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Riemann–Hilbert problem for the defocusing Lakshmanan–Porsezian–Daniel equation with fully asymmetric nonzero boundary conditions

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The Riemann–Hilbert approach is demonstrated to investigate the defocusing Lakshmanan–Porsezian–Daniel equation under fully asymmetric nonzero boundary conditions. In contrast to the symmetry case, this paper focuses on the branch points related to the scattering problem rather than using the Riemann surfaces. For the direct problem, we analyze the Jost solution of lax pairs and some properties of scattering matrix, including two kinds of symmetries. The inverse problem at branch points can be presented, corresponding to the associated Riemann–Hilbert. Moreover, we investigate the time evolution problem and estimate the value of solving the solutions by Jost function. For the inverse problem, we construct it as a Riemann–Hilbert problem and formulate the reconstruction formula for the defocusing Lakshmanan–Porsezian–Daniel equation. The solutions of the Riemann–Hilbert problem can be constructed by estimating the solutions. Finally, we work out the solutions under fully asymmetric nonzero boundary conditions precisely via utilizing the Sokhotski–Plemelj formula and the square of the negative column transformation with the assistance of Riemann surfaces. These results are valuable for understanding physical phenomena and developing further applications of optical problems.

Keywords: Riemann–Hilbert problem, defocusing Lakshmanan–Porsezian–Daniel equation, inverse scattering transform, asymmetric nonzero boundary conditions

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1. Introduction

The nonlinear Schrödinger equation (NLS) has a variety of applications in physics and engineering.[1,2] In optics, it plays a crucial role in describing light propagation,[3] especially in nonlinear optics, such as fiber communication[4] and nonlinear optics.[5] In plasma physics, the NLS equation can describe shock wave propagation, solitons and other nonlinear waves,[6–10] impacting plasma physics, heating, and diagnostics. Additionally, it is used in condensed matter physics, particularly in describing superfluid phenomena such as Bose–Einstein condensates.[11,12] Overall, the NLS equation plays a key role in describing nonlinear effects and wave propagation in various physical phenomena and has important implications in many practical applications. The solutions of NLS-type equations can be obtained by the Riemann–Hilbert (RH) approach,[13–18] Darboux transform (DT),[19,20] and other methods. These solutions are valuable for describing the propagation of optical pulses in optical fibers.[21]

In optics, the NLS equation is mainly used to describe the picosecond short pulse. However, for subpicosecond or femtosecond short pulses, additional properties become indispensable. Hence, higher-order equations incorporating effects such as higher-order dispersion, nonlinearity, self-steepening, and self-frequency shift, as discussed in Ref. [22], are derived to capture these nuances. The Lakshmanan–Porsezian–Daniel (LPD) equation is an higher-order equation which describes the evolution of the light field in nonlinear optics, and it is a variant of the nonlinear Schrödinger equation. The LPD equation is often used to describe nonlinear effects in optical resonators, including phenomena such as optical soliton[23] and mode formation. The solutions of the LPD equation provide an in-depth understanding of complex phenomena such as solitons,[24] periodic waves,[25] and the dynamical behaviors associated with them. These higher-order equations involve derivatives that offer more precise descriptions of light wave propagation phenomena.

So far, the solutions to the Gerdjikov–Ivanov equation and defocusing nonlocal modified Korteweg–de Vries (mKdV) equations under nonzero boundary conditions have been studied.[26,27] Moreover, for fully asymmetric nonzero boundary conditions about the solutions of mKdV equations have been reported.[28] In physics, fully asymmetric nonzero boundary conditions in Refs. [29–31] arise in a variety of practical problems, such as wave propagation in inhomogeneous media, and analyses of the response of structures under inhomogeneous loads. Additionally, the RH problem for the defocusing NLS equation with NZBCs was firstly presented in Ref. [14].

Based on this, in this work, we focus on the defocusing

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LPD equation[22,32,33]

$$\mathrm{i}q_t+\alpha(-q_{xx}+2|q|^2q)+\beta(-q_{xxxx}-6q|q|^4+4|q_x|^2q+6q_x^2q^*+2q^2q_{xx}^*+8q_{xx}|q|^2)=0, \tag{1}$$

under the fully asymmetric nonzero boundary conditions

$$\lim_{x\to\pm\infty}q(x,t)=q_\pm(t),\ \ |q_+(t)|\neq|q_-(t)|, \tag{2}$$

where $q=q(x,t)$ denotes the complex envelope, x and t denote the temporal and spatial coordinates, respectively, the subscripts indicate the partial derivatives, the asterisk signifies the complex conjugation, α denotes the coefficient of the second-order derivatives, and β represents the strength of high-order linear and nonlinear effects. In particular, the symmetric situation is $|q_+(t)|=|q_-(t)|$. There has been many fundamental studies on Eq. (1) in physics, e.g., the interaction of optical dark bound state solitons has been investigated.[22] Furthermore, utilizing finite-gap integration theory, researchers have studied the periodic solutions and investigated solitons of harmonic limits under certain conditions.[32] In optics, analytical solutions have been constructed by applying the DT. However, to our knowledge, the RH approach for Eq. (1) under the fully asymmetric nonzero boundary conditions, i.e., conditions (2), has not been investigated up to date. Unlike in literature, the Riemann surfaces are used to solve the LPD equation problem, and we study the properties of the solution in an asymmetric case, not in a symmetric case.

When $x\to\pm\infty$, the spatial derivative of $q(x,t)$ approaches zero, which implies $|q_\pm(t)|=|q_\pm(0)|$. Therefore, we take the boundary conditions (BCs) in the form of

$$q_\pm(t)=\nu_\pm\,\mathrm{e}^{\mathrm{i}\gamma_\pm+2\mathrm{i}(\nu_\pm^2\alpha-3\nu_\pm^4\beta)t}, \tag{3}$$

where $0\le\gamma_\pm<2\pi$. According to the symmetry $x\to-x$ of the LPD equation, we choose $\nu_->\nu_+>0$.

The framework of this paper is as follows: In Section 2, the direct problem will be introduced to discuss the analyticity, symmetry and distribution of discrete spectrum. In Section 3, the equation of time evolution is shown. The evolution of scattering data, reflection coefficients, and norming constants will be presented. In Section 4, we focus on computing solutions in the construction and recombination of jump matrix by the RH approach. In Section 5, we give the discussions and conclusions.

