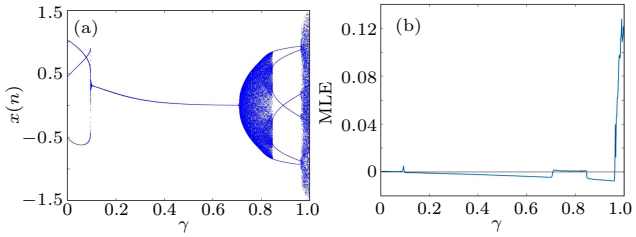


Fig. 6. (a) Bifurcation of commensurate financial map (14) for $\gamma \in (0, 1]$. (b) The corresponding MLE.

4. The incommensurate fractional financial map

In this section we analyze the behavior of the incommensurate fractional financial map. The principle is based on the use of diverse $\gamma_1, \gamma_2, \gamma_3$ fractional orders for each individual equation of the system. The characterization of the incommensurate fractional financial map can be defined as

$$\begin{cases} {}^{c}\Delta_t^{\gamma_1} x(s) = \delta(z(s-1+\gamma_1) \\ \qquad\qquad + (y(s-1+\gamma_1) - a)x(s-1+\gamma_1)), \\ {}^{c}\Delta_t^{\gamma_2} y(s) = \delta(1 - by(s-1+\gamma_2) - x^2(s-1+\gamma_2)), \\ {}^{c}\Delta_t^{\gamma_3} z(s) = \delta(-x(s-1+\gamma_3) - cz(s-1+\gamma_3)). \end{cases} \tag{22}$$

4.1. Stability analysis

Here, to investigate the stability of equilibrium E_1^* of the financial map (22) for $b = 0.6$ via incommensurate order utilizing the condition stated in Lemma 3, we assess the following cases as examples.

Case 1 Let us consider $(\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.7, 0.8)$. Notice that $L = 10$, and so

$$\det\left(\begin{pmatrix} \lambda^6 & 0 & 0. \\ 0 & \lambda^7 & 0 \\ 0 & 0 & \lambda^8 \end{pmatrix} - (1-\lambda^{10}) \begin{pmatrix} -0.03 & 0 & -0.9 \\ 0 & -0.54 & 0 \\ 0.9 & 0 & -0.18 \end{pmatrix}\right) = 0$$

$$\begin{aligned} \Leftrightarrow \quad & 0.440316\lambda^{30} - 0.0162\lambda^{28} + 0.8154\lambda^{27} + 0.0972\lambda^{26} - 0.03\lambda^{25} + 0.54\lambda^{24} - 0.18\lambda^{23} + \lambda^{21} \\ & - 1.32095\lambda^{20} + 0.0324\lambda^{18} - 1.6308\lambda^{17} + 0.1944\lambda^{16} + 0.03\lambda^{15} - 0.54\lambda^{14} + 0.18\lambda^{13} \\ & + 1.32095\lambda^{10} - 0.0162\lambda^{8} + 0.8154\lambda^{7} - 0.0972\lambda^{6} - 0.440316 = 0. \end{aligned} \tag{23}$$

The solutions of Eq. (23) are

$$\boldsymbol{\lambda} = \begin{pmatrix} -1.26849 \\ -0.980576 \pm 0.109963\mathrm{i} \\ -0.944221 \pm 0.605882\mathrm{i} \\ -0.795128 \pm 0.439379\mathrm{i} \\ -0.728739 \pm 0.547445\mathrm{i} \\ -0.342607 \pm 0.871722\mathrm{i} \\ -0.222415 \pm 0.85497\mathrm{i} \\ -0.194083 \pm 1.0402\mathrm{i} \\ 0.194083 \pm 1.0402\mathrm{i} \\ 0.342607 \pm 0.871722\mathrm{i} \\ 0.562193 \pm 1.08285\mathrm{i} \\ 0.651547 \pm 0.716927\mathrm{i} \\ 0.728739 \pm 0.547445\mathrm{i} \\ 0.944221 \pm 0.605882\mathrm{i} \\ 0.980576 \pm 0.109963\mathrm{i} \\ 0.876096 \end{pmatrix}. \tag{24}$$

Thus, one can see that we obtain some solutions λ_j of the system (23) that satisfy $|\lambda_j| > \left(2\cos\left(10|\arg\lambda_j|\right)\right)^{\frac{1}{10}}$, while the other solutions λ_j satisfy $|\arg\lambda_j| > \pi/20$. This signifies $\lambda_j \in \mathbb{C}/K^{\frac{1}{10}}$, $(j = \overline{1,30})$.

According to Lemma 3, E_1^* for the equilibrium point $b = 0.6$ of the incommensurate financial map (22) is an asymptotically stable point.

