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Dynamical distribution of continuous service time model involving non-Maxwellian collision kernel and value functions

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The distribution of continuous service time in call centers is investigated. A non-Maxwellian collision kernel combining two different value functions in the interaction rule are used to describe the evolution of continuous service time, respectively. Using the statistical mechanical and asymptotic limit methods, Fokker–Planck equations are derived from the corresponding Boltzmann-type equations with non-Maxwellian collision kernels. The steady-state solutions of the Fokker–Planck equation are obtained in exact form. Numerical experiments are provided to support our results under different parameters.

Keywords: kinetic theory, service time, Fokker–Planck equation, value function

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1. Introduction

Many scholars apply the Boltzmann equation in statistical mechanics to investigate various problems in multidisciplines. Scholars compare agents in a social system as gas molecules. The service time of agents is regarded as the energy of molecules and the service process between agents is regarded as collisions between molecules. Thus, the Boltzmann equation, which describes the change in the velocity distribution of gas molecules, is applied to investigate the distribution of service time in Refs. [1–8]. The distribution of service time satisfies the correponding Boltzmann equation. Brown *et al.*[1] found that the statistical distribution of service time is close to lognormal distribution, but not completely lognormal distribution. Excoffier *et al.*[2] used the data of call centers to discuss queuing problems. Methods based on statistical mechanics are proposed to optimize the trade-off about labor cost and quality of customer service. In order to consider a call center system with Poisson arrival and unlimited waiting areas, Nazarov *et al.*[3] employed the model of non-exponential service time to explore the stationary distribution of calls for recurrent service. The circumstances where customers might decide to leave before the service time were analyzed in Garnett *et al.*[4] Using the length of waiting time as an indicator of customers' patience, it was discovered in Ref. [4] that the function of customer's patience is exponential. Stepanov and Stepanov[5] proposed a measurement method based on the general characteristics of call services in call centers and estimated the strength of the call arrival. Aksin *et al.*[6] investigated the amount of customers who are willing to wait and discuss the percentage of customers who leave the queue before the service time in call center. It was found in Ref. [6] that the agent's psychological behavior has a significant impact on the distribution of service time. Gualandi and Toscani[7] applied statistical mechanical methods to derive that the service time density function is a lognormal distribution. The value function in Ref. [8] plays an important role to discuss the density function of service time. Thus, it leads to the question of whether there are different distributions of service time in call centers under different value functions.

Regarding the psychological characteristics of the agent, Kahneman and Tversky[9,10] showed that agents have the ability to perceive risk in the process of interaction and react irrational behavior. Kahneman and Tversky[9,10] provided a suitable value function to describe this behavior of the agent. Hu and Chen[11] used the kinetic theory to analyze influence of agent competence and psychological factors on investment decision-making. Naldi *et al.*[12] made use of statistical mechanical methods to describe macro-behavior as well as the psychological characteristics of agents in society. Burini *et al.*[13] proposed a systematic approach to study the perception and learning capacity of the population and describe the psychological characteristics of the behavior of the agent.

Recently, the theory on dilute dynamics has been applied to various aspects in society. For example, Du *et al.*[14] analyzed an asymmetric payoff distribution mechanism and used the Gini coefficient to investigate the effect of asymmetric payoff distribution about wealth. Quevedoa and Quimbay[15] constructed a Boltzmann-type equation that describes the evolution of wealth distribution. Zhou and Lai[16] studied the interaction between the accumulation of knowledge and the growth

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of personal wealth. Cordier *et al.*[17] used the asymptotic limit method and the Fokker–Planck equation to discuss the change of wealth. It is derived that the wealth density function behaves an inverse gamma distribution. Using the kinetic theory, Pareschi and Toscani[18] investigated changes of wealth depending on knowledge and wealth. Toscani *et al.*[19] investigated the behavior of online gamblers who invest their wealth in the gambling process. Dimarco and Toscani[20] considered a model of social climbing by utilizing collision-like kinetic theory.

Consider the following three value functions in Ref. [20]

$$\Phi_1(s) = \gamma\left(1-\frac{1}{s}\right),$$
$$\Phi_2(s) = \gamma\frac{s^\delta - 1}{s^\delta + 1 - \gamma},$$
$$\Psi(s) = \gamma\frac{s^\delta - 1}{s^\delta + 1}.$$

Comparing with the works in Gualandi and Toscani,[8] in which the density function of service time is discussed, our main contributions are as follows.

(i) We choose two value functions $\Phi_1(s)$ and $\Phi_2(s)$ in Ref. [20] that differ from value function $\Psi(s)$ in Ref. [8]. The two value functions $\Phi_1(s)$ and $\Phi_2(s)$ conform to prospect theory in Kahneman and Tversky[9,10] and approach the reference point with different speeds. When the agent exceeds the maximum service time, comparing the effect of value function $\Psi(s)$ in Ref. [8], the value functions $\Phi_1(s)$ and $\Phi_2(s)$ lead to different speeds that the agent returns to the ideal service time.

(ii) Gualandi and Toscani[8] used the Maxwellian collision kernel, which is a constant collision kernel. In this paper, we choose the non-Marxwellian collision kernel $k\omega^\alpha$ (ω denotes the service time). This collision kernel illustrates that a long service time level for an agent makes the interaction frequency faster, while a short service time level causes its interaction frequency to slow down.

(iii) The steady-state solution in Ref. [8] is a lognormal distribution. Our steady-state solution is an inverse gamma distribution in the case of value function $\Phi_1(s)$ and possesses a lognormal distribution in the case of value function $\Phi_2(s)$.

The rest of this paper is arranged as follows. In Section 2, a micro-interaction mechanism for service time is given, which contains unpredictable risks and value function. In Section 3, we discuss the properties of value functions. In Section 4, a Fokker–Planck equation is obtained by using an appropriate asymptotic limit. The steady-state solution of the equation is obtained. In Section 5, taking advantage of numerical simulation, we discuss the evolution of steady-state solution under different parameters.

