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Noise-induced phase transition in the Vicsek model through eigen microstate methodology

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This paper presents a comprehensive framework for analyzing phase transitions in collective models such as the Vicsek model under various noise types. The Vicsek model, focusing on understanding the collective behaviors of social animals, is known due to its discontinuous phase transitions under vector noise. However, its behavior under scalar noise remains less conclusive. Renowned for its efficacy in the analysis of complex systems under both equilibrium and non-equilibrium states, the eigen microstate method is employed here for a quantitative examination of the phase transitions in the Vicsek model under both vector and scalar noises. The study finds that the Vicsek model exhibits discontinuous phase transitions regardless of noise type. Furthermore, the dichotomy method is utilized to identify the critical points for these phase transitions. A significant finding is the observed increase in the critical point for discontinuous phase transitions with escalation of population density.

Keywords: Vicsek model, phase transitions, eigen microstate method, noise

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1. Introduction

Collective phenomena are ubiquitous in nature, referring to the gathering of living or non-living entities. Examples include clusters of bacteria,[1] schools of fish,[2,3] flocks of birds,[4,5] and even inanimate molecules and galaxies.[6,7] In recent years, researchers have continuously attempted to study swarm and collective behaviors, and its underlying mechanisms.[8–11] Inspired by these phenomena, drone cluster control has been extensively studied, which has significant applications in agriculture and the military. Swarming theory, particularly the phase transition from disorder to order, is vital for research of coordination control of large-scale drones.

So far, researchers have developed various models to simulate natural collective phenomena, such as the Boid model,[12] Vicsek model,[13] Cucker–Smale model,[14,15] and strictly metric-free (SMF) model.[16] The Boid model sets three rules for replicating natural clusters, i.e., velocity alignment, center approach, and collision avoidance.[12] In 1995, Vicsek *et al.* introduced the Vicsek model from a statistical mechanics perspective. This model features individuals moving under a two-dimensional periodic boundary condition and analyzes the phase transition among these self-driven particles.[13] Cucker and Smale developed the so-called Cucker–Smale model, introducing an attenuation index and demonstrating its relationship with model convergence.[14,15] In 2014, Pearce and Turner proposed the SMF three-dimensional model, which simplifies algorithms by focusing on boundary location and extreme neighbor identification instead of distance comparisons.[16]

The Vicsek model, known for its simplicity, classic nature, and wide application, has been extensively studied due to its phase transition from disorder to order in the presence of noise.[17–20] Phase transition was initially studied in chemical systems. For example, the mutual transformation between liquid water and solid water is the most classic example of phase transition. Later, this concept was adopted in mathematics and physics to describe system behaviors in two-phase transitions. Phase transition was usually classified into either discontinuous (first-order) or continuous (second-order) case. A transition is categorized based on whether the order parameters undergo a sudden change. Abrupt changes indicate a first-order transition, while the absence of such changes signifies a second-order transition.

A commonly used method to identify phase transition types is plotting the probability histogram of order parameters' occurrences.[21] Two peaks in the histogram indicate a first-order phase transition, while a single clear peak suggests a second-order phase transition. Another method is the Binder cumulant method, utilizing the fourth-order cumulant of the order parameter and its scatter plot against various parameters to determine the type of phase transition.[22] As a rule, a distinct minimum at the transition point indicates a discontinuous transition, while absence of such a minimum suggests a continuous transition. This method provides a nuanced approach for analyzing transitions, particularly useful in complex systems where traditional methods might not yield clear distinctions between different types of phase transitions.

The Vicsek model under vector noise exhibits a discontinuous phase transition and has been widely accepted.[23–25]

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However, the nature of the phase transition under scalar noise in the Vicsek model remains uncertain. In their seminal work, Chate and colleagues elucidated a discontinuous phase transition within the Vicsek model under the influence of scalar noise, employing the Binder cumulant method.[23] However, they utilized disparate order parameters to delineate particle velocity uniformity and density fluctuations. This divergence necessitates the adoption of an alternative methodology to provide a cohesive analytical framework for complex system phase transitions. The eigen microstate method emerges as a potential candidate to fulfill this requirement, promising a unified approach to comprehensively capture the multifaceted nature of such transitions.