Case 2 when $(\gamma_1, \gamma_2, \gamma_3) = (0.92, 1, 1)$ note that $L = 50$, and so

$$\det\left(\begin{pmatrix} \lambda^{46} & 0 & 0. \\ 0 & \lambda^{50} & 0 \\ 0 & 0 & \lambda^{50} \end{pmatrix} - (1-\lambda^{10}) \begin{pmatrix} -0.03 & 0 & -0.9 \\ 0 & -0.54 & 0 \\ 0.9 & 0 & -0.18 \end{pmatrix}\right) = 0$$

$$\Leftrightarrow \quad 0.361284\lambda^{150} + 0.3772\lambda^{146} - 0.312252\lambda^{100} + 0.5256\lambda^{96} - 0.489348\lambda^{50} + 0.0972\lambda^{46} + 0.440316 = 0. \tag{25}$$

Consequently, by solving Eq. (25) it can be inferred that there exist solutions $\lambda_j = (0.9934 \pm 0.0184366\mathrm{i})$ that imply $|\arg\lambda_j| \leq \left(2\cos\left(50|\arg\lambda_j|\right)\right)^{\frac{1}{50}}$ and $|\arg\lambda_j| \leq \frac{\pi}{100}$, meaning that $\lambda_j \in K^{\frac{1}{50}}$.

Hence, according to Lemma 3, the equilibrium point E_1^* for $b = 0.6$ is unstable. Therefore, in this case the fractional financial map (22) provides the conditions needed for exhibiting chaotic behavior.

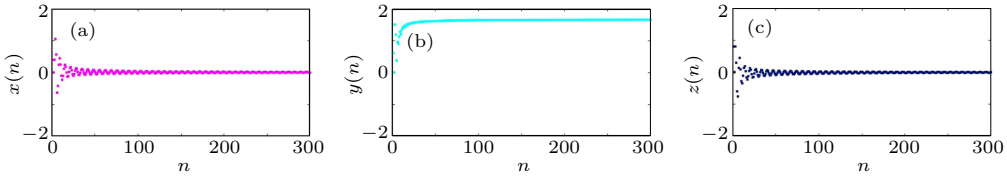


Fig. 7. Time evolution of the incommensurate fractional financial map (22) with $(\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.7, 0.8)$.

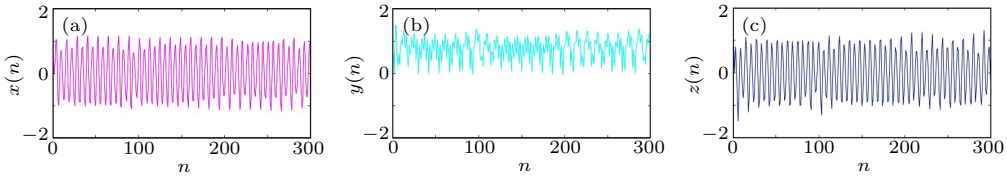


Fig. 8. Time evolution of the incommensurate financial map (22) with $(\gamma_1, \gamma_2, \gamma_3) = (0.92, 1, 1)$.

4.2. Bifurcation and Lyapunov exponents

Here we present the numerical formula of the incommensurate fractional financial map (22) as follows:

$$\begin{cases} x(n) = x(0) + \sum_{j=0}^{n-1} \dfrac{\Gamma(n-1-j+\gamma_1)}{\Gamma(\gamma_1)\Gamma(n-j)} \left(\delta(z(j) + (y(j)x(j) - ax(j))\right), \\ y(n) = y(0) + \sum_{j=0}^{n-1} \dfrac{\Gamma(n-1-j+\gamma_2)}{\Gamma(\gamma_2)\Gamma(n-j)} \left(\delta(1 - by(j) - x^2(j))\right), \\ z(n) = z(0) + \sum_{j=0}^{n-1} \dfrac{\Gamma(n-1-j+\gamma_3)}{\Gamma(\gamma_3)\Gamma(n-j)} \left(\delta(-cz(j) - x(j))\right). \end{cases} \tag{26}$$