2. Kinetic description of service time

In this part, we describe the psychological characteristics of agents and give an appropriate interaction rule. Using the prospect theory in Refs. [9,10], we discuss a model that is helpful to consider the evolution of service time. We assume that agents are indistinguishable (see Ref. [18]). Thus, the state of the system is fully represented by the service time density function $f(\omega,t)$. Namely, for an given interval $D \subseteq R_+$, the integral

$$\int_D f(w,t)\,\mathrm{d}w$$

means the number of persons depicted by service time $\omega \in D$ at the moment $t \geq 0$. Usually, we assume that the density function satisfies

$$\int_{\mathbb{R}_+} f(\omega,t)\,\mathrm{d}w = 1.$$

Indeed, it is essential to consider that no negative service time arises. This coincides with reality, implying that it is impossible to cross the boundary at $\omega = 0$. Agents in the system continuously update their states through micro-interactions. An agent spends a certain amount of time to complete his work in any single operation. A change in the future service time leads to a change of density function $f(\omega,t)$.

We discuss the nature of agents' psychological characteristics. Firstly, we assume that the agents can receive a precise instruction from the service manager that the call must end within the ideal service time, denoted by $\bar{\omega}$, which possesses the maximum service time $\bar{\omega}_\mathrm{L}$, meaning that all service time below it in any case, including the ideal time $\bar{\omega} < \bar{\omega}_\mathrm{L}$. Usually, in a company or organization, it is stipulated that exceeding the maximum service time $\bar{\omega}_\mathrm{L}$, the agents are penalized to pay money or extend working hours. We assume that the agents recognize $\bar{\omega}_\mathrm{L}$ as an important time rather than ideal time $\bar{\omega}$. Indeed, it is only when $\omega > \bar{\omega}_\mathrm{L}$ that the agent fears the consequences of exceeding the maximum service time $\bar{\omega}_\mathrm{L}$. Similarly, when an agent completes a task quickly within $\bar{\omega}_\mathrm{L}$, he is comfortable and faces the next task in a better state. Obviously, these two situations are very different since the former worries about the consequences of exceeding the maximum service time $\bar{\omega}_\mathrm{L}$, while the latter finishes a task in a better way. Following the ideas in Ref. [17], we characterize the agent's modification of the time service as follows:

$$\omega_* = \omega - \Phi\left(\omega/\bar{\omega}_\mathrm{L}\right)\omega + \eta\,\omega, \tag{1}$$

where ω and ω_* denote the service time before and after the agent interaction, respectively. $\Phi\left(\omega/\bar{\omega}_\mathrm{L}\right)$ is a function about $\omega/\bar{\omega}_\mathrm{L}$, which has positive or negative values. The psychological and behavioral characteristics of agents are expressed by $\Phi\left(\omega/\bar{\omega}_\mathrm{L}\right)$. η is a random variable that describes how service time is affected by uncertainty. We usually assume that its expectation is 0 with limited variance. So we have $\langle\eta\rangle = 0$ and

$\langle \eta^2 \rangle = \lambda$, where $\lambda > 0$. Here and after, $\langle \cdot \rangle$ denotes mathematical expectation.

3. Descriptions of value functions

In this section, we introduce two value functions. We discuss the bounds of value functions as well as asymmetry. Assume that two agents are on either side of the reference point and possess the same distance from the reference point. If the two agents approach the reference point at different speeds, then we say that the reference point is asymmetric.

Gualandi and Toscani[8] chose the following value function:

$$\Psi(s) = \gamma \frac{s^\delta - 1}{s^\delta + 1}, \quad s \geq 0,$$

where $0 < \delta < 1$ and $0 < \gamma < 1$ are constants. γ represents the maximum change in service time that an agent performs in a single task. The value function $\Psi(s)$ describes the agent's psychological behavior (see Ref. [8]).

3.1. Value function $\Phi_1(s)$

For the interaction (1), we assume that

$$\Phi_1(s) = \gamma \left(1 - \frac{1}{s}\right),$$

where $0 < \gamma < 1$ is a suitable constant to describe the behavior of the agent. The value function $\Phi_1(s)$ is discussed in Ref. [20]. The difference between the value function $\Phi_1(s)$ and value function $\Psi(s)$ is that value function $\Phi_1(s)$ has no lower bounds. This implies that when the random term of interaction rule (1) is removed, the value function $\Phi_1(s)$ has a sudden increase in service time after interaction. Observing that these two functions which are located in the right side of the reference point $s = 1$, we see that the curve of $\Phi_1(s)$ is much smoother than the curve of $\Psi(s)$. This indicates that for the value function $\Phi_1(s)$, once the actual service time of the agent exceeds the maximum service time, it becomes challenging to return to the ideal service time.

We know that $\Phi_1(s)$ is a concave and increasing function on R_+. Function $\Phi_1(1) = 0$ and $\Phi_1(s)$ possesses asymmetric about the point $s = 1$. Using the prospect theory proposed in Refs. [9,10], the function $\Phi_1(s)$ satisfies the following two properties.

(i) For all Δs, $0 < \Delta s \leq 1$, we have

$$-\Phi_1(1 - \Delta s) > \Phi_1(1 + \Delta s),$$
$$\Phi_1'(1 + \Delta s) < \Phi_1'(1 - \Delta s).$$

Function $\Phi_1(s)$ possesses the main property of the prospect theory. The asymmetry of the value function about the reference point possesses the features that when two agents start at the same distance from the reference point, agent from left side arrives at the reference point more easily than the agent starting from right.

(ii) For $0 < \gamma < 1$, $s \geq 0$, we have

$$\Phi_1(s) \leq \gamma.$$

It is important to note that this value function $\Phi_1(s)$ have not lower bounds. This means that when the random term of the interaction rule (1) is eliminated, it leads to a sudden increase in service time after interaction.