The eigen microstate method, a tool for studying complex systems, has been effectively applied in fields like global weather forecasting and stock market predictions. Recently, this rapidly evolving method was used to study ferromagnetic phase transitions in the Ising model under equilibrium conditions.[26] Proving its versatility, the eigen microstate method unifies the study of critical behaviors in both equilibrium and non-equilibrium complex systems. Unlike the standard Vicsek model, which primarily considers particle direction uniformity, the eigen microstate method also accounts for particle density fluctuations. This approach provides a more comprehensive understanding of Vicsek model's internal changes under both vector and scalar noise. Li *et al.* implemented this methodology in their examination of the Vicsek model, extending their analysis to incorporate a variant encompassing leaders.[27] Their investigation revealed the occurrence of discontinuous density transitions alongside continuous velocity transitions in the presence of scalar noise. Notably, they assigned a particle speed of 0.5,[27] a deviation from the standard Vicsek model where speeds within the range of [0.003, 0.3] are typically employed, as these do not significantly influence simulation outcomes.[13] This adjustment prompts a reevaluation of the model's parameters to ensure the robustness and applicability of the findings.

Therefore, this paper builds on the standard Vicsek model, proposing a theoretical framework to address phase transitions in collective models. Utilizing the eigen microstate method, the study examines phase transitions in the Vicsek model under both scalar and vector noise. By analyzing the order parameter's response to these noises, it discerns whether the Vicsek model exhibits continuous or discontinuous phase transitions under different noise types. After confirming discontinuous transitions, the dichotomy method is used to identify critical noise values at varying densities, revealing the correlation between noise criticality and density. The study finds that the Vicsek model undergoes discontinuous phase transitions under both noise types, with critical noise values increasing alongside population density.

The structure of this paper is organized as follows. Section 2 introduces the standard Vicsek model, while Section 3 explains the eigen microstate method and phase transition. Sections 4 and 5 detail the experimental design, simulation results, and quantitative analysis of the phase transition in the Vicsek model under scalar and vector noise. The last section presents the conclusions and the future work.

2. The standard Vicsek model

2.1. Fundamental statement

Under a two-dimensional periodic boundary condition, particles initially have random velocities and positions. All particles maintain a constant speed v_0 and interact within a radius r. Interaction occurs only among neighbors. Neighbors are defined as the two particles whose straight-line distance is less than r. The movement direction of a particle at any given time t aligns with the average direction of all particles within its interaction radius.

The position of a particle at the next time step is determined by its current position, the constant speed v_0, and the calculated average direction of movement, i.e.,

$$x_i(t+\Delta t) = x_i(t) + v_0 \hat{v}_i(t)\Delta t, \quad i = 1, \ldots, N, \tag{1}$$

where $x_i(t)$ denotes the position of particle i at time t, and $v_i(t)$ denotes the velocity of particle i at time t. The symbol $\wedge$ denotes vector normalization. The time interval between two iterations is represented as Δt, setting to 1 in this model. According to the standard Vicsek model, speeds within [0.003, 0.3] do not affect simulation outcomes. Thus, the constant speed v_0 is set to 0.3 in this study.

2.2. Scalar noise

Under scalar noise, the velocity direction of particle i at the next time step in the standard Vicsek model is determined by a specific angular value:

$$\theta_i(t+\Delta t) = \langle \theta_i(t) \rangle_{\mathrm{r}} + \xi_i(t)\eta_{\mathrm{s}}, \tag{2}$$

where $\xi_i(t)$ represents a random value uniformly distributed between $[-1/2, 1/2]$, and η_{s} denotes the magnitude of scalar noise, with $\eta_{\mathrm{s}} \in [0, 2\pi]$. The term $\xi_i(t)\eta_{\mathrm{s}}$ indicates the angle by which particle i deviates at time t. The angle bracket $\langle\ \rangle$ signifies averaging, thus term $\langle \theta_i(t) \rangle_{\mathrm{r}}$ signifies the average value of velocity direction of all neighbors of particle i at time t. This formulation integrates the randomness introduced by scalar noise into the model's directional computation.