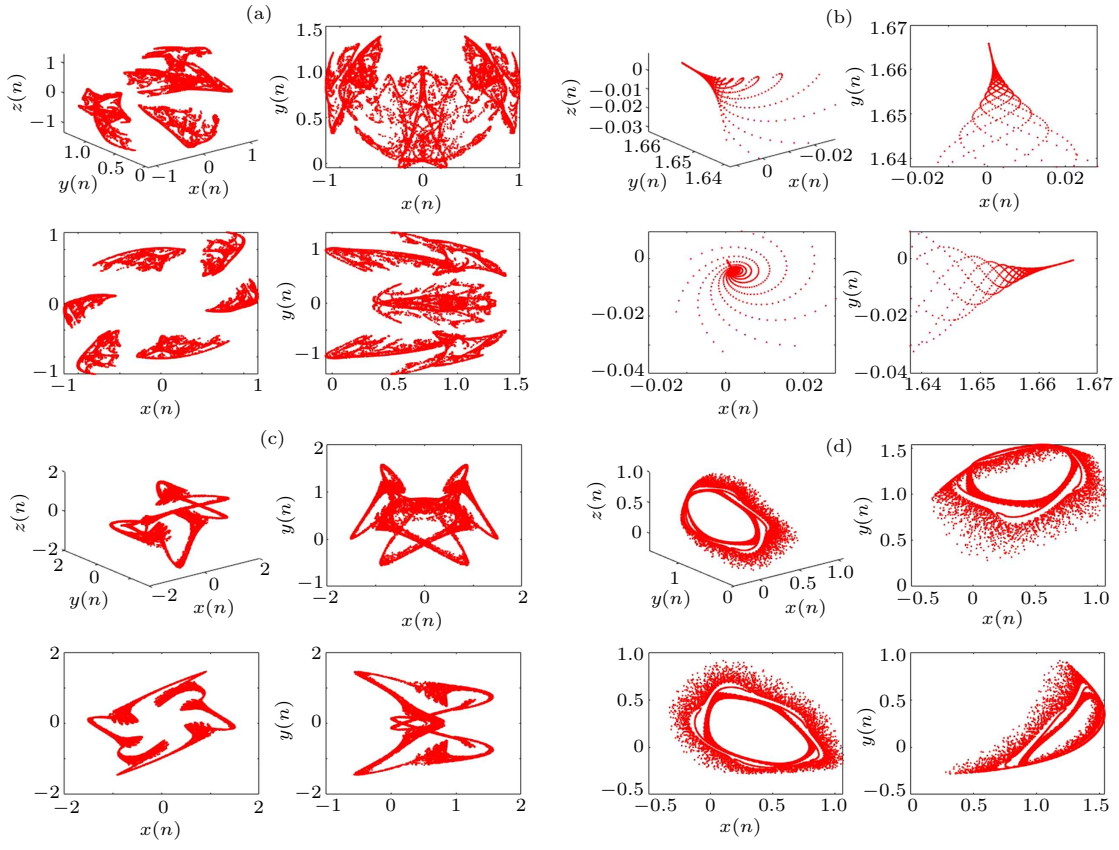


Fig. 9. Diagrams of the incommensurate fractional financial map (22) for different $(\gamma_1, \gamma_2, \gamma_3)$: (a) $(\gamma_1, \gamma_2, \gamma_3) = (0.92, 1, 1)$, (b) $(\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.7, 0.8)$, (c) $(\gamma_1, \gamma_2, \gamma_3) = (1, 0.05, 1)$, (d) $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 0.05)$.

Therefore, we look at the behavior of the incommensurate fractional financial map. Figure 9 shows the phase portraits of the incommensurate order values for γ_1, γ_2, γ_3. It can seen that there is a difference between the chaotic attractors of the commensurate financial map and incommensurate order. In Fig. 10 the bifurcation and its corresponding MLE are carried out using the incommensurate order fractional financial map (22). With the parameter a varying within $[1,2]$ the simulations were conducted with $a = 1.7$, $c = 0.2$, $b = 0.6$ and $\delta = 0.9$ and the initial conditions designated by $x(0) = 0.4$, $y(0) = 0.6$ and $z(0) = 0.8$. Clearly, the figure shows various trajectories, meaning that the change in fractional order $(\gamma_1, \gamma_2, \gamma_3)$ has a noticeable effect on the behavior of the fractional financial map with incommensurate order (22). For example, when $(\gamma_1, \gamma_2, \gamma_3) = (0.92, 1, 1)$, chaos is present for $a \in [1.56, 1.72]$, and in the event that a increases, the trajectories of the proposed model (22) are stable with the appearance of two-, four- and six-period orbits. On the other hand, when $(\gamma_1, \gamma_2, \gamma_3) = (1, 0.05, 1)$, we can see that when $a \in [1.6, 1.72]$, the trajectories become chaotic and remain stable for large values of a, while as $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 0.1)$ the states of the model change if the parameter a varies from periodic to chaotic behavior accompanied by doubling bifurcation, provided that a increases, which is clear from the positive or negative values of MLE. These results confirm the effect of changes in parameter a and the incommensurate values γ_1, γ_2 and γ_3 on the sensitivity of the incommensurate financial map (22).