3.2. Value function $\Phi_2(s)$

In order to avoid the above situation, we consider a function that is different from function $\Phi_1(s)$

$$\Phi_2(s) = \gamma \frac{s^\delta - 1}{s^\delta + 1 - \gamma}, \quad s \geq 0.$$

The value function $\Phi_2(s)$ is also discussed in Ref. [20]. The main difference among value functions $\Phi_2(s)$, $\Phi_1(s)$, and $\Psi(s)$ is the part which is located to the right side of the reference point $s = 1$. Observing the part which is located in the right of the reference point, we find that the curve of $\Phi_2(s)$ is smoother relative to the curve of $\Psi(s)$, but steeper than $\Phi_1(s)$. Therefore, when the actual service time exceeds the maximum service time, if the agent wants to return the reference point location, the value function $\Phi_2(s)$ is easier to reach the reference point than $\Phi_1(s)$ does, but more difficult to reach the reference point than the value function $\Psi(s)$. However, function $\Phi_2(s)$ is just a special form of the following expression:

$$\Phi_\delta^{\mathrm{e}}(s) = \gamma \frac{\mathrm{e}^{\varepsilon(s^\delta - 1)/\delta} - 1}{\mathrm{e}^{\varepsilon(s^\delta - 1)/\delta} + 1 - \gamma}, \quad \varepsilon \geq 0,$$

when $\delta \to 0$, namely,

$$\Phi_\delta^{\mathrm{e}}(s) \to \Phi_2^{\varepsilon}(s) = \gamma \frac{s^\varepsilon - 1}{s^\varepsilon + 1 - \gamma}, \quad s \geqslant 0.$$

Thus, when $\delta \to 0$, the function $\Phi_\delta^{\mathrm{e}}(s)$ is $\Phi_2(s)$. Indeed, function $\Phi_\delta^{\mathrm{e}}(s)$ reaches the reference point more slowly than $\Phi_2(s)$ from the right side. It is clear that $\Phi_2(s)$ is a concave and increasing function. It is equal to zero at the reference point $s = 1$. Function $\Phi_2(s)$ satisfies the following properties.

(iii) For all $\Delta s, 0 < \Delta s \leq 1$, it has

$$-\Phi_2(1 - \Delta s) > \Phi_2(1 + \Delta s), \quad \Phi_2'(1 + \Delta s) < \Phi_2'(1 - \Delta s).$$

Similarly, the asymmetry point $s = 1$ is consistent with the characterization of the value function in Refs. [9,10].

(iv) For $0 < \gamma < 1$, we obtain

$$-\gamma \leq \Phi_2(s) \leq \gamma.$$

We know that function $\Phi_2(s)$ is bounded, which does not lead to an infinite maximum service time and is more in line with the real situation.

4. The kinetic model of service time

If the interaction rule (1) is chosen, the model on service time resorts to a kinetic model. The density function $f(\omega,t)$ satisfies the Boltzmann equation. Namely, for all smooth functions $\varphi(\omega)$ with supported set in R_+, the density function $f(\omega,t)$ is the weak solution of the equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\varphi(\omega)f(\omega,t)\,\mathrm{d}\omega\\&=\frac{1}{\tau}\left\langle\int_{\mathbb{R}_+}k\omega^{\alpha}\left(\varphi\left(\omega_*\right)-\varphi(\omega)\right)f(\omega,t)\,\mathrm{d}\omega\right\rangle.\end{aligned}\tag{2}$$

Here $k\omega^{\alpha}$ is the collision kernel where the constant $k>0$ and $0<\alpha\leq 1$, and ω denotes the service time of the agent. Equation (2) takes into account the presence of a random parameter η. Note that the basic mutual interaction rule (1). The right side of Eq. (2) quantifies the change in the observable measure of φ for the agent whose value is modified from ω to ω_*. In Eq. (2), if $\varphi(\omega)=1$, it means that equation (2) is conserved, implying that $f(\omega,t)$ is a probability density function and remains a probability density function for all subsequent times $t>0$.

4.1. Stationary solution corresponding to value function $\Phi_1(s)$

We consider the value function

$$\Phi_1(s)=\gamma\left(1-\frac{1}{s}\right),\quad s\geq 0.$$

Let the mean

$$m(t)=\int_{\mathbb{R}_+}\omega f(\omega,t)\,\mathrm{d}w.$$

Since

$$\begin{aligned}&\langle\varphi\left(\omega_*\right)-\varphi(\omega)\rangle=\langle\omega_*-\omega\rangle\\&=\left\langle-\Phi_1\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)\omega+\eta\omega\right\rangle\\&=\left\langle-\gamma\left(1-\frac{\bar{\omega}_{\mathrm{L}}}{\omega}\right)\omega+\eta\omega\right\rangle\\&=-\gamma(\omega-\bar{\omega}_{\mathrm{L}}),\end{aligned}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}m(t)=\frac{\gamma k}{\tau}\int_{\mathbb{R}_+}\omega^{\alpha}(\bar{\omega}_{\mathrm{L}}-\omega)f(\omega,t)\,\mathrm{d}\omega.\tag{3}$$

Note that equation (3) has no explicit solution. However, we know in the previous section that the value function possesses upper bounds. Thus, we know that the $m(t)$ has also upper bounds. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m(t)&=-\frac{k}{\tau}\int_{R+}\Phi_1\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)\omega^{\alpha+1}f(\omega,t)\,\mathrm{d}\omega\\&\leqslant-\frac{k\gamma}{\tau}\int_{R_+}\omega^{\alpha+1}f(\omega,t)\,\mathrm{d}\omega.\end{aligned}$$

Letting

$$m_{\alpha}=\int_{R_+}\omega^{\alpha}f(\omega,t)\,\mathrm{d}\omega,$$

$$m_{\alpha+1}=\int_{R_+}\omega^{\alpha+1}f(\omega,t)\,\mathrm{d}\omega.$$

Utilizing the Hölder's inequality yields

$$\begin{aligned}m_{\alpha}&=\int_{R_+}w^{\alpha}f(w,t)\,\mathrm{d}w\\&=\int_{R_+}(wf(w,t))^{\alpha}f(w,t)^{1-\alpha}\,\mathrm{d}w\\&\leq\left[\int_{R+}wf(w,t)\,\mathrm{d}w\right]^{\alpha}\left[\int_{R+}f(w,t)\,\mathrm{d}w\right]^{1-\alpha}\\&=m^{\alpha}(t).\end{aligned}$$

We obtain

$$\begin{aligned}m_{a+1}&\leq m^{\alpha+1}(t),\\\frac{\mathrm{d}}{\mathrm{d}t}m(t)&\leq-\frac{k\gamma}{\tau}m^{\alpha+1}(t).\end{aligned}$$