2.3. Vector noise

Under vector noise, the velocity direction of particle i at the next time step in the standard Vicsek model is represented by a specific unit vector. This vector incorporates the influence of vector noise, fundamentally different from scalar noise, in

determining the particle's movement direction. The exact formulation of this vector typically integrates both the average direction of the particle's neighbors and the random perturbation caused by the vector noise:

$$v_i(t+\Delta t) = \langle \hat{v}_i(t) \rangle_{\mathrm{r}}(1-\eta_{\mathrm{v}}) + \overline{\xi_i(t)}\eta_{\mathrm{v}}, \tag{3}$$

where η_{v} represents the intensity of the vector noise, with values ranging from 0 to 1 ($\eta_{\mathrm{v}} \in [0,1]$). The term $\overline{\xi_i(t)}$ is a normalized vector, randomly generated with a magnitude of 1. The term $\overline{\xi_i(t)}\eta_{\mathrm{v}}$ signifies the influence of vector noise on the velocity direction of particle i.

2.4. Order parameter

In the standard Vicsek model, the order parameter P is set to represent the degree of alignment of velocity directions of these self-driven particles within the system:

$$P = \Big| \frac{1}{N} \sum_{i=1}^{N} v_i(t) \Big|. \tag{4}$$

The order parameter P quantifies the collective motion's coherence, indicating how similarly the particles are moving in terms of direction.

Order parameter P is commonly used to gauge the level of uniformity or coherence in collective behaviors, $P \in [0,1]$. The higher the value of P, the greater the alignment of velocities, signifying a more ordered state. Conversely, a lower value of P reflects a more disordered state with less directional coherence among the particles. Therein, $P = 0$ signifies complete randomness and disorder in particle movements, and $P = 1$ indicates perfect alignment of all particle velocities.

3. Eigen microstate and phase transition

3.1. Microstate

In this system, each of the N particles is characterized by M microstates. M microstates mean M samples of the model state during the iteration process. This framework allows for a detailed analysis of the system's state and the dynamics of particle interactions over time.[28] The microstate of the entire system at time t can be exrepssed as

$$S(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_N(t) \end{bmatrix}, \tag{5}$$

where $s_i(t)$ denotes the microstate of particle i at time t. The definition of $s_i(t)$ can be described by

$$s_i(t) = \begin{bmatrix} \cos\theta_i(t) \\ \sin\theta_i(t) \\ \delta_{n_i}(t) \end{bmatrix}, \tag{6}$$

where $\theta_i(t)$ represents the angle corresponding to the velocity direction of particle i at time t. The term $\delta_{n_i}(t)$ reflects the distribution of particles around particle i at time t, whose definition is

$$\delta_{n_i}(t) = [n_i(t) - \overline{n}]/\overline{n}, \tag{7}$$

where $n_i(t)$ represents the local particle density around particle i at time t, and its expression is

$$n_i(t) = N_i(t)/(2r)^2, \tag{8}$$

where $N_i(t)$ denotes the number of particles within a square lattice centered around particle i at time t, with a side length of $2r$. Therein, $r = 1$ is the interaction radius, which has been defined in the standard Vicsek model. Meanwhile, $\overline{n}$ denotes the average distribution of all the particles at time t with the definition

$$\bar{n} = \frac{1}{MN} \sum_{t=1}^{M} \sum_{i=1}^{N} | \, n_i(t) \, |. \tag{9}$$

Parameters $\delta_{n_i}(t)$ can be mathematically formulated to quantify the local density around each specific particle i compared to the overall density of particles in the system. This comparison helps in understanding the spatial distribution of particles, which is crucial for analyzing collective behavior in the standard Vicsek model. A smaller $\delta_{n_i}(t)$ indicates that the distribution of particle i is more similar to the global particle distribution at time t. This can be mathematically represented to quantify the degree of similarity between a particle's local environment and the overall distribution of particles in the system.