2. Direct problem

The Lax pair associated with Eq. (1) is

$$\Gamma_x=U\Gamma,\ \ \Gamma_t=V\Gamma, \tag{4}$$

with

$$U=-\mathrm{i}z\sigma_3-\mathrm{i}\sigma_3Q,\quad V=\sum_{j=0}^{j=4}z^jV_j,$$

$$\begin{aligned}
V_4&=-8\mathrm{i}\beta\sigma_3,\ \ V_3=8\mathrm{i}\beta Q\sigma_3,\\
V_2&=2\mathrm{i}\alpha\sigma_3-4\mathrm{i}\beta\sigma_3Q^2+4\beta Q_x,\\
V_1&=-2\beta QQ_x+2\beta Q_xQ+2\mathrm{i}\alpha\sigma_3Q-4\mathrm{i}\beta\sigma_3Q^3-\mathrm{i}\beta Q_{xx}\sigma_3,\\
V_0&=-\alpha Q_x+6\beta Q^2Q_x-\beta Q_{xxx}-3\mathrm{i}\beta\sigma_3Q^4-\mathrm{i}\beta\sigma_3Q_x^2\\
&\quad+\mathrm{i}\beta\sigma_3QQ_{xx}+\mathrm{i}\alpha\sigma_3Q^2+\mathrm{i}\beta\sigma_3Q_{xx}Q,\\
Q&=\begin{pmatrix}0&q\\q^*&0\end{pmatrix},\ \ \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},
\end{aligned}$$

where z is the spectral parameter, and $\Gamma=\binom{\phi_1}{\phi_2}$ is the vector eigenfunction with ϕ_1 and ϕ_2 being eigenfunctions of x and t. The zero curvature equation $U_t-V_x+UV-VU=0$ of lax pair (4) corresponds to Eq. (1).

As $x\to\pm\infty$, the first expression of lax pair (4) is reduced to

$$\Gamma_x=U_\pm\Gamma, \tag{5}$$

where

$$U_\pm=\mathrm{i}z\Lambda+\mathrm{i}\Lambda Q_\pm,\ \ Q_\pm=\begin{pmatrix}0&q_\pm(t)\\q_\pm^*(t)&0\end{pmatrix}.$$

Thus we can obtain in calculation that the eigenvalues of $U_\pm$ are $\pm\mathrm{i}\lambda_\pm(z)$, with

$$\lambda_\pm^2=z^2-\nu_\pm^2. \tag{6}$$

Obviously, the eigenvalues have branch points. In the symmetric case, $\lambda_+=\lambda_-$, Riemann surfaces are introduced to deal with this problem. In this work, we define $\lambda_\pm$ as a single-valued function to avoid the use of Riemann surfaces.

2.1. Jost solutions and scattering matrix

For the sake of further discussion, we introduce the following notations:

$$Z_\pm=(-\infty,-\nu_\pm]\cup[\nu_\pm,\infty],\ \ Z_0=[-\nu_-,-\nu_+]\cup[\nu_+,\nu_-]. \tag{7}$$

Similarly, we can define their internal as

$$\mathring{Z}_\pm=(-\infty,-\nu_\pm)\cup(\nu_\pm,\infty),\ \ \mathring{Z}_0=(-\nu_-,-\nu_+)\cup(\nu_+,\nu_-). \tag{8}$$

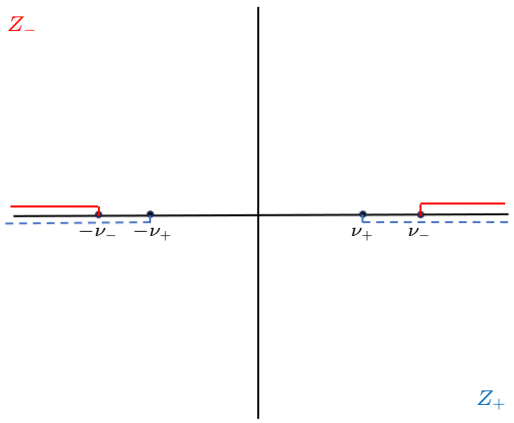


Fig. 1. The branch cuts Z_+ and Z_- of the complex z-plane.

As $x\to\pm\infty$, branch points arise at the values of z where $\lambda_\pm=0$, namely, $z=\pm\nu_\pm$. We construct the branch cuts of $\lambda_\pm$ on $Z_\pm$ within the complex z-plane, as depicted in Fig. 1. Notably, $\lambda_\pm$ are analytic in $\mathbb{C}\backslash Z_\pm$ and remain continuous as z approaches $Z_\pm$ from above.

Similar to the case mentioned in Refs. [34,35], we derive the eigenvector of $U_\pm$ as

$$E_\pm(z,t)=\begin{pmatrix}1 & \frac{-q_\pm}{\lambda_\pm+z}\\ \frac{-q_\pm^*}{\lambda_\pm+z} & 1\end{pmatrix}=I-\frac{1}{\lambda_\pm+z}Q_\pm. \tag{9}$$

According to expression (9), the Jost function can be written as

$$\Gamma_\pm(x,t,z)=E_\pm(z,t)\,\mathrm{e}^{-\mathrm{i}\lambda_\pm(z)x\sigma_3}(I+o(1)),\ \ z\in\mathring{Z}_\pm,\ \ x\to\pm\infty. \tag{10}$$

We may define and write the determinant of $E_\pm(z,t)$ as

$$d_\pm:=\det(E_\pm(z,t))=\frac{2\lambda_\pm(z)}{z+\lambda_\pm}. \tag{11}$$

From expression (9), we obtain the inverse of $E_\pm(z,t)$:

$$E_\pm(z,t)^{-1}=\frac{1}{d_\pm}\begin{pmatrix}1 & \frac{q_\pm}{\lambda_\pm+z}\\ \frac{q_\pm^*}{\lambda_\pm+z} & 1\end{pmatrix}. \tag{12}$$

To remove the oscillations, we take the following modified eigenfunctions:

$$\psi_\pm(x,t,z)=\Gamma_\pm(x,t,z)\,\mathrm{e}^{\mathrm{i}\lambda_\pm(z)x\sigma_3}. \tag{13}$$