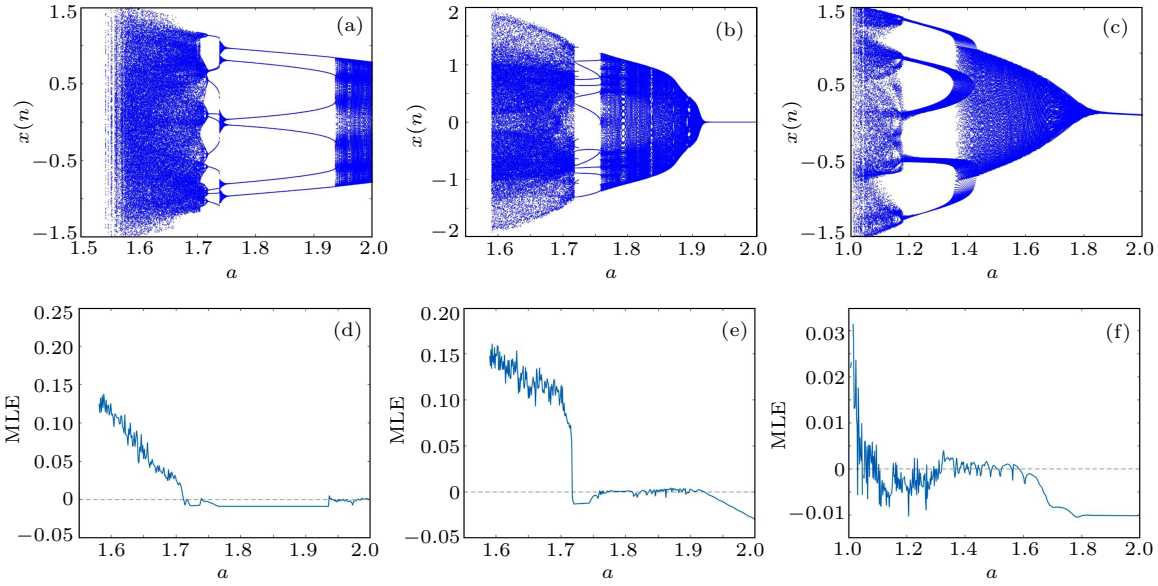


Fig. 10. Bifurcations of the incommensurate financial map (22) versus the parameter system a for different $(\gamma_1, \gamma_2, \gamma_3)$: (a) $(\gamma_1, \gamma_2, \gamma_3) = (0.92, 1, 1)$, (b) $(\gamma_1, \gamma_2, \gamma_3) = (1, 0.05, 1)$ and (c) $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 0.1)$. Panels (d)–(f) show the corresponding MLE.

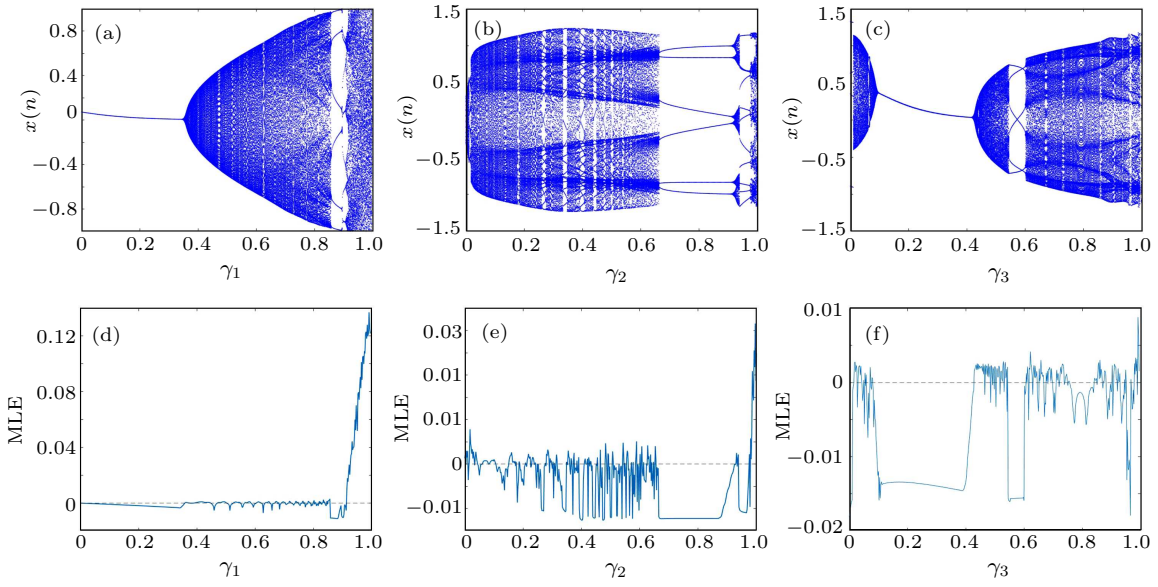


Fig. 11. Bifurcation of the incommensurate financial map (22) and the MLE for (a) $\gamma_2 = \gamma_3 = 1$, (b) $\gamma_1 = 0.92$, $\gamma_3 = 1$ and (c) $\gamma_1 = 1$, $\gamma_2 = 0.9$. Panels (d)–(f) show the corresponding MLE.

Additionally, to offer further clarification of the effect of the incommensurate orders on the behavior of the fractional financial map (22), Figs. 11(a) and 11(d) elucidate the alteration of γ_1 from 0 to 1 with $\Delta\gamma_1 = 0.001$. A bifurcation diagram and the equivalent maximum Lyapunov exponents of the incommensurate fractional financial map (22) are established for $\gamma_2 = 1$ and $\gamma_3 = 1$. So in Fig. 11(a), for the smallest values of γ_1, at the interval $(0, 0.88)$ the map (22) displays a periodic attitude in accordance with negative Lyapunov exponents as portrayed in Fig. 11(d). Conversely, as $\gamma_1 \in (0.89, 1)$, chaotic behaviors emerge with positive values of MLE. In Figs. 11(b) and 11(e) we see the variation of $\gamma_2 \in (0, 1)$ with $\Delta\gamma_2 = 0.001$. The model is chaotic at the start and end of this interval, However, it appears and disappears for the rest of the interval, as displayed by positive or negative MLE in Fig. 11(e). Furthermore, where the parameter γ_3 is varied $\in (0, 1)$, presented in Figs. 11(c) and 11(f), we can see that when the incommensurate order $\gamma_3 \in (0.18, 0.4) \cup (0.55, 0.6)$ the trajectories becomes stable and the MLE values are negative, Additionally, for the rest of the field, the variation of MLE variation between positive and negative signifies the emergence and vanishing of chaos.