Solving the above differential inequality yields

$$m(t)\leq\left(\frac{\tau}{\alpha k\gamma t}\right)^{1/\alpha}.$$

We assume that the service time is only modified by a very small amount. This is achieved by a very small value $\varepsilon>0$ and $\varepsilon\ll 1$. We have

$$\Phi_1(s)\to\Phi_{1\varepsilon}(s)=\varepsilon\gamma\left(1-\frac{1}{s}\right),\quad\eta\to\sqrt{\varepsilon}\eta,$$

and rewrite Eq. (2) as

$$\frac{\mathrm{d}}{\mathrm{d}t}m(t)=\varepsilon\frac{\gamma k}{\tau}\int_{\mathbb{R}_+}\frac{1}{\varepsilon}\omega^{\alpha}(\bar{\omega}_{\mathrm{L}}-\omega)f(\omega,t)\,\mathrm{d}\omega.$$

For any time $t>0$, we know that there is a small change in the mean value $m(t)$. In this case, we observe the change in the mean value independent of τ with $\tau\to\varepsilon\tau$. Then equation (2) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\omega f(\omega,t)\,\mathrm{d}\omega=\frac{\gamma}{\tau}\int_{\mathbb{R}_+}\frac{1}{\varepsilon}(\bar{\omega}_{\mathrm{L}}-\omega)f(\omega,t)\,\mathrm{d}\omega.$$

For a smooth function $\varphi(\omega)$, using interaction rule (1), expanding in Taylor series $\varphi(\omega_*)$ around $\varphi(\omega)$, and using the properties of random variables η, we obtain

$$\begin{aligned}\langle\omega_*-\omega\rangle&=\left\langle-\Phi_{1\varepsilon}\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)w+\eta\omega\right\rangle\\&=-\varepsilon\gamma\left(1-\frac{\bar{\omega}_{\mathrm{L}}}{\omega}\right)\omega\\&=-\varepsilon\gamma(\omega-\bar{\omega}_{\mathrm{L}}),\\&\left\langle(\omega_*-\omega)^2\right\rangle\\&=\left\langle\left(-\Phi_{1\varepsilon}\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)\omega+\eta\omega\right)^2\right\rangle\\&=\left\langle\Phi_{1\varepsilon}^2\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)\omega^2-2\Phi_{1\varepsilon}\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)\omega^2\eta+\eta^2\omega^2\right\rangle\\&=\varepsilon^2\gamma^2(\omega-\bar{\omega}_{\mathrm{L}})^2+\varepsilon\lambda\omega^2.\end{aligned}$$

Therefore, we acquire

$$\begin{aligned}&\langle\varphi(\omega_*)-\varphi(\omega)\rangle\\&=\varphi'(\omega)\langle\omega_*-\omega\rangle+\frac{\varphi''(\omega)}{2}\langle(\omega_*-\omega)^2\rangle+R_\varepsilon(\omega)\\&=\varepsilon\left(-\varphi'(\omega)\gamma(\omega-\bar{\omega}_{\rm L})+\varphi''(\omega)\frac{\lambda}{2}\omega^2\right)+R_\varepsilon(\omega),\end{aligned}$$

in which the term $R_\varepsilon(\omega)$ satisfies

$$\begin{aligned}R_\varepsilon(\omega)\ =\ &\frac{\varepsilon^2}{2}\varphi''(\omega)\gamma^2(\omega-\bar{\omega}_{\rm L})^2\\&+\frac{1}{6}\left\langle\varphi'''(\omega+\theta(\omega_*-\omega))(\omega_*-\omega)^3\right\rangle\end{aligned}$$

for $0\le\theta\le1$. We have

$$\begin{aligned}&\left\langle(\omega_*-\omega)^3\right\rangle\\&=-\varepsilon^3r^3(\omega-\bar{w}_{\rm L})^3+3\varepsilon^2r(\omega-\bar{\omega}_{\rm L})\lambda\omega^3.\end{aligned}$$

Thus, we obtain the Boltzmann-type equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_t}\varphi(\omega)f(\omega,t)\mathrm{d}\omega\\&=\frac{1}{\tau}\int_{R_+}k\omega^\alpha\left[-\gamma(\omega-\bar{\omega}_{\rm L})\varphi'(\omega)+\frac{\lambda}{2}\omega^2\varphi''(\omega)\right]f(\omega,t)\mathrm{d}\omega\\&+\frac{1}{\varepsilon\tau}\mathcal{R}_\varepsilon(\omega,t),\end{aligned}$$

in which

$$\mathcal{R}_\varepsilon(\omega,t)=\int_{\mathbb{R}_+}k\omega^\alpha R_\varepsilon(\omega)f(\omega,t)\mathrm{d}\omega.$$

Letting $\varepsilon\to0$, we have $\frac{1}{\varepsilon}R_\varepsilon(\omega)\to0$ and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_t}\varphi(\omega)f(\omega,t)\mathrm{d}\omega\\&=\frac{1}{\tau}\int_{R_+}k\omega^\alpha\Big[-\gamma(\omega-\bar{\omega}_{\rm L})\varphi'(\omega)\\&+\frac{\lambda}{2}\omega^2\varphi''(\omega)\Big]f(\omega,t)\mathrm{d}\omega.\end{aligned}\tag{4}$$

Note that the smooth function φ possesses the supported set on R_+. Utilizing integration by parts, we obtain

$$\begin{aligned}&\int_{R_+}-\gamma k\omega^\alpha(\omega-\bar{\omega}_{\rm L})\varphi'(\omega)f(\omega,t)\mathrm{d}\omega\\&=\int_{R_+}-\gamma k\omega^\alpha(w-\bar{w}_{\rm L})f(w,t)\mathrm{d}\varphi(w)\\&=\int_{R_+}\varphi(\omega)\frac{\mathrm{d}\left(\gamma k\omega^\alpha(w-\bar{w}_{\rm L})f(w,t)\right)}{\mathrm{d}\omega}\mathrm{d}\omega,\end{aligned}\tag{5}$$

$$\begin{aligned}&\int_{R_+}\frac{\lambda}{2}k\omega^{\alpha+2}\varphi''(\omega)f(\omega,t)\mathrm{d}\omega\\&=-\int_{R_+}\varphi'(\omega)\frac{\mathrm{d}\left(\frac{\lambda}{2}k\omega^{\alpha+2}f(\omega,t)\right)}{\mathrm{d}\omega}\mathrm{d}\omega\\&=\int_{R_+}\varphi(\omega)\frac{\mathrm{d}^2\left(\frac{\lambda}{2}k\omega^{\alpha+2}f(\omega,t)\right)}{\mathrm{d}\omega^2}\mathrm{d}\omega.\end{aligned}\tag{6}$$