Thus,

$$\lim_{x\to\pm\infty}\psi_\pm=E_\pm(t). \tag{14}$$

Using the modified eigenfunctions (13), the Volterra integral equation can be written as

$$\begin{aligned}\psi_-(x,t,z)&=E_--\int_{-\infty}^{x}E_-\,\mathrm{e}^{-\mathrm{i}\lambda_-(z)(x-\zeta)\hat{\sigma}_3}\\&\quad\times[E_-^{-1}\mathrm{i}\sigma_3\Delta Q_-(\zeta,t)\psi_-(\zeta,t,z)]\,\mathrm{d}\zeta,\\ \psi_+(x,t,z)&=E_++\int_{x}^{+\infty}E_+\,\mathrm{e}^{-\mathrm{i}\lambda_+(z)(x-\zeta)\hat{\sigma}_3}\\&\quad\times[E_+^{-1}\mathrm{i}\sigma_3\Delta Q_+(\zeta,t)\psi_+(\zeta,t,z)]\,\mathrm{d}\zeta,\end{aligned} \tag{15}$$

where $\mathrm{e}^{-f\hat{\sigma}_3}P:=\mathrm{e}^{-f\sigma_3}P\mathrm{e}^{f\sigma_3}$ and $\Delta Q_\pm(x,t):=Q(x,t)-Q_\pm(t)$. For simplicity, to calculate the solution, we define $\psi_\pm$ in forms of $\psi_\pm=(\psi_{\pm,1},\psi_{\pm,2})$; $\psi_{-,1}$ is well defined on Z_- and analytic in $\mathbb{C}\backslash Z_-$; $\psi_{+,2}$ is well defined on Z_+ and analytic in $\mathbb{C}\backslash Z_+$. Specifically, at the branch points of $z=\pm\nu_\pm$, we have $\lambda_\pm(z)=0$, so $\det(E_\pm(z,t))=0$. This means that $E_\pm(z,t)$ does not have an inverse on $z=\pm\nu_\pm$. Thus, when $z\to\pm\nu_+$ and $z\to\pm\nu_-$, we can get the expressions of $\psi_+(x,t,z)$ and $\psi_-(x,t,z)$:

$$\begin{aligned}\psi_+(x,t,\pm\nu_+)&=I\mp\frac{Q_+}{\nu_+}+\int_{x}^{+\infty}[(x-\zeta)U_+(\pm\nu_+,t)+I]\\&\quad\times\mathrm{i}\sigma_3\Delta Q_+(\zeta,t)\psi_+(\zeta,t,\pm\nu_-)\,\mathrm{d}\zeta,\\ \psi_-(x,t,\pm\nu_-)&=I\mp\frac{Q_-}{\nu_-}-\int_{-\infty}^{x}[(x-\zeta)U_-(\pm\nu_-,t)+I]\\&\quad\times\mathrm{i}\sigma_3\Delta Q_-(\zeta,t)\psi_-(\zeta,t,\pm\nu_-)\,\mathrm{d}\zeta,\end{aligned} \tag{16}$$

where $\psi_{+,1}$ and $\psi_{-,2}$ are analytically extended to the domains $\mathbb{C}\backslash Z_+$ and $\mathbb{C}\backslash Z_-$, respectively. As $x\to\pm\infty$, we derive $\det(\Gamma_\pm)$ as follows:

$$\det(\Gamma_\pm)=\det(\psi_\pm)=d_\pm. \tag{17}$$

Both $\Gamma_\pm$ are the matrix solutions of first-order scattering ODE, so the two solutions satisfy the linear scattering relation

$$\Gamma_-(x,t,z)=\Gamma_+(x,t,z)S(z,t), \tag{18}$$

where $S(z,t)$ is a scattering matrix, and satisfies $\det(S)=\frac{d_-}{d_+}$.

It can be noted that $S(z,t)=(s_{ij}(z,t))_{1\le i,j\le 2}$ is independent of x, and $\det(S)\neq 1$, which is different from the situations of symmetric NZBC. Notably, $\det(S(\pm\nu_-,t))=0$. Employing the scattering relation (18), we have

$$\begin{aligned}s_{11}(z,t)&=\det(\Gamma_{-,1},\Gamma_{+,2})(x,t,z)/d_+,\\ s_{12}(z,t)&=\det(\Gamma_{-,2},\Gamma_{+,2})(x,t,z)/d_+,\\ s_{21}(z,t)&=\det(\Gamma_{+,1},\Gamma_{-,1})(x,t,z)/d_+,\\ s_{22}(z,t)&=\det(\Gamma_{+,1},\Gamma_{-,2})(x,t,z)/d_+.\end{aligned} \tag{19}$$

Through the definition of the determinant, we can analyze their continuous discrete spectrum problem.

For subsequent discussions on the inverse problem, we introduce the reflection coefficient

$$\begin{aligned}r(z,t)&:=s_{21}(z,t)/s_{11}(z,t),\\ \hat{r}(z,t)&:=s_{12}(z,t)/s_{22}(z,t),\ \ z\in Z_-.\end{aligned} \tag{20}$$

2.2. Symmetries

Based on lax pair (4), we find two kinds of symmetries about the LPD equation, i.e., $z\to z^*$ and $\lambda_\pm(z)\to-\lambda_\pm(z)$. Distinct symmetries imply different properties of the eigenfunctions and scattering data, which are related to the diverse behaviors about z and $\lambda_\pm(z)$.

(i) For first symmetry: $z\to z^*$.

Considering the asymptotic behaviors of $\Gamma_\pm(x,t,z)$ as $x\to\pm\infty$, we have

$$\Gamma_\pm(x,t,z)=\sigma_1\Gamma_\pm^*(x,t,z)\sigma_1,\ \ z\in Z_\pm, \tag{21}$$

with $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Substituting expression (21) into Eq. (18), we can obtain $S(z,t)=\sigma_1S(z,t)^*\sigma_1$ for $z\in Z_-$, which is equivalent to

$$s_{22}(z,t)=s_{11}(z,t)^*,\ \ s_{12}(z,t)=s_{21}(z,t)^*,\ \ z\in Z_-, \tag{22}$$

and get $\hat{r}(z,t)=r(z,t)^*$, $z\in Z_-$. From the value of $\det(S)$ and expression (20), we conclude that

$$\det(S)=|s_{22}|^2-|s_{12}|^2=\frac{d_-}{d_+},\ \ z\in Z_-. \tag{23}$$

Since $\frac{d_-}{d_+} > 0$ when $z \in Z_-$, it can be found that s_{22} has no zeros on $z \in Z_-$.

When $z \notin Z_+$ and $x \to -\infty$, we derive

$$\sigma_1 \Gamma^*_{-,2}(x,t,z^*) = -\frac{q^*_-(t)}{z-\lambda_-(z)} \Gamma_{-,2}(x,t,z), \quad z \notin Z_+. \tag{24}$$

Similarly, when $x \to \infty$, we get

$$\sigma_1 \Gamma^*_{+,1}(x,t,z^*) = -\frac{q_+(t)}{z-\lambda_+(z)} \Gamma_{+,1}(x,t,z), \quad z \notin Z_-. \tag{25}$$

Utilizing expression (19), we can draw the conclusion

$$s^*_{22}(z^*,t) = \frac{q_+}{q_-}\frac{z+\lambda_-}{z+\lambda_+} s_{22}, \quad z \notin Z_-. \tag{26}$$

(ii) For second symmetry: $\lambda_\pm(z) \to -\lambda_\pm(z)$.