5. Chaotic test and complexity

5.1. The 0–1 test

An analysis of the behavior and dynamics of the systems via the 0–1 test technique[35] is also employed for the confirmation of chaotic regions and regular behavior of the financial map. This technique can be represented as follows:

$$p_\tau(l) = \sum_{\bar{k}=1}^{l} x(\bar{k}) \cos(\bar{k}\tau), \quad q_\tau(l) = \sum_{\bar{k}=1}^{l} x(\bar{k}) \sin(\bar{k}\tau), \quad (27)$$

where τ is a randomly chosen constant in $(0, \pi)$, and the time series $x(\bar{k}), \bar{k} = \overline{1, N}$ is used for the translation variables. Plotting of p_τ, q_τ in the p_τ–q_τ plane is employed to find if chaos exists or not.

We use the following definition of mean square displacement:

$$M_\tau(l) = \lim_{N\to\infty} \frac{1}{N} \sum_{\bar{k}=1}^{N} \Big[\big(p_\tau(\bar{k}+l) - p_\tau(\bar{k})\big)^2 + \big(q_\tau(\bar{k}+l) - q_\tau(\bar{k})\big)^2 \Big], \quad l \le \frac{N}{10}. \quad (28)$$

Further, K_τ, the asymptotic growth rate, is denoted by

$$K_\tau = \lim_{l\to\infty} \frac{\log M_\tau(l)}{\log l}. \quad (29)$$

Therefore, the dynamics of the fractional financial system will be chaotic provided that the plot of p_τ and q_τ in the p_τ–q_τ plane presents a Brownian like behavior when $K = \text{median}(K_\tau)$ is close to 1, while it becomes regular as K is close 0 and p_τ–q_τ shows bounded-like behavior.

So, Fig. 12 describes the p–q plots for balanced and chaotic dynamics of the fractional financial map for different commensurate and incommensurate values. When $\gamma = 0.9$ and $(\gamma_1, \gamma_2, \gamma_3) = (0.85, 0.7, 0.6)$, Figs. 12(b) and 12(e) display bounded-like trajectories, which implies that the financial map is periodic, otherwise Brownian like trajectories are shown when $\gamma = 0.97$, $\gamma = 0.99$, $(\gamma_1, \gamma_2, \gamma_3) = (0.92, 1, 1)$ and $(\gamma_1, \gamma_2, \gamma_3) = (1, 0.05, 1)$ (Figs. 12(a), 12(c), 12(d) and 12(f)) which confirms that the fractional financial map is chaotic. Consequently, the 0–1 test results are in accordance with the MLE and bifurcation diagram.

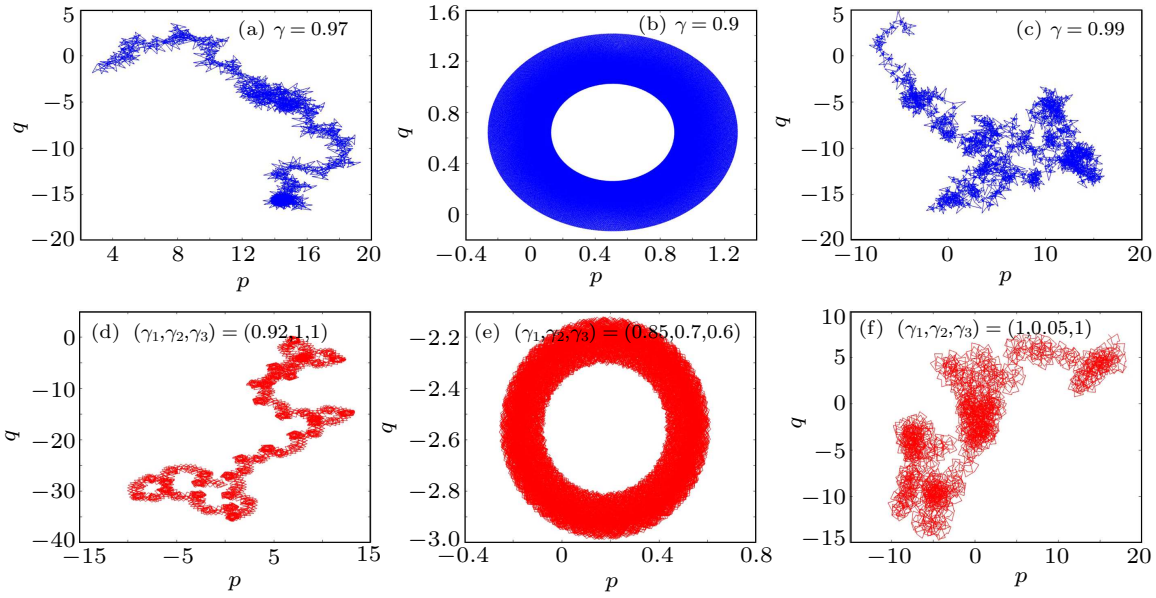


Fig. 12. The 0–1 test of the commensurate (14) and incommensurate (22) fractional financial maps.

5.2. The approximate entropy

In the following, we utilize the ApEn mechanism to evaluate the amount of complexity in the commensurate financial map (14) and the incommensurate financial map (22) created by a time series. A high ApEn reflects complexity of the map. According to Pincus,[36] ApEn can be calculated by

$$\mathrm{ApEn} = \wp^r(\nu) - \wp^{r+1}(\nu), \tag{30}$$

where $\wp^r(\nu)$ is given by

$$\wp^r(\nu) = \frac{1}{n-r+1}\sum_{j=1}^{n-r+1} \log C_j^r(\nu). \tag{31}$$

We put $\nu = 0.2\mathrm{std}(x)$, where ν is the tolerance and $\mathrm{std}(x)$ expresses the standard deviation.