Substituting Eqs. (5) and (6) into Eq. (4), we have

$$\begin{aligned}&\int_{R_+}\varphi(w)\frac{\mathrm{d}}{\mathrm{d}t}f(\omega,t)\mathrm{d}\omega\\&=\frac{1}{\tau}\int_{R_+}\varphi(\omega)\frac{\mathrm{d}\left(\gamma k\omega^\alpha(\omega-\bar{\omega}_{\rm L})f(\omega,t)\right)}{\mathrm{d}\omega}\mathrm{d}\omega\\&+\frac{1}{\tau}\int_{R_+}\varphi(\omega)\frac{\mathrm{d}^2\left(\frac{\lambda}{2}k\omega^{\alpha+2}f(\omega,t)\right)}{\mathrm{d}\omega^2}\mathrm{d}\omega,\end{aligned}$$

from which we obtain the Fokker–Planck-type equation

$$\begin{aligned}\frac{\partial f(\omega,t)}{\partial t}\ =\ &\frac{\gamma k}{\tau}\frac{\partial\left(\omega^\alpha(\omega-\bar{\omega}_{\rm L})f(\omega,t)\right)}{\partial\omega}\\&+\frac{\lambda k}{2\tau}\frac{\partial^2\left(\omega^{\alpha+2}f(\omega,t)\right)}{\partial\omega^2}.\end{aligned}\tag{7}$$

Equation (7) describes the evolution of the density function. We obtain the ordinary differential equation for the equilibrium state

$$\gamma\omega^\alpha(\omega-\bar{\omega}_{\rm L})f_\infty(\omega)+\frac{\lambda}{2}\frac{\partial\left(\omega^{\alpha+2}f_\infty(\omega)\right)}{\partial\omega}=0.\tag{8}$$

We solve Eq. (8) by separating the variables. A smooth solution $f_\infty(\omega)$ is obtained

$$\begin{aligned}f_\infty(w)=&\frac{\left(\frac{2\gamma\bar{w}_{\rm L}}{\lambda}\right)^{\alpha+(2\gamma/\lambda)+1}}{\Gamma\left(\alpha+\frac{2\gamma}{\lambda}+1\right)}\omega^{-2-\alpha-(2\gamma/\lambda)}\\&\times\exp\left(-\frac{2\gamma\bar{\omega}_{\rm L}}{\lambda w}\right).\end{aligned}$$

The steady state density $f_\infty(\omega)$ exhibits an inverse gamma distribution.

4.2. Stationary solution corresponding to value function $\Phi_2(s)$

We know that the value function $\Phi_1(s)$ has no lower bounds, implying that there is a sharp increase in post-trade service time. To avoid this, we use the value function $\Phi_2(s)$. We discuss the mathematical modeling of the underlying interactions for service time based on the value function $\Phi_2(s)$.

$$\Phi_2(s)=\gamma\frac{s^\delta-1}{s^\delta+1-\gamma},\quad s\ge0.$$

Using the value function $\Phi_2(s)$ and the interaction rule (1), we obtain

$$\begin{aligned}\langle\omega_*-\omega\rangle&=\left\langle-\Phi_2\left(\frac{\omega}{\bar{\omega}_{\rm L}}\right)\omega+\eta\omega\right\rangle\\&=-\gamma\left[\left(\frac{\omega}{\bar{\omega}_{\rm L}}\right)^\delta-1\right]\frac{1}{(\omega/\bar{\omega}_{\rm L})^\delta+1-\gamma}\omega.\end{aligned}$$

If $\varphi(x)=\omega$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}m(t)\ =\ \frac{\gamma k}{\tau}\int_{\mathbb{R}_+}-\left[\left(\frac{\omega}{\bar{\omega}_{\rm L}}\right)^\delta-1\right]$$

$$\times \frac{\omega^{1+\alpha} f(\omega,t)}{(\omega/\bar{\omega}_{\mathrm{L}})^{\delta}+1-\gamma}\mathrm{d}\omega. \tag{9}$$

Note that equation (9) is not explicitly solvable. However, we know that the value function $\Phi_2(s)$ is bounded. Thus, we know that the $m(t)$ has also upper bounds for time $t>0$.

We consider the Boltzmann Eq. (2) and interaction rules (1). Obviously, the Boltzmann equation seems to be difficult to be solved. The general approach is to use numerical simulations to solve Eq. (2). Alternatively, the scaling rule is used appropriately to obtain the Fokker–Planck equation, which would give the steady-state solution of the equation. This article makes use of the latter approach. We assume that there is a very small variable $0<\varepsilon\ll 1$. We have

$$\Phi_2(s)\to\Phi_{2\varepsilon}(s)=\gamma\frac{s^{\varepsilon}-1}{s^{\varepsilon}+1-\gamma},\quad \eta\to\sqrt{\varepsilon}\eta. \tag{10}$$

The scaling assumptions in Eq. (10) preserve the effects of all parameters in the interaction rule (1) in the asymptotic limit. An exhaustive discussion of these scaling assumptions is found in Ref. [21].

We write Eq. (9) in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m(t) =& \frac{\gamma k}{\tau}\int_{\mathbb{R}_+}-\frac{1}{\varepsilon}\left[\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)^{\varepsilon}-1\right]\\ &\times\frac{\omega^{1+\alpha}f(\omega,t)}{(\omega/\bar{\omega}_{\mathrm{L}})^{\varepsilon}+1-\gamma}\mathrm{d}\omega\end{aligned}$$

for $\omega\geq\bar{\omega}_{\mathrm{L}}$. We have

$$\frac{1}{\varepsilon}\left[\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)^{\varepsilon}-1\right]\leq\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right).$$

For $\omega\leq\bar{\omega}_{\mathrm{L}}$, we obtain

$$\frac{1}{\varepsilon}\left[\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)^{\varepsilon}-1\right]\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)\geq -1.$$

Clearly, for any time $t>0$, we choose such ε which makes the mean value $m(t)$ have a small change. Letting $s=\omega/\bar{\omega}_{\mathrm{L}}$ and $\varepsilon\to 0$, we have

$$A_{\varepsilon}(\omega)=\frac{1}{\varepsilon}\frac{(s^{\varepsilon}-1)}{s^{\varepsilon}+1-\gamma}\to\frac{1}{2-\gamma}\log\frac{\omega}{\bar{\omega}_{\mathrm{L}}}.$$

As illustrated in Ref. [21], when $\varepsilon\to 0$, a finite change in the average density is observed provided that each agent in the system experiences a large number of interactions over a fixed period of time. In order to observe the evolution of the mean, it suffices to increase the service frequency. Applying the scale in Eq. (10), we obtain the evolution model for the second-order moments of $f(\omega,t)$.