The solution of the scattering problem $\Gamma_\pm$ can be expressed as $\hat{\Gamma}_\pm(x,t,z) = (I - \frac{1}{z-\hat{\lambda}_\pm} Q)\, \mathrm{e}^{\mathrm{i}\hat{\lambda}_\pm x \sigma_3}(I + o(1))$ when $\lambda_\pm(z) \to -\lambda_\pm(z)$. For convenience, we set that $\hat{\lambda}_\pm(z) := -\lambda_\pm(z)$. By means of the expression of $\Gamma_\pm(x,t,z,\lambda)$, we find that $\hat{\Gamma}_\pm(x,t,z) := \Gamma_\pm(x,t,z,\hat{\lambda})$ is also a solution to the lax pair (4) when $x \to \pm\infty$. Specifically,

$$\begin{aligned}
\hat{\Gamma}_\pm(x,t,z) &= \Gamma_\pm(x,t,z)\frac{-1}{z-\lambda_\pm} Q_\pm(t), \quad z \in \mathring{Z}_\pm, \\
\Gamma_\pm(x,t,z) &= \hat{\Gamma}_\pm(x,t,z)\frac{-1}{z+\lambda_\pm} Q_\pm(t), \quad z \in \mathring{Z}_\pm.
\end{aligned} \tag{27}$$

Defining $\hat{S}(z,t)$ as the scattering matrix of $\hat{\Gamma}_\pm(x,t,z)$, we can infer that

$$S = \frac{z-\lambda_-}{z-\lambda_+} Q_+ \hat{S} Q_-^{-1}, \quad z \in Z_-. \tag{28}$$

Therefore, elements of $S(z,t)$ are correlated with those in $\hat{S}(z,t)$, namely,

$$s_{22} = \frac{q^*_+}{q^*_-}\frac{z-\lambda_-}{z-\lambda_+}\hat{s}_{11}, \quad s_{21} = \frac{q^*_+}{q_-}\frac{z-\lambda_-}{z-\lambda_+}\hat{s}_{12}, \quad z \in Z_-. \tag{29}$$

Furthermore, to investigate $\lambda_\pm$, we define

$$\begin{aligned}
\lambda^+_\pm &:= \lim_{\sigma\downarrow 0} \lambda_\pm(z+\mathrm{i}\sigma) = \lambda_\pm(z), \quad z \in Z_-, \\
\lambda^-_\pm(z) &:= \lim_{\sigma\uparrow 0} \lambda_\pm(z+\mathrm{i}\sigma) = -\lambda_\pm(z), \quad z \in Z_-, \\
\lambda^-_-(z) &:= \lim_{\sigma\uparrow 0} \lambda_-(z+\mathrm{i}\sigma) = \lambda_-(z), \quad z \in Z_0, \\
\lambda^-_+(z) &:= \lim_{\sigma\uparrow 0} \lambda_+(z+\mathrm{i}\sigma) = -\lambda_+(z), \quad z \in Z_0.
\end{aligned} \tag{30}$$

According to $\psi_{-,1}$ and $\psi_{+,2}$ mentioned above, $\Gamma_{-,1}(x,t,z)$ and $\Gamma_{+,2}(x,t,z)$ are continuous on Z_- and Z_+, and analytic in $\mathbb{C}\backslash Z_-$ and $\mathbb{C}\backslash Z_+$, respectively. Hence, in Z_+, Z_- and Z_0, we have

$$\begin{aligned}
\Gamma^-_{+,1}(x,t,z) &:= \lim_{\sigma\uparrow 0} \Gamma_{+,1}(x,t,z+\mathrm{i}\sigma) = \hat{\Gamma}_{+,1}(x,t,z), \quad z \in Z_+, \\
\Gamma^-_{-,2}(x,t,z) &:= \lim_{\sigma\uparrow 0} \Gamma_{-,2}(x,t,z+\mathrm{i}\sigma) = \hat{\Gamma}_{-,2}(x,t,z), \quad z \in Z_-,
\end{aligned}$$

$$\Gamma^-_{-,2}(x,t,z) := \lim_{\sigma\uparrow 0} \Gamma_{-,2}(x,t,z+\mathrm{i}\sigma) = \Gamma_{-,2}(x,t,z), \quad z \in Z_0. \tag{31}$$

Therefore, using expression (27), we find

$$\begin{aligned}
\Gamma^-_{-,2}(x,t,z) &= -\frac{q_-(t)}{z-\lambda_-}\Gamma_{-,1}(x,t,z), \quad z \in Z_-, \\
\Gamma^-_{+,1}(x,t,z) &= -\frac{q^*_+(t)}{z-\lambda_+}\Gamma_{+,2}(x,t,z), \quad z \in Z_+.
\end{aligned} \tag{32}$$

Making use of expression (19), we obtain the expression of the final result as

$$s^-_{22}(z,t) := \lim_{\sigma\uparrow 0} s_{22}(z+\mathrm{i}\sigma,t) = \begin{cases} \frac{q_-}{q_+}\frac{z-\lambda_+}{z-\lambda_-} s_{11}(z,t), & z \in Z_-, \\ -\frac{q^*_+}{z+\lambda_+} s_{12}(z,t), & z \in \mathring{Z}_0. \end{cases} \tag{33}$$

2.3. Behavior of the scattering data at the branch points

As mentioned above, the branch points in the inverse problem are $\pm\nu_-$ and $\pm\nu_+$. Based on expressions of Γ_+ and Γ_- given above, we deduce that Γ_+ and Γ_- are also defined at branch points. $\Gamma_\pm(x,t,\pm\nu)$ are also the solutions to the problem. Making use of their asymptotics, we acquire

$$\begin{aligned}
s_{12}(\pm\nu_-,t) &= \mp \mathrm{e}^{-\mathrm{i}\gamma_- - 2\mathrm{i}(\nu_-^2\alpha - 3\nu_-^4\beta)t} s_{11}(\pm\nu_-,t), \\
s_{22}(\pm\nu_-,t) &= \mp \mathrm{e}^{-\mathrm{i}\gamma_- - 2\mathrm{i}(\nu_-^2\alpha - 3\nu_-^4\beta)t} s_{21}(\pm\nu_-,t).
\end{aligned} \tag{34}$$