3.2. Eigen microstate and phase emergence

In order to provide a measure related to the system's overall state, a statistical ensemble matrix $A_{3N\times M} = [a_{it}]$ is introduced, where $i \in [1,N]$, $t \in [1,M]$, and a_{it} is defined as

$$a_{it} = \frac{s_i(t)}{\sqrt{C_0}}, \tag{10}$$

where $C_0 = \sum_{t=1}^{M} \sum_{i=1}^{N} s_i^2(t)$. The correlations between microstates and states of particles can be expressed by

$$C = A^{\mathrm{T}} \cdot A, \tag{11}$$

$$K = A \cdot A^{\mathrm{T}}, \tag{12}$$

where C and K are matrices of dimensions $M \times M$ and $3N \times 3N$, respectively. Using the eigenvectors of C and K, two unitary matrices can be constructed as

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_M \end{bmatrix}, \tag{13}$$

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_{3N} \end{bmatrix}, \tag{14}$$

where V and U are matrices formed by arranging the eigenvectors of matrices C and K, respectively, in the order corresponding to their eigenvalues sorted from the largest to the smallest.

According to the singular value decomposition (SVD),[29] the ensemble matrix A is factorized as

$$A = U \cdot \Sigma \cdot V^{\mathrm{T}}, \tag{15}$$

where Σ is an $3N \times M$ diagonal matrix with elements

$$\Sigma_{IJ} = \begin{cases} \sigma_I, & I = J \leq Y, \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

where $Y = \min(M, 3N)$ and $\sum_{I=1}^{Y} \sigma_I^2 = 1$, with $\sigma_1^2 > \sigma_2^2 > \cdots > \sigma_Y^2$. Here, $\sigma_I \in [-1, 1]$ represents the probability amplitude of the eigen microstate u_I, while $W_I^{\mathrm{E}} = \sigma_I^2$ indicates the probability of the eigen microstate u_I. At the limits $M \to \infty$ and $N \to \infty$, if all probability amplitudes σ_I approach 0, then all W_I^{E}'s also tend towards 0, indicating a completely disordered state. If there is a nonzero σ_I, it signifies an emergent phase described by the eigen microstate u_I. More than one emergent phase can exist in a system.

3.3. Finite-size scaling and phase transition

It has been proven that thermodynamic functions of finite system are proposed to satisfy finite-size scaling relations in the region near a critical point.[30–32] Suppose that there is a phase transition with order-parameter described by σ_I and new phase characterized by u_I. The external conditions (such as scalar noise or vector noise) is characterized by a parameter η, one gets $\eta = \eta_{\mathrm{c}}$ at the phase transition point. Let $h = (\eta - \eta_{\mathrm{c}})/\eta_{\mathrm{c}}$ denote the distance from the phase transition point. Therefore, a finite-size scaling form of σ_I in the asymptotic region with $h \ll 1$ is proposed as[33]

$$\sigma_I(\eta, L) = L^{(-\beta/\nu)} f_I(hL^{1/\nu}), \tag{17}$$

where L is the system's linear length under periodic boundary conditions, β represents the critical exponent of the order parameter W_I^{E}, ν denotes the critical exponent of correlation length, and f_I is the finite-size scaling function expressed as

$$f_I(x) = \frac{2}{x+1} \cdot \frac{\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}}, \tag{18}$$

where x is the scaling variable.

For $\beta = 0$, there is a jump from $\sigma_I(\eta, \infty) = 0$ at $\eta > \eta_{\mathrm{c}}$ to $\sigma_I(\eta_{\mathrm{c}}, L) = f_I(0) \neq 0$. This indicate a discontinuous phase transition.