For a smooth function $\varphi(\omega)$, expanding in Taylor series $\varphi(\omega_*)$ around $\varphi(\omega)$, we obtain

$$\begin{aligned}&\langle\omega_*-\omega\rangle\\ &=\left\langle-\Phi_{2\varepsilon}\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)\omega+\eta\omega\right\rangle\\ &=\left\langle-\gamma\varepsilon\frac{1}{\varepsilon}(\omega(/\bar{\omega}_{\mathrm{L}})^{\varepsilon}-1)\frac{1}{(\omega/\bar{\omega}_{\mathrm{L}})^{\varepsilon}+1-\gamma}\omega+\eta\omega\right\rangle\\ &=-\varepsilon\gamma A_{\varepsilon}(\omega)\omega,\\ &\langle(\omega_*-\omega)^2\rangle\\ &=\left\langle\Phi_{2\varepsilon}^2\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)\omega^2-2\Phi_{2\varepsilon}\left(\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)\omega^2\eta+\eta^2\omega^2\right\rangle\\ &=\varepsilon^2\gamma^2A_{\varepsilon}^2(\omega)\omega^2+\varepsilon\lambda\omega^2.\end{aligned}$$

Since

$$\begin{aligned}\varphi(\omega_*)=&\varphi(\omega)+\varphi'(\omega)(\omega_*-\omega)\\ &+\frac{\phi''(\omega)}{2}(\omega_*-\omega)^2+R_{\varepsilon}(\omega),\end{aligned}$$

we have

$$\begin{aligned}&\langle\varphi(\omega_*)-\varphi(\omega)\rangle\\ &=\varphi'(\omega)\langle\omega_*-\omega\rangle+\frac{\varphi''(\omega)}{2}\langle(\omega_*-\omega)^2\rangle+R_{\varepsilon}(\omega)\\ &=\varepsilon\left(-\varphi'(\omega)\frac{\gamma}{2-\gamma}\log\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\omega+\varphi''(\omega)\frac{\lambda}{2}\omega^2\right)+R_{\varepsilon}(\omega),\end{aligned}$$

where the $R_{\varepsilon}(\omega)$ is given by

$$\begin{aligned}R_{\varepsilon}(\omega)=&\frac{\varepsilon^2}{2}\varphi''(\omega)\gamma^2A_{\varepsilon}^2(\omega)\omega^2\\ &+\frac{1}{6}\left\langle\varphi'''(\omega+\theta(\omega_*-\omega))(\omega_*-\omega)^3\right\rangle,\end{aligned}$$

in which $0\leq\theta\leq 1$. We obtain the equation for the service time

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+}\varphi(\omega)f(\omega,t)\mathrm{d}\omega\\ &=\frac{1}{\tau}\left\langle\int_{R_+}k\omega^{\alpha}(\varphi(\omega*)-\varphi(\omega))f(\omega,t)\mathrm{d}\omega\right\rangle\\ &=\frac{k\gamma}{\tau(2-\gamma)}\int_{R_+}-\omega^{\alpha+1}\log\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\varphi'(\omega)f(\omega,t)\mathrm{d}\omega\\ &\quad+\frac{k\lambda}{2\tau}\int_{R_+}w^{\alpha+2}\varphi''(\omega)f(\omega,t)\mathrm{d}\omega\\ &\quad+\frac{\mathcal{R}_{\varepsilon}(t)}{\varepsilon},\end{aligned}\tag{11}$$

where

$$\mathcal{R}_{\varepsilon}(t)=\int_{\mathbb{R}_+}k\omega^{\alpha}R_{\varepsilon}(\omega)f(\omega,t)\mathrm{d}\omega,$$

in which the remainder term $\mathcal{R}_{\varepsilon}(t)$ satisfies $(1/\varepsilon)\mathcal{R}_{\varepsilon}(t)\to 0$ as $\varepsilon\to 0$. When the residual term tends to zero, we ignore the last term of Eq. (11) and obtain the equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+}\varphi(\omega)f(\omega,t)\mathrm{d}\omega\\ &=\frac{k\gamma}{\tau(2-\gamma)}\int_{R_+}-\omega^{\alpha+1}\log\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\varphi'(\omega)f(\omega,t)\mathrm{d}\omega\\ &\quad+\frac{k\lambda}{2\tau}\int_{R_+}w^{\alpha+2}\varphi''(\omega)f(\omega,t)\mathrm{d}\omega.\end{aligned}$$

We consider that the boundary value of the density function is zero. The specific steps are as follows. If the boundary term arising from the integral vanishes, we have