From the above formula and expression (22), we get

$$|s_{22}(\pm\nu_-,t)| = |s_{12}(\pm\nu_-,t)| \neq 0. \tag{35}$$

By means of symmetries of expression (20), we derive $|\hat{r}(\pm\nu_-,t)| = |\frac{s_{12}}{s_{22}}| = 1$. Then, recalling the definition of d_+, as $z \to \pm\nu_+$, we gain

$$\frac{1}{d_+(z)} = \frac{1}{2} + \frac{\sqrt{(\pm\nu_+)}}{2\sqrt{2}(z \mp \nu_+)^{1/2}} + O((z \mp \nu_+)^{1/2}). \tag{36}$$

Through the determinant relation of the matrix and scattering coefficient, we have

$$\begin{aligned}
s_{12}(z,t) &= (\det(\Gamma_{-,2},\Gamma_{+,2})(x,t,\pm\nu_+) + o(1))/d_+, \quad z \to \pm\nu_+, \\
s_{22}(z,t) &= (\det(\Gamma_{+,1},\Gamma_{-,2})(x,t,\pm\nu_+) + o(1))/d_+, \quad z \to \pm\nu_+.
\end{aligned} \tag{37}$$

Comparing the asymptotics of $\Gamma_{+,1}$ and $\Gamma_{-,2}$ when $x \to \infty$, we find $\lim_{z\to\pm\nu_+} \hat{r}(z,t) = \pm \mathrm{e}^{-\mathrm{i}\gamma_+ - 2\mathrm{i}(\nu_+^2\alpha - 3\nu_+^4\beta)t}$, namely, $z \to \pm\nu_-$, $|r(\pm\nu_+,t)| = 1$.

2.4. Discrete spectrum

According to the reflection of the wave in the scattering problem, the spectral singularity is important in studying the inverse scattering problem, reflecting the influence of nonlinear effect on the wave propagation. The generation of spectral singularities has an important effect on soliton solutions to be equal to 0, denoted as $s_{11}(z,t) = 0$ or $s_{22}(z,t) = 0$. However, when $z \in \mathring{Z}_0$, there has no zeros in the scattering problem, i.e.,

$$s_{12}(z,t)s_{22}(z,t) \neq 0, \quad z \in \mathring{Z}_0. \tag{38}$$

Thus, the singularity of s_{22} at $z=\pm\nu_+$ is zeros in $(-\nu_+,\nu_+)$. To verify the above formula, we suppose that $z_1,z_2,\ldots,z_N$ are the zeros of s_{22} in $(-\nu_+,\nu_+)$. When $z=z_n$, we have

$$\Gamma_{-,2}(x,t,z_n)=b_n(t)\Gamma_{+,1}(x,t,z_n),\ \ n=1,2,3,\ldots,N, \tag{39}$$

where b_n is scalar and independent of x and t, the norming constants of the inverse problem are defined as

$$c_n(t)=b_n(t)/s'_{22}(z_n,t), \tag{40}$$

with the prime denoting the differentiation of z. Recalling Eqs. (24)–(26), we deduce that

$$b_n=\frac{q_+}{q_-^*}\frac{z_n-\lambda_{-,n}}{z_n-\lambda_{+,n}}b_n^*,\ \ n=1,2,\ldots,N, \tag{41}$$

$$s'_{22}(z_n^*,t)^*=\frac{q_+}{q_-}\frac{z_n+\lambda_{-,n}}{z_n+\lambda_{+,n}}s'_{22}(z_n,t),\ \ n=1,2,\ldots,N. \tag{42}$$

It follows that

$$c_n^*(t)=\frac{q_+^*}{q_+}c_n,\ \ n=1,2,\ldots,N. \tag{43}$$

3. Time evolution

As mentioned above, $\Gamma_+(x,t,z)$ and $\Gamma_-(x,t,z)$ are not the solutions of the second expression of lax pair (4). In order to derive solutions to Eq. (1), we need to find the solution that satisfies the scattering problem and time evolution of lax pair (4). Next, similar to Ref. [36], we define that $X(x,t,z)$ is a solution of lax pair (4), expressed as

$$X(x,t,z)=\Gamma_\pm(x,t,z)W_\pm(z,t),\ \ z\in\mathring{Z}_\pm, \tag{44}$$

where $W_\pm(z,t)$ are independent of x. Taking the partial derivative of expression (44) with respect to t, we gain

$$(W_\pm)_t=Y_\pm W_\pm,\ \ z\in\mathring{Z}_\pm, \tag{45}$$

where $Y_\pm(z,t)=\Gamma_\pm^{-1}[V\Gamma_\pm-(\Gamma_\pm)_t],\ \ z\in\mathring{Z}_\pm$. As $x\to\pm\infty$, we get

$$Y_\pm=\lim_{x\to\pm\infty}\Gamma_\pm^{-1}[V\Gamma_\pm-(\Gamma_\pm)_t]=\mathrm{i}h_\pm(z)\sigma_3,\ \ z\in\mathring{Z}_\pm, \tag{46}$$

with $h_\pm(z)=\frac{-15}{2}\beta z^4-3\beta\lambda_\pm z^3+(\frac{3}{2}\alpha+5\beta\lambda_\pm^2)z^2+(\frac{1}{2}\alpha\lambda_\pm+\frac{1}{2}\beta\lambda_\pm^3)z-\frac{3}{2}\beta\lambda_\pm^4-\frac{1}{2}\alpha\lambda_\pm^2+(\nu_\pm^2\alpha-3\nu_\pm^2\beta)$. Subsequently, we have

$$(\Gamma_\pm)_t=V\Gamma_\pm-\Gamma_\pm Y_\pm,\ \ z\in Z_\pm. \tag{47}$$

By employing expressions (18) and (46), we obtain

$$S_t=Y_+-SY_-,\ \ z\in\mathring{Z}_-. \tag{48}$$

Substituting expression (46) into expression (50), we gain

$$\begin{aligned}&s_{12}(z,t)=s_{12}(z,0)\,\mathrm{e}^{\mathrm{i}t(h_+(z)+h_-(z))},\ \ z\in\mathring{Z}_-,\\&s_{22}(z,t)=s_{22}(z,0)\,\mathrm{e}^{-\mathrm{i}t(h_+(z)-h_-(z))},\ \ z\in\mathring{Z}_-.\end{aligned} \tag{49}$$

As a result, the first formula of expression (49) can be extended in the cases that $s_{22}(z,0)$ is analytic. Similarly, the second formula of expression (49) can be extended to $z\in\mathring{Z}_+$. Through expressions (47) and (19), we derive

$$\hat{r}(z,t)=\hat{r}(z,0)\,\mathrm{e}^{2\mathrm{i}th_+(z)},\ \ z\in Z_+. \tag{50}$$