The outcomes of ApEn for the financial maps (14) and (22) are shown in Fig. 13. The results for ApEn for $a = 1.7$, $c = 0.2$, $b = 0.6$, $\delta = 0.9$, and $x(0) = 0.4$, $y(0) = 0.6$, $z(0) = 0.8$ are shown in Fig. 13(a) versus γ, Fig. 13(b) versus γ_1 for $\gamma_2 = \gamma_3 = 1$, Fig. 13(c) versus γ_2 for $\gamma_1 = 0.92$, $\gamma_3 = 1$ and Fig. 13(d) versus γ_3 for $\gamma_1 = 1$, $\gamma_2 = 0.9$. This confirms the relation between high ApEn values and complexity.

Hence, the above results are in accordance with the previously displayed MLE, confirming that the financial system is exposed to chaos.

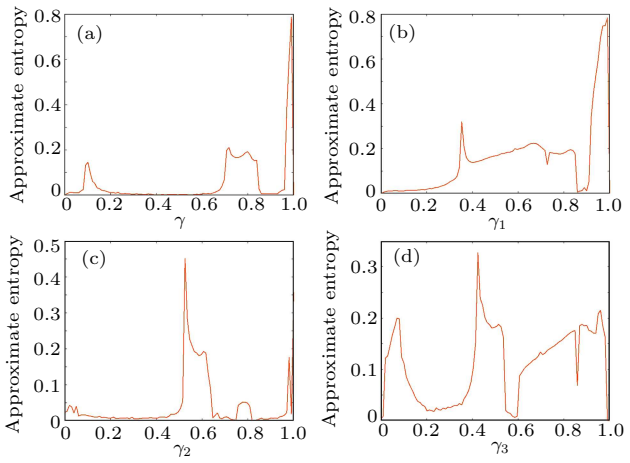


Fig. 13. The approximate entropy test of the commensurate (14) and incommensurate (22) fractional financial maps.

6. Chaotic control of the fractional financial maps

In this section, attention is paid to our ability to control and stabilize the fractional financial maps (14) and (22).

6.1. Control of the commensurate financial map

To this end we use the stability Lemma 2, which specifies the zero solution asymptotic stability criteria for the fractional financial map.

So, the controlled fractional financial map with the controller $(C_1, C_2, C_3)^{\mathrm{T}}$ is given by

$$\begin{cases} {}^c\Delta_t^{\gamma} x(s) = \delta(z(s-1+\gamma) + (y(s-1+\gamma) - a) \\ \qquad\qquad \times x(s-1+\gamma)) + C_1(s-1+\gamma), \\ {}^c\Delta_t^{\gamma} y(s) = \delta(1 - by(s-1+\gamma) \\ \qquad\qquad - x^2(s-1+\gamma)) + C_2(s-1+\gamma), \\ {}^c\Delta_t^{\gamma} z(s) = \delta(-x(s-1+\gamma) \\ \qquad\qquad - cz(s-1+\gamma)) + C_3(s-1+\gamma). \end{cases} \tag{32}$$

The ensuing theorem defines the control law establishing the new commensurate financial map.

Theorem 2 The fractional financial map is stabilized to the ensuing control law

$$\begin{cases} C_1(s-1+\gamma) = -\delta z(s-1+\gamma) \\ \qquad\qquad - \delta(y(s-1+\gamma)x(s-1+\gamma)), \\ C_2(s-1+\gamma) = -\delta + \delta x^2(s-1+\gamma), \\ C_3(s-1+\gamma) = \delta x(s-1+\gamma). \end{cases} \tag{33}$$

Proof Substituting Eq. (33) into Eq. (32) produces a linear system that follows

$${}^c\Delta_t^{\gamma} \bar{X} = B\bar{X}(s-1+\gamma), \tag{34}$$

where $\bar{X} = (x, y, z)^{\mathrm{T}}$ and

$$B = \begin{pmatrix} -\delta a & 0 & 0 \\ 0 & -\delta b & 0 \\ 0 & 0 & -\delta c \end{pmatrix}. \tag{35}$$

The λ_j of matrix B satisfies the following conditions:

$$|\lambda_j| < \left(2\cos\frac{|\arg\lambda_j| - \pi}{2-\gamma}\right)^{\gamma} \text{ and } |\lambda_j| > \frac{\gamma\pi}{2}, \quad j = 1, 2, 3.$$

Then, the controlled fractional financial map (33) is asymptotically stable.