When $\beta > 0$, we have $\sigma_I(\eta, \infty) = 0$ at $\eta > \eta_{\mathrm{c}}$ and $\sigma_I(\eta, \infty) = 0$ at $\eta < \eta_{\mathrm{c}}$. Meanwhile, one gets $\sigma_I(\eta_{\mathrm{c}}, \infty) = 0$. This is a continuous phase transition.

In order to identify the phase transition point and type of phase transition, we investigate the size dependence of $W_I^{\mathrm{E}}(\eta, L)$. Equation (17) can be rewritten as[27]

$$\ln W_I^{\mathrm{E}}(\eta, L) = -(2\beta/\nu)\ln L + 2\ln f_I(hL^{1/\nu}), \tag{19}$$

where there appears to be a linear dependence of $\ln W_I^{\mathrm{E}}$ on $\ln L$ at $h = 0$. This can be used to determine the transition point and critical exponent ratio β/ν at the same time.

4. Phase transition in the standard Vicsek model with scalar noise

This section investigates the phase transition in the two-dimensional standard Vicsek model under scalar noise. There are N particles labeled as $1, 2, \ldots, N$ and placed randomly on a two-dimensional domain with size L under periodic boundary conditions. They move synchronously at discrete time steps by a fixed distance $v_0 \Delta t$, where $\Delta t = 1$.

To study the relationship between the critical noise value and phase transition under different densities $\rho = N/L^2$, analyses are conducted for densities $\rho = 0.5$, $\rho = 1$, and $\rho = 2$. Experiments are performed with boundary lengths $L = 2^2, 2^3, 2^4, 2^5$, resulting in $N = 2^3, 2^5, 2^7, 2^9$ for $\rho = 0.5$, $N = 2^4, 2^6, 2^8, 2^{10}$ for $\rho = 1$, and $N = 2^5, 2^7, 2^9, 2^{11}$ for $\rho = 2$.

Each experiment iterates 510000 steps for the sake of enhancing the simulation accuracy in the standard Vicsek model. The first 10000 iterations of each experiment are disregarded to ensure the model reaches a stable state. Sampling is then conducted on the remaining 500000 iterations. Due to high similarity between samples in consecutive time steps, sampling occurs every 50 iterations, totaling 10000 samples ($M = 10000$). Each data point is averaged over 50 experimental runs to minimize error. The settings of relevant parameters in the experiment are listed in Table 1.

Table 1. Simulation experiment parameter settings.

v_0	ρ	r	Δt	L	Simulation times	M
0.3	0.5,1,2	1	1	$2^2, 2^3, 2^4, 2^5$	510000	10000

4.1. Order parameter P

First of all, we investigate the relationship of the order parameter P to scalar noise η_{s} and population size N at different densities. Figure 1 reveals that under scalar noise, the system's order parameter P increases as scalar noise decreases, indicating a transition from disorder to order. The consistency in velocity direction decreases with increasing population size, aligning with the Vicsek model's original findings.[13] The parameter P quantifies the uniformity of particle velocity, yet it omits any assessment of the distribution or density fluctuations inherent within the colony. Nonetheless, as delineated by Chate *et al.*, the emergence of a discontinuous phase transition in the Vicsek model, induced by scalar noise, is fundamentally ascribed to these density fluctuations.[23] Consequently, the eigen microstate method is employed as a quantitative analytical instrument to investigate the system's phase transition. This approach holistically considers both the average velocity and particle density fluctuation post-stabilization, thereby providing a more comprehensive understanding of the dynamic behavior under examination.