In expression (49), we find that, at $z=z_n$,

$$s'_{22}(z_n,t)=s'_{22}(z_n,0)\,\mathrm{e}^{-\mathrm{i}t(h_+-h_-)},\ \ n=1,2,\ldots,N. \tag{51}$$

Based on expression (47), we note

$$\begin{aligned}&(\Gamma_{-,2})_t=V\Gamma_{-,2}+\mathrm{i}h_-(z)\Gamma_{-,2},\\&(\Gamma_{+,1})_t=V\Gamma_{+,1}-\mathrm{i}h_+(z)\Gamma_{+,1},\ \ z\in\mathring{Z}_-.\end{aligned} \tag{52}$$

Then, substituting expression (52) to expression (39) yields

$$b_n(t)=b_n(0)\,\mathrm{e}^{\mathrm{i}t[h_+(z_n)+h_-(z_n)]},\ \ n=1,2,\ldots,N. \tag{53}$$

According to the above formula combined with expression (40), we get the norming constant as follows:

$$c_n(t)=c_n(0)\,\mathrm{e}^{2\mathrm{i}th_+},\ \ n=1,2,\ldots,N. \tag{54}$$

In fact, $s_{22}(z,t)$ is independent of t, and for a fixed value z, $s_{22}(z,t)\to0$ and $s_{22}(z,t)\to\infty$ when $t\to\infty$. On the other hand, for a fixed value t, we have

$$\lambda_\pm(z)=z-\frac{\nu_\pm^2}{2z}+O\Big(\frac{1}{z^3}\Big),\ \ z\to\infty. \tag{55}$$

It follows that $h_+(z)-h_-(z)=(\alpha-3\beta)(\nu_+^2-\nu_-^2)-\frac{3}{2}\beta(\lambda_+^4-\lambda_-^4)+\frac{1}{2}\beta z(\lambda_+^3-\lambda_-^3)+(5\beta-\frac{1}{2}\alpha)(\lambda_+^2-\lambda_-^2)+(\frac{1}{2}\alpha z-3\beta z^3)(\lambda_+-\lambda_-)$.

4. Inverse problem

In this section, based on the lax pair (4), we will construct the corresponding RH problem, and reconstruct the solution for Eq. (1) with fully NZBCs.

4.1. Formulation of the Riemann–Hilbert problem

According to the analysis mentioned above, we consider a meromorphic matrix function $M(x,t,z)$

$$M(x,t,z)=(\psi_{+,1}(x,t,z),\frac{\psi_{-,2}(x,t,z)}{s_{22}(z,t)}),\ \ z\notin Z_-. \tag{56}$$

Next, we define $M(x,t,z)$ onto the cut, that is,

$$\begin{aligned}&M^+(x,t,z):=\lim_{\sigma\downarrow0}M(x,t,z+\mathrm{i}\sigma)=\Big(\psi_{+,1},\frac{\psi_{-,2}}{s_{22}}\Big),\ \ z\in Z_+,\\&M^-(x,t,z):=\lim_{\sigma\uparrow0}M(x,t,z+\mathrm{i}\sigma)=\Big(\psi_{+,1}^-,\frac{\psi_{-,2}^-}{s_{22}^-}\Big),\ \ z\in Z_-,\\&M^-(x,t,z):=\lim_{\sigma\uparrow0}M(x,t,z+\mathrm{i}\sigma)=\Big(\psi_{+,1}^-,\frac{\psi_{-,2}}{s_{22}^-}\Big),\ \ z\in Z_0.\end{aligned} \tag{57}$$

Obviously, S is defined by $(s_{ij}(z,t))_{1\le i,j\le 2}$, and ϕ is continuous on the cut. Introducing the jump matrix $J(x,t,z)$, we have

$$M^+(x,t,z)=M^-(x,t,z)J(x,t,z),\ \ z\in Z_+, \tag{58}$$

with

$$J=\begin{cases}J_{Z_-}(x,t,z),\ \ z\in Z_-,\\ J_{Z_0}(x,t,z),\ \ z\in Z_0.\end{cases} \tag{59}$$

From expression (27), we obtain

$$\begin{aligned}&M(x,t,z)=O(1),\ \ z\to\pm v_-,\\ &M(x,t,z)=(O(z\mp v_+)^{1/2},O(1)),\ \ z\to\pm v_+.\end{aligned} \tag{60}$$

The corresponding RH problem associated with jump matrix is in different fields. Then, we discuss the jump matrix separately.

Jump matrix for $z\in Z_-$. In row form, employing expressions (31)–(33) to express $\Gamma_{+,1}$ and $\Gamma_{-,2}$ in the form of $\hat{\Gamma}_{+,1}$ and $\hat{\Gamma}_{-,2}$, we have

$$-\frac{q_-^*}{z+\lambda_-}\hat{\Gamma}_{-,2}=s_{11}\Gamma_{+,1}-\frac{q_+}{z+\lambda_+}\hat{\Gamma}_{+,1}s_{21}, \tag{61}$$

$$\Gamma_{-,2}=-\frac{q_+}{z+\lambda_+}\hat{\Gamma}_{+,1}s_{22}+\Gamma_{+,1}s_{12}. \tag{62}$$

Thus the meromorhpic matrix can be expressed as

$$\left(\Gamma_{+,1},\frac{\Gamma_{-,2}}{s_{22}}\right)=\left(\Gamma_{+,1}^-,\frac{\Gamma_{-,2}^-}{s_{22}^-}\right)\frac{1}{z+\lambda_+}\sigma_3Q_+J_r, \tag{63}$$

where

$$J_r=\begin{pmatrix}1&\hat{r}\\ \hat{r}^*&|r|^2-1\end{pmatrix}.$$

Then, through expressions (58) and (59), we obtain

$$J_{Z_-}=(E_+-I)\sigma_3\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\lambda_+x}&0\\0&\mathrm{e}^{\mathrm{i}\lambda_-x}\end{pmatrix}J_r\begin{pmatrix}\mathrm{e}^{\mathrm{i}\lambda_-x}&0\\0&\mathrm{e}^{-\mathrm{i}\lambda_+x}\end{pmatrix}. \tag{64}$$

Jump matrix $z\in Z_0$. Rewriting $\Gamma_{+,1}$ and $\frac{\Gamma_{-,2}}{s_{22}}$ in terms of $\hat{\Gamma}_{+,1}$ and $\frac{\Gamma_{-,2}}{s_{22}^-}$, we get

$$\frac{\Gamma_{-,2}}{s_{22}}=-\frac{\Gamma_{-,2}}{s_{22}^-}\frac{q_+^*}{z+\lambda_+}\hat{r},\ \ z\in\mathring{Z}_0. \tag{65}$$