$$\begin{aligned}
&\frac{k\gamma}{\tau(2-\gamma)}\int_{R_+} -\omega^{\alpha+1}\log\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\varphi'(\omega)f(\omega,t)\,\mathrm{d}\omega\\
&=\frac{k\gamma}{\tau(2-\gamma)}\int_{R_+} -\omega^{\alpha+1}\log\frac{\omega}{\bar{\omega}_{\mathrm{L}}}f(\omega,t)\,\mathrm{d}\varphi(\omega)\\
&=\frac{k\gamma}{\tau(2-\gamma)}\int_{R_+}\varphi(\omega)\frac{\mathrm{d}\left(\omega^{\alpha+1}\log\frac{\omega}{\bar{\omega}_{\mathrm{L}}}f(\omega,t)\right)}{\mathrm{d}\omega}\mathrm{d}\omega,\\
&\frac{k\lambda}{2\tau}\int_{R_+}\omega^{\alpha+2}\varphi''(\omega)f(\omega,t)\,\mathrm{d}\omega\\
&=\frac{k\lambda}{2\tau}\int_{R_+}\omega^{\alpha+2}f(\omega,t)\,\mathrm{d}\varphi'(\omega)\\
&=-\frac{k\lambda}{2\tau}\int_{R_+}\varphi'(\omega)\frac{\mathrm{d}\left(\omega^{\alpha+2}f(\omega,t)\right)}{\mathrm{d}\omega}\mathrm{d}\omega\\
&=-\frac{k\lambda}{2\tau}\int_{R_+}\frac{\mathrm{d}\left(\omega^{\alpha+2}f(\omega,t)\right)}{\mathrm{d}\omega}\mathrm{d}\varphi(\omega)\\
&=\frac{k\lambda}{2\tau}\int_{R_+}\varphi(\omega)\frac{\mathrm{d}^2\left(\omega^{\alpha+2}f(\omega,t)\right)}{\mathrm{d}\omega^2}\mathrm{d}\omega,
\end{aligned}$$

from which we obtain

$$\begin{aligned}
&\int_{R_+}\varphi(\omega)\frac{\mathrm{d}}{\mathrm{d}t}f(\omega,t)\,\mathrm{d}\omega\\
&=\frac{k\gamma}{\tau(2-\gamma)}\int_{R_+}\varphi(\omega)\frac{\mathrm{d}\left(\omega^{\alpha+1}\log\frac{\omega}{\bar{\omega}_{\mathrm{L}}}f(\omega,t)\right)}{\mathrm{d}\omega}\mathrm{d}\omega\\
&+\frac{k\lambda}{2\tau}\int_{R_+}\varphi(\omega)\frac{\mathrm{d}^2\left(\omega^{\alpha+2}f(\omega,t)\right)}{\mathrm{d}\omega^2}\mathrm{d}\omega.
\end{aligned}$$

Thus, we acquire the Fokker–Planck-type equations in the form

$$\begin{aligned}
\frac{\partial f(\omega,t)}{\partial t} &= \frac{k\gamma}{\tau(2-\gamma)}\frac{\partial}{\partial\omega}\left(\omega^{\alpha+1}\log\frac{\omega}{\bar{\omega}_{\mathrm{L}}}f(\omega,t)\right)\\
&+\frac{k\lambda}{2\tau}\frac{\partial^2}{\partial\omega^2}\left(\omega^{\alpha+2}f(\omega,t)\right).
\end{aligned}\tag{12}$$

Equation (12) describes the evolution of the density function $f(\omega,t)$ for a service time $\omega\in R_+$ in the asymptotic limit.

When $t\to+\infty$, we have

$$\frac{\gamma}{(2-\gamma)}\omega^{\alpha+1}\log\frac{\omega}{\bar{\omega}_{\mathrm{L}}}g_\infty(\omega)+\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega^{\alpha+2}g_\infty(\omega)\right)=0.\tag{13}$$

From Eq. (13), we obtain

$$\begin{aligned}
&\left(\frac{\lambda(\alpha+2)}{2}+\frac{\gamma}{2-\gamma}\log\frac{\omega}{\bar{\omega}_{\mathrm{L}}}\right)g_\infty(\omega)\\
&+\frac{\lambda}{2}\omega\frac{\partial}{\partial\omega}g_\infty(\omega)=0,
\end{aligned}$$

which has the steady state solution

$$g_\infty(\omega)=\frac{1}{\sqrt{2\pi\sigma}\omega}\exp\left\{-\frac{(\log\omega-\mu)^2}{2\sigma}\right\},$$

where

$$\sigma=\frac{\lambda(2-\gamma)}{2\gamma},\quad \mu=\log\bar{\omega}_{\mathrm{L}}-\sigma-\sigma\alpha,\tag{14}$$

where λ is variance of random variables η, γ is a constant, $0<\gamma<1$, $\bar{\omega}_{\mathrm{L}}$ is the maximum service time.

5. Numerical experiments

In this part, numerical analysis is carryed out to analyze the distribution of service time. We provide several numerical results, which depict the changes of the service-time of steady-state solution under different parameters. When μ and σ are modified, the final steady state of the distribution function also changes. We perform numerical analysis on the changes of the value functions $\Phi_1(s)$ and $\Phi_2(s)$ under different parameters. Using value functions, we analyze the psychological characteristics of the agent. When the agent exceeds the maximum service time, we observe that the rate at which the agent returns to the ideal service time is different.

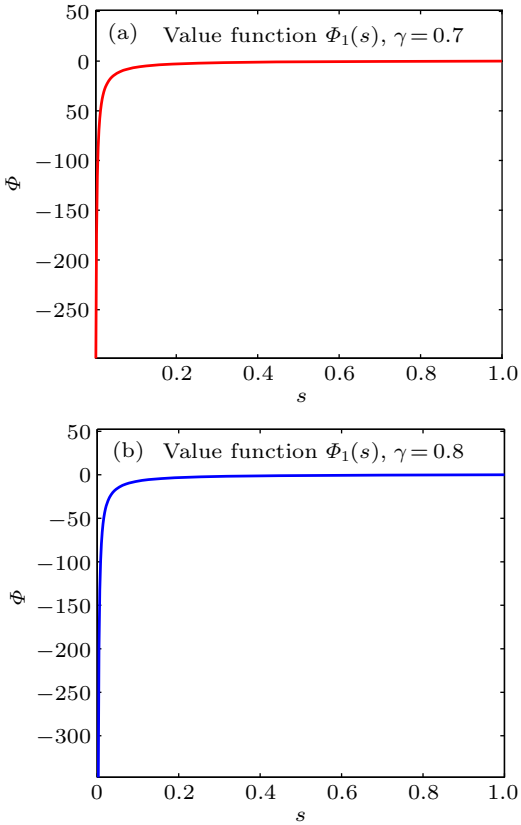


Fig. 1. Image of value function $\Phi_1(s)$.