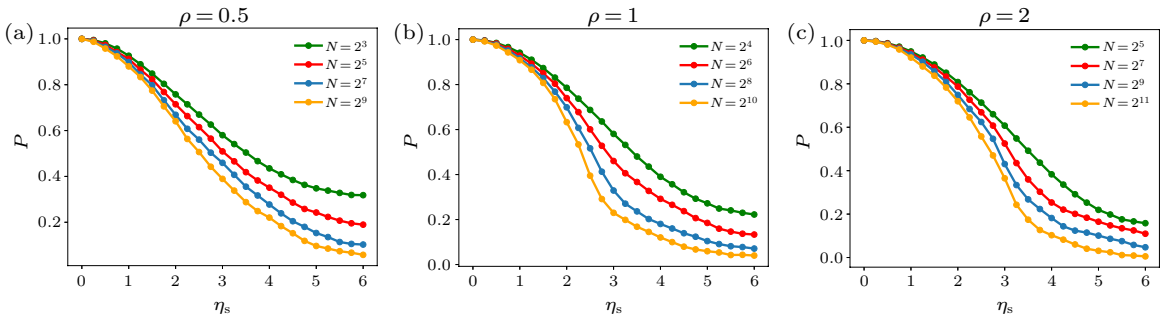


Fig. 1. Order parameter P versus scalar noise η_s and population size N in the standard Vicsek model: (a) $\rho = 0.5$, (b) $\rho = 1$, (c) $\rho = 2$.

4.2. Eigen microstates and phase transition

In order to make a thorough investigation on the type of phase transition in the standard Vicsek model, this section analyzes the available eigen microstates of the system under scalar noise and further elucidate how the criticality of these phase transitions is influenced by changes in the density of the system.

Under scalar noise, the velocity $v_i(t+1)$ of a particle in the next time step can be described by Eq. (2). Figure 2 illustrates the probabilities of eigen microstates in the standard Vicsek model with $\rho = 1$, $N = 256$, and $L = 2^4$. These microstates satisfy $W_1^{\mathrm{E}} > W_2^{\mathrm{E}} > W_3^{\mathrm{E}} > W_4^{\mathrm{E}} > \cdots > W_{\mathrm{r}}^{\mathrm{E}}$. Since W_4^{E} approaches zero as $M \to \infty$, Fig. 2 indicates the existence of three phase transitions (corresponding to W_1^{E}, W_2^{E}, and W_3^{E}). Under strong noises, no eigen microstate is dominant and the system is in disorder. With the decrease of η_s, σ_1 becomes finite and a phase u_1 emerges at first. Further, two degenerate eigen microstates u_2 and u_3 appear simultaneously.

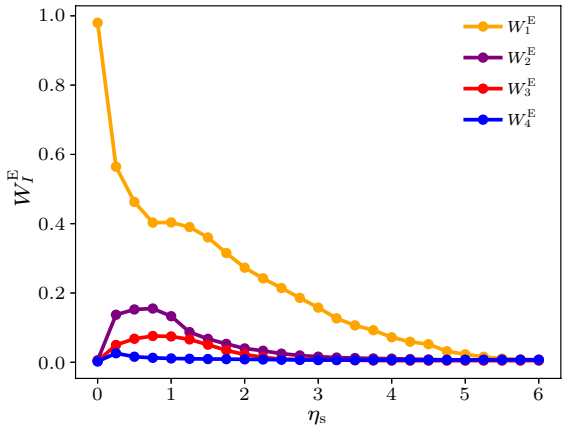


Fig. 2. The top four eigen microstates W_1^{E}, W_2^{E}, W_3^{E}, and W_4^{E} under scalar noise when $\rho = 1$ and $N = 256$ ($L = 2^4$).

Figure 3(a) presents the maximum eigen microstate W_1^{E} with scalar noise under different sizes when $\rho = 1$. Under smaller noises, W_1^{E} increases with the growing population size. However, W_1^{E} decreases as the population size increases when noises are large enough. Therefore, W_1^{E} has a jump at a certain noise threshold, called phase transition point. The existence of the critical point indicates a discontinuous phase transition of u_1. In order to calculate the precise value of this critical noise threshold, a dichotomy method is employed. According to the data in Fig. 3(a), the critical threshold η_{sc} lies within [3, 3.25]. First, a midpoint value $\eta_s = 3.13$ is selected, where $\beta > 0$ under this scalar noise level. Then, continuous interpolation and repeated experiments were conducted, until the noise value making $\beta = 0$ is found, where $\eta_s = \eta_{\mathrm{sc}}$.