As mentioned above, we gain expression (63) when $\Gamma_{-,2}^-(x,t,z)=\Gamma_{-,2}(x,t,z)$, where

$$J_r=\begin{pmatrix}1&\hat{r}\\ \frac{1}{\hat{r}}&0\end{pmatrix},\ \ z\in Z_0. \tag{66}$$

Then, employing expressions (58) and (59), we derive

$$J_{Z_0}=(E_+-I)\sigma_3\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\lambda_+x}&0\\0&\mathrm{e}^{-\mathrm{i}\lambda_-x}\end{pmatrix}J_r\begin{pmatrix}\mathrm{e}^{\mathrm{i}\lambda_-x}&0\\0&\mathrm{e}^{-\mathrm{i}\lambda_+x}\end{pmatrix}. \tag{67}$$

Based on expression (63), we rewrite the expression of J_r as

$$J_r=\begin{pmatrix}1&\hat{r}\\ r&r\hat{r}-1\end{pmatrix},\ \ z\in Z_0, \tag{68}$$

where we define r as $r=\frac{1}{\hat{r}}$, $z\in Z_0$. Furthermore, employing $\hat{r}(z,t)=r^*(z,t)$ for $z\in Z_-$, we derive that $r^*(\pm v_-,t)=\frac{1}{r(\pm v_-,t)}$, where $r(z,t)$ and $\hat{r}(z,t)$ are continuous at $z=\pm v_-$ for all $z\in Z_+$.

In conclusion, by means of expression (58), we sum up $M(x,t,z)$ as

$$M^+(x,t,z)=M^-(x,t,z)(E_+-I)\sigma_3[I-J_0(x,t,z)],\ \ z\in Z_+, \tag{69}$$

where

$$\begin{aligned}J_0&=\begin{pmatrix}1-\mathrm{e}^{-\mathrm{i}(\lambda_+-\lambda_-)x}&-\mathrm{e}^{-2\mathrm{i}\lambda_+x}\hat{r}\\ -\mathrm{e}^{2\mathrm{i}(\lambda_-)x}\hat{r}^*&1-\mathrm{e}^{-\mathrm{i}(\lambda_+-\lambda_-)x}(|r|^2-1)\end{pmatrix},\ z\in Z_-,\\ J_0&=\begin{pmatrix}1-\mathrm{e}^{-\mathrm{i}(\lambda_+-\lambda_-)x}&-\hat{r}\mathrm{e}^{-2\mathrm{i}\lambda_+x}\\ -\frac{1}{\hat{r}}&1\end{pmatrix},\ z\in Z_0.\end{aligned} \tag{70}$$

4.2. Solutions to the RH problem and reconstruction formula

In this subsection, we turn the original RH problem into a solvable problem, and finally get the solution as $z\to\infty$ under fully asymmetric nonzero boundary conditions. Through calculation, we have

$$\lambda_-(z)=\begin{cases}z-\frac{v_-^2}{2z}+o(\frac{1}{z}),\ \ z\to\infty\ \wedge\ \mathrm{Im}z\ge 0,\\ -z+\frac{v_-^2}{2z}+o(\frac{1}{z}),\ \ z\to\infty\ \wedge\ \mathrm{Im}z<0.\end{cases} \tag{71}$$

Then, we derive the asymptotic behaviors of $M(x,t,z)$ as

$$M(x,t,z)=\begin{cases}I+O(\frac{1}{z}),\ \ z\to\infty\ \wedge\ \mathrm{Im}z>0,\\ \frac{1}{z+\lambda_+(z)}\sigma_3Q_+(t)+O(1),\ \ z\to\infty\ \wedge\ \mathrm{Im}z<0.\end{cases} \tag{72}$$

Thus, we get a new RH problem about $M_*(x,t,z)$ with reflectionless potential of the jump matrix

$$M_*^+(x,t,z)=M_*^-(x,t,z)(E_+(z,t)-I),\ \ z\in Z_+, \tag{73}$$

under the jump condition

$$E_+(z,t)=E_+^-(z,t)(E_+(z,t)-I),\ \ z\in Z, \tag{74}$$

where $M_*(x,t,z)=E_+(z,t)$. Moreover, we modify the initial RH problem by introducing the matrix $P(x,t,z)$ defined as

$$M(x,t,z)=P(x,t,z)E_+(z,t), \tag{75}$$

with $P(x,t,z)=I+O(1/z)$, $z\to\infty$. Therefore, we have

$$P^+(x,t,z)=P^-(x,t,z)\hat{J}(x,t,z), \tag{76}$$

where $\hat{J}(x,t,z)=E_+^-(z,t)J(x,t,z)E_+^{-1}(z,t)$, $z\in Z_+$. Using expressions (58), (69), and (74), we find

$$\hat{J}(x,t,z)=E_+(z,t)\sigma_3(I-J_0(x,t,z))E_+^{-1}(z,t),\ \ z\in Z_+, \tag{77}$$

where J_0 is defined in expression (70).

Expression (60) implies that

$$P=\begin{cases}O(1), & z\to\pm\nu_-,\\ O(z\mp\nu_+)^{1/2}, & z\to\pm\nu_+.\end{cases}\tag{78}$$

From expression (39), we have $\psi_{-,2}(x,t,z_n)=b_n\psi_{+,1}(x,t,z_n)\,\mathrm{e}^{-\mathrm{i}(\lambda_{+j}+\lambda_{-j})x}$, $n=1,2,\ldots,N$. Noting that $\det(\Gamma_{+1},\Gamma_{-2})(x,t,z_n)$ differentiation about t is nonzero, we can find that the $s'_{22}(z_j,t)$ is nonzero. Since the zeros of the $s_{22}(z_j,t)$ are simple, we obtain $\mathrm{Res}[\frac{\psi_{-,2}(x,t,z)}{s_{22}(z,t)}]=\frac{\psi_{-,2}(x,t,z_n)}{s'_{22}(z_n,t)}=c_n(t)\,\mathrm{e}^{-\mathrm{i}(\lambda_{-,n}+\lambda_{+,n})x}\psi_{+,1}(x,t,z_n)$, $n=1,2,\ldots,N$, where $\lambda_{\pm,n}=\lambda_\pm(z_n)$. Thus

$$\operatorname*{Res}_{z=z_n}M(x,t,z)=(0,c_n(t)\,\mathrm{e}^{-\mathrm{i}(\lambda_{-,n}+\lambda_{+,n})x}M_1(x,t,z_n)),\quad n=1,2,\ldots,N.\tag{79}$$