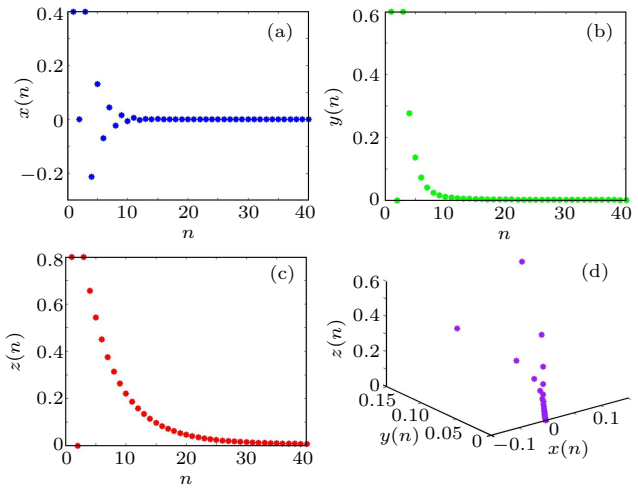


Fig. 14. Time evolution and attractor of the controller fractional financial map (32) for $\gamma = 0.97$, $a = 1.7$, $b = 0.2$, $c = 0.6$, $\delta = 0.9$.

In order to approve the results of Theorem 2, numerical simulations are carried out. Figures 14 and 15 indicate the time series of the controlled map (32), where $\gamma = 0.97$ and $\gamma = 0.99$. The figures confirm the stabilization results by showing how the system approaches $(0,0,0)$ asymptotically with $(x(0),y(0),z(0)) = (0.4,0.6,0.8)$.

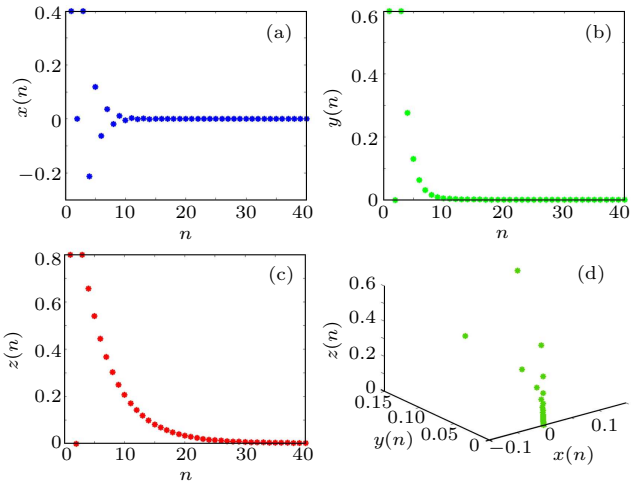


Fig. 15. Time evolution and attractor of the controller fractional financial map (32) for $\gamma = 0.99$, $a = 1.7$, $b = 0.2$, $c = 0.6$, $\delta = 0.9$.

6.2. Control of the incommensurate financial map

Here, we use the stability Lemma 3. The main objective is to ensure stability conditions that asymptotically move all states of the map towards zero.

The controlled system of the fractional financial map is as follows:

$$\begin{cases} {}^{c}\Delta_t^{\gamma_1} x(s) = \delta(z(s-1+\gamma_1) + (y(s-1+\gamma_1) - a) \\ \qquad\qquad \times x(s-1+\gamma_1)) + C_1(s-1+\gamma_1), \\ {}^{c}\Delta_t^{\gamma_2} y(s) = \delta(1 - by(s-1+\gamma_2) \\ \qquad\qquad - x^2(s-1+\gamma_2)) + C_2(s-1+\gamma_2), \\ {}^{c}\Delta_t^{\gamma_3} z(s) = \delta(-x(s-1+\gamma_3) \\ \qquad\qquad - cz(s-1+\gamma_3)) + C_3(s-1+\gamma_3). \end{cases} \tag{36}$$

We start by introducing the theorem of stability for the incommensurate financial map.

Theorem 3 Stabilization of the fractional financial map is governed by the following control:

$$\begin{cases} C_1(s-1+\gamma_1) = -\delta z(s-1+\gamma_1) \\ \qquad\qquad - \delta(y(s-1+\gamma_1)x(s-1+\gamma_1)), \\ C_2(s-1+\gamma_2) = -\delta + \delta x^2(s-1+\gamma_2), \\ C_3(s-1+\gamma_3) = \delta x(s-1+\gamma_3). \end{cases} \tag{37}$$

If $-2^{\gamma_i} < \lambda_j < 0, j = 1,2,3, \forall i$ the fractional financial map (36) must be stabilized at the solution $(0,0,0)$.

Proof Substituting Eq. (37) into Eq. (36) yields the following system:

$$\begin{cases} {}^{c}\Delta_t^{\gamma_1} x(s) = -\delta a(x(s-1+\gamma_1)), \\ {}^{c}\Delta_t^{\gamma_2} y(s) = -\delta b(y(s-1+\gamma_2)), \\ {}^{c}\Delta_t^{\gamma_n} z(s) = -\delta c(z(s-1+\gamma_3)). \end{cases} \tag{38}$$

This, however, implies

$$\det(\mathrm{diag}(\lambda^{K\gamma_1},\ldots,\lambda^{K\gamma_n}) - (1-\lambda^K)B) = 0,$$

where $K = 50$ and

$$B = \begin{pmatrix} -\delta a & 0 & 0 \\ 0 & -\delta b & 0 \\ 0 & 0 & -\delta c \end{pmatrix}, \tag{39}$$

$$\begin{aligned} &\det\left(\begin{pmatrix} \lambda^{46} & 0 & 0. \\ 0 & \lambda^{50} & 0 \\ 0 & 0 & \lambda^{50} \end{pmatrix} - \left(1-\lambda^{10}\right)\right. \\ &\left.\begin{pmatrix} -1.53 & 0 & 0 \\ 0 & -0.54 & 0 \\ 0 & 0 & -0.18 \end{pmatrix}\right) = 0 \end{aligned} \tag{40}$$

$$\begin{aligned} \Leftrightarrow\ & -0.577116\lambda^{150} + 0.3772\lambda^{146} - 0.227052\lambda^{100} \\ & +0.5256\lambda^{96} + 0.655452\lambda^{50} \\ & +0.0972\lambda^{46} + 0.148716 = 0. \end{aligned} \tag{41}$$