From Figs. 1 and 2, the value function has a value of 0 at $s=1$, and this function is a convex and increasing function for $s>0$. It is found that function $\Phi_1(s)$ and function $\Phi_2(s)$ are asymmetric on both sides of the reference point $s=1$. The slope on both sides of the reference point is not the same, meaning that if two agents are on both sides of the reference point and the distance from the reference point is the

same, agents to the left of the reference point reach the ideal service time faster than agents to the right of the reference point. This is the asymmetry of value functions. In addition, by observing the value function $\Phi_1(s)$, we see that the curve of the value function $\Phi_1(s)$ after the inflection point is very flat, indicating that once the agent exceeds the maximum service time, it is almost impossible to return to the ideal service time. By observing that if value functions $\Phi_1(s)$ and function $\Phi_2(s)$ are located to the right of the reference point and are at the same distance from the reference point, the agent with the value function $\Phi_2(s)$ is faster than the agent with the value function $\Phi_1(s)$ to reach the ideal service time.

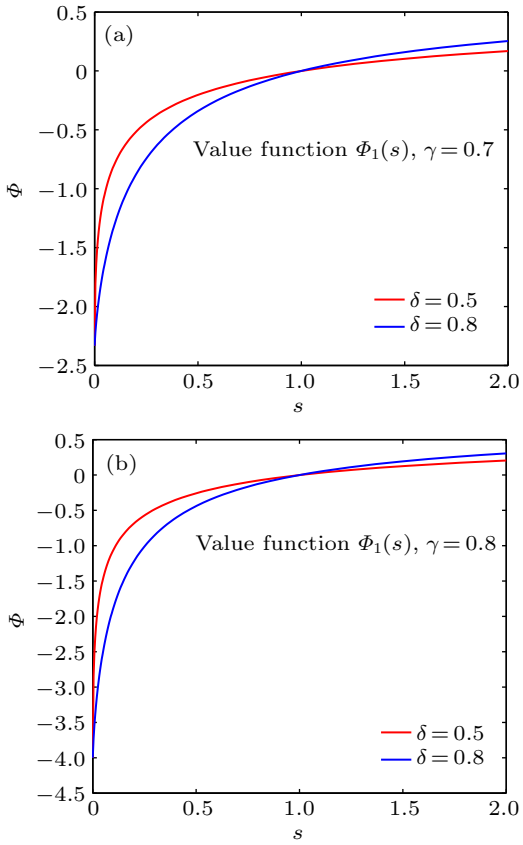


Fig. 2. Image of value function $\Phi_2(s)$.

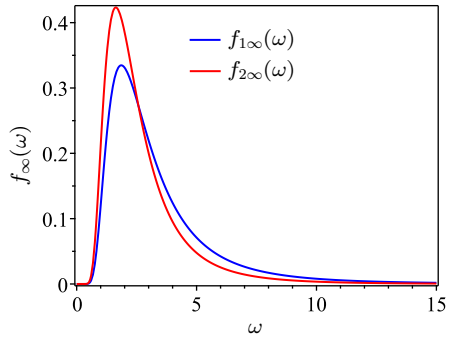


Fig. 3. Image of density function for steady-state solution.

The specific parameters in Fig. 3: $f_{1\infty}(\omega)$: $\alpha=0.3$, $2\gamma/\lambda=2$, $\bar{\omega}_{\rm L}=4$, and $f_{2\infty}(\omega)$: $\alpha=0.9$, $2\gamma/\lambda=2$, $\bar{\omega}_{\rm L}=4$. Figure 3 shows an image of the density function with the value function $\Phi_1(s)$. It can be seen that the maximum value of the density function changes with the change of α.

Figures 4 and 5 show the distribution of steady-state solutions with different values $\mu=1$ and $\mu=2$. From the plot, we see the distribution pattern of lognormal distribution under different parameters. Noticing Eq. (14), it is illustrated that the variance of equilibrium distribution relies only on the variance λ of the random service time and the rate of change in service time γ. Thus, when the σ changes, the essence is that the ratios of λ and γ are changing. The mean of the equilibrium distribution depends not only the variance λ of the random service time and the rate of change in service time γ, but also on the value of α. Gualandi and Toscani[8] obtained the real data of a call center through a special platform, which cope with its theoretical analysis. Our conclusions are in line with those in Ref. [8].

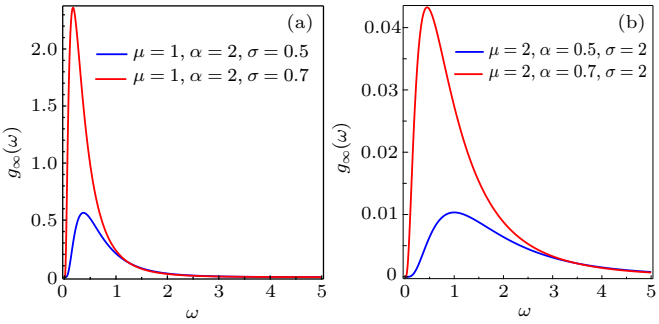


Fig. 4. Image of density function for steady-state solution.

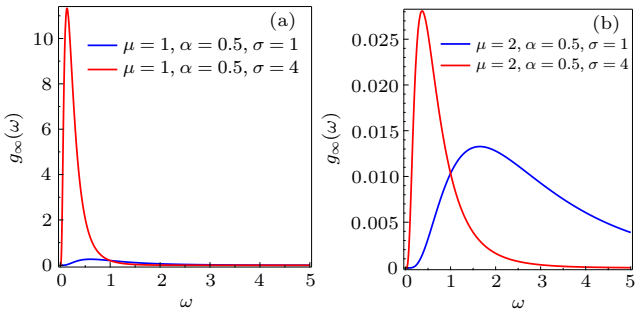


Fig. 5. Image of density function for steady-state solution.

6. Conclusion

In this paper, we use statistical mechanical methods to investigate the distribution of continuous service time in call centers. Based on the prospect theory in Refs. [9,10], we use microscopic interaction rules and two different value functions in multi-agent systems to describe the evolution of continuous service time, respectively. Considering the frequency of interactions between agents, we introduce a non-Maxwellian collision kernel. This collision kernel causes the frequency of interactions between agents to change. Using the asymptotic limit method, we discuss the evolution of service time. The steady-state solution are obtained in exact form. The properties of the two value functions are numerically analyzed and

compared. The steady-state solutions under different parameters are investigated.

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