Since E_+ is reversible, from expression (75), we derive

$$\begin{aligned}\operatorname*{Res}_{z=z_n}P(x,t,z)&=\operatorname*{Res}_{z=z_n}M(x,t,z)E_+^{-1}\\&=(0,c_n(t)\,\mathrm{e}^{-\mathrm{i}(\lambda_{-,n}+\lambda_{+,n})x}M_1(x,t,z_n))E_+^{-1},\\&\quad n=1,2,\ldots,N.\end{aligned}\tag{80}$$

Obviously, to derive the solution of the RH problem, the residue conditions of $P(x,t,z)$ can be expressed as

$$\begin{aligned}&\operatorname*{Res}_{z=z_n}[P_1(x,t,z)-\frac{q_+^*(t)}{z+\lambda_+(z)}P_2(x,t,z)]=0,\\&\operatorname*{Res}_{z=z_n}[P_2(x,t,z)+\frac{q_+(t)}{z+\lambda_+(z)}P_1(x,t,z)]\\&=c_n(t)\,\mathrm{e}^{-\mathrm{i}(\lambda_{-,n}+\lambda_{+,n})x}\Big[P_1(x,t,z_n)-\frac{q_+^*(t)}{z_n+\lambda_{+,n}}P_2(x,t,z_n)\Big].\end{aligned}\tag{81}$$

Eventually, the solution is

$$\begin{aligned}P(x,t,z)=I&+\sum_{n=1}^{N}\frac{1}{z-z_n}\operatorname*{Res}_{z=z_n}[P(x,t,z)]\\&-\frac{1}{2\pi\mathrm{i}}\int_{Z_+}\frac{[P^-(I-\hat{J})](x,t,\theta)}{\theta-z}\mathrm{d}\theta,\quad z\in\mathbb{C}\backslash Z_-.\end{aligned}\tag{82}$$

From expressions (74), (75), and (77), we obtain the meromorphic matrix function $M(x,t,z)$ as

$$\begin{aligned}&M(x,t,z)\\&=E_+(z,t)+\sum_{n=1}^{N}\frac{1}{z-z_n}\operatorname*{Res}_{z=z_n}[M(x,t,z)]E_+^{-1}(z,t)E_+(z,t)\\&\quad-\frac{1}{2\pi\mathrm{i}}\int_{Z_+}\Big\{\frac{1}{\theta-z}[M^-(E_+-I)E_+^{-1}(I-E_+\sigma_3E_+^{-1}\\&\quad+E_+\sigma_3J_0E_+^{-1})](x,t,\theta)E_+(z,t)\Big\}\mathrm{d}\theta,\quad z\in\mathbb{C}\backslash Z_-.\end{aligned}\tag{83}$$

Taking into account expressions (80) and (83), we have

$$\begin{aligned}&\Big(1-\frac{q_+^*}{2\lambda_{+,n}(z+\lambda_{+,n}(z_n))}c_n(t)\,\mathrm{e}^{-\mathrm{i}(\lambda_{-,n}+\lambda_{+,n})x}\Big)M_1(x,t,z_n)\\&=I-\frac{1}{2\pi\mathrm{i}}\int_{Z_+}\Big\{\frac{1}{\theta-z_n}[M^-(E_+-I)E_+^{-1}(I-E_+\sigma_3E_+^{-1}\\&\quad+E_+\sigma_3J_0E_+^{-1})](x,t,\theta)\Big\}\mathrm{d}\theta+\sum_{n'=1}^{N}\frac{1}{z_n-z_{n'}}c'_n\,\mathrm{e}^{-\mathrm{i}(\lambda_{-,n'}+\lambda_{+,n'})x}\\&\quad\times(0,M_1(x,t,z_{n'})E_+^{-1}(z_{n'},t))E_{+,1}(z_n,t),\quad n=1,2,\ldots,N.\end{aligned}\tag{84}$$

Similarly to Ref. [36], the solution to the problem is derived as

$$q^*(x,t)=2\mathrm{i}\lim_{z\to\infty,\mathrm{Im}z>0}zM_{21}(x,t,z).\tag{85}$$

Additionally, considering expression (83), we have

$$\begin{aligned}&M(x,t,z)\\&=I-\frac{1}{2z}Q_++\frac{1}{z}\sum_{n=1}^{N}c_n(t)\,\mathrm{e}^{-\mathrm{i}(\lambda_{-,n}+\lambda_{+,n})x}\\&\quad\times(0,M_1(x,t,z_n))E_+^{-1}(x,t,\theta)\\&\quad+\frac{1}{2\pi\mathrm{i}z}\int_{Z_+}[M^-(E_+-I)E_+^{-1}(I-E_+\sigma_3E_+^{-1}+E_+\sigma_3J_0E_+^{-1})]\\&\quad\times(x,t,\theta)E_+(z,t)\mathrm{d}\theta+O(\frac{1}{z^2}),\quad z\to\infty\wedge\mathrm{Im}z>0.\end{aligned}\tag{86}$$

Computing the component of expression (86) separately and utilizing the solution in expression (85), we deduce the final result as

$$\begin{aligned}&q^*(x,t)\\&=q_+^*(t)\Big(1-\sum_{n=1}^{N}\frac{c_n(t)}{\lambda_{+,n}}\,\mathrm{e}^{-\mathrm{i}(\lambda_{-,n}+\lambda_{+,n})x}M_{21}(x,t,z_n)\Big)\\&\quad+\frac{1}{2\pi\mathrm{i}}\int_{Z_-\cup Z_0}\frac{1}{\lambda_+(k)}\Big[M_{21}^-(x,t,k)\Big(\frac{q_+}{2}J_{0,21}(x,t,k)\\&\quad+\frac{(k-\lambda_+)}{2}J_{0,22}-(k-\lambda_+)\Big)-\Big(\frac{q_+^*}{2}J_{0,11}(x,t,k)\\&\quad+\frac{(q_+^*)^2}{2(k+\lambda_+(k))}J_{0,12}(x,t,k)\Big)M_{22}^-(x,t,k)\Big]\mathrm{d}k.\end{aligned}\tag{87}$$

5. Discussion and conclusion

In conclusion, the RH approach is employed to derive the solution of the defocusing LPD equation under fully asymmetric nonzero boundary conditions. For the analyticality of the upper and lower planes of the work, it can be concluded that the solutions of the Jost function are still analytical. With respect to the Jost function, we derive certain properties from the equation. The inverse scattering matrix is formulated to represent the solutions of the defocusing LPD equation, and it holds significance in various aspects, allowing us to gain a better understanding of the final solutions. Finally, we derive the representation formula (87) of $q(x,t)$, and use it to prove that it is the only solution in some appropriate function space. For subsequent research, the results obtained in this study are valuable for understanding physical phenomena and further developing applications of optical problems to other fields.

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