According to Lemma 3, and as a result of $\lambda_j \in \mathbb{C}/K^{\frac{1}{50}}$, $j = \overline{1,150}$. Consequently, the map (36) is asymptotically stable toward the solution $(0,0,0)$. Figures 16 and 17 indicate the stabilization of the controlled fractional financial map (36) for $(\gamma_1,\gamma_2,\gamma_3) = (0.92,1,1)$ and $(\gamma_1,\gamma_2,\gamma_3) = (1,0.9,1)$ by showing how the system approaches $(0,0,0)$ asymptotically with $(x(0),y(0),z(0)) = (0.4,0.6,0.8)$.

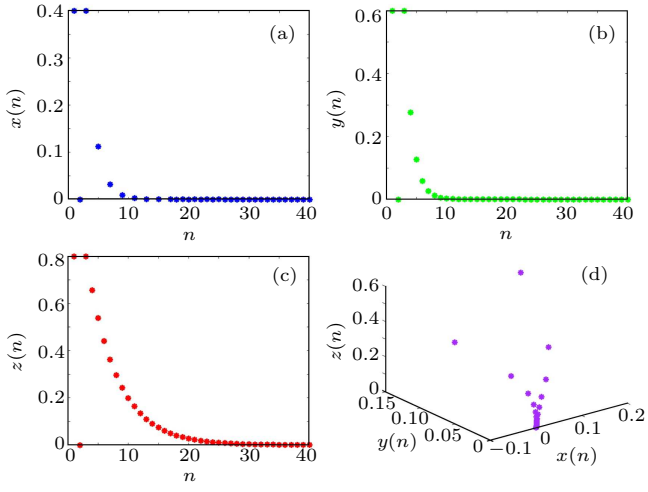


Fig. 16. Time evolution and attractor of the controller fractional financial map (36) for $(\gamma_1,\gamma_2,\gamma_3) = (0.92,1,1)$, $a = 1.7$, $b = 0.2$, $c = 0.6$, $\delta = 0.9$.

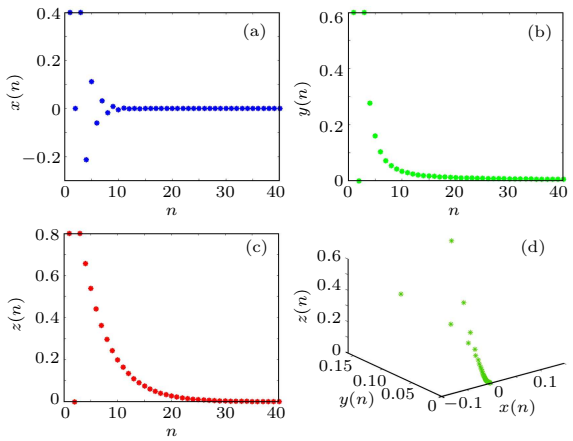


Fig. 17. Time evolution and attractor of the controller fractional financial map (36) for $(\gamma_1,\gamma_2,\gamma_3)=(1,0.9,1)$, $a=1.7$, $b=0.2$, $c=0.6$, $\delta=0.9$.

7. Conclusion and perspectives

This work introduced a 3D fractional financial map derived from the Caputo-like fractional difference operator that is dependent on commensurate and incommensurate orders. The behavior of the financial system is also affected by these values. Furthermore, using commensurate and incommensurate fractional orders in modeling allows the capture of the complex and multidimensional dynamics of markets, providing deeper insights into how different factors interact and affect the financial map. Our findings prove that chaos exists in this fractional financial map and that the range of chaotic attractors depends on these fractional values. The existence of chaos in the proposed map and the general dynamics are examined experimentally by creating phase portraits, plotting bifurcation charts and computing the maximum Lyapunov exponent. Moreover, the intricacy of the model was gauged using the entropy ApEN and the 0–1 test, which validated chaos in the financial map. Eventually, in order to generate the linear fractional financial map and converge toward zero equilibrium, we proposed a control technique that stabilizes the map by adding a controller term.

The stability region of the financial map can be significantly increased by including fractional orders, either commensurate or incommensurate. This expansion of stability enhances the ability of the model to predict future changes with greater accuracy. Economists and financiers need these predictions to make wise choices and take the correct measures. For instance, they can be utilized to establish the best times to buy or sell in commercial markets and create financial stability and economic strategies to promote economic growth. The financial map thus perfectly illustrates how to explore and then control chaos to improve our understanding of complex systems in various economic and financial disciplines such as macroeconomics, microeconomics and other branches. The theoretical framework provided by this study can help with future research in economics and finance of both commensurate and incommensurate fractional order.

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