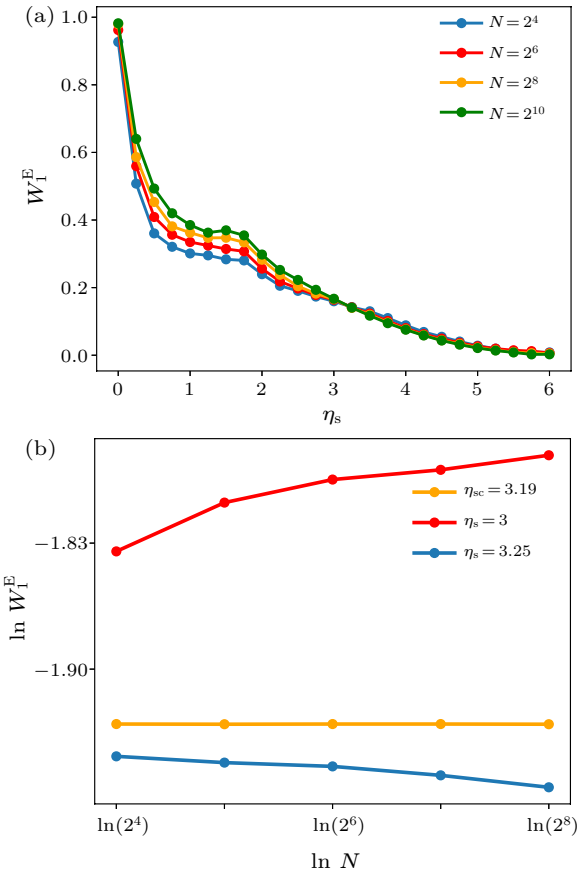


Fig. 3. Phase transition of eigen microstate W_1^{E} in the standard Vicsek model under scalar noise when $\rho = 1$: (a) W_1^{E} versus scalar noise η_s under different population sizes, (b) a logarithmic plot of W_1^{E} against N to highlight the phase transition point $\eta_{\mathrm{sc}} = 3.19$.

Table 2 summarizes the results of all the experiments conducted to approximate the critical threshold of scalar noise η_s using the dichotomy method in the standard Vicsek model when $\rho = 1$. This methodical approach allows for a precise and fast determination of the critical point where the sys-

tem's behavior changes significantly, providing valuable insights into the dynamics of the model under specified conditions. Figure 3(b) is constructed by these data in Table 2, where each line's slope corresponds to $-2\beta/\nu$ as defined in Eq. (19). Since $\ln L = \ln\sqrt{N/\rho} = \frac{1}{2}(\ln N - \ln\rho)$, choosing N as the abscissa does not affect the experimental outcomes. A slope tending towards zero signifies a discontinuous phase transition at this critical point. According to Fig. 3(b), β equals 0 when $\eta_s = 3.19$. Consequently, under scalar noise, the standard Vicsek model exhibits a discontinuous phase transition, occurring at the critical value $\eta_{sc} = 3.19$ when $\rho = 1$.

Table 2. The relationship between W_1^E and N facilitates the determination of phase transition point η_{sc} in the standard Vicsek model under scalar noise η_s when $\rho = 1$.

W_1^E	$N = 16$	$N = 64$	$N = 256$	$N = 1024$
$\eta_s = 3$	0.159670	0.164071	0.166181	0.167072
$\eta_s = 3.13$	0.150691	0.155956	0.158388	0.162686
$\eta_s = 3.16$	0.148189	0.149111	0.150075	0.159130
$\eta_s = 3.19$	0.145073	0.145058	0.145074	0.145077
$\eta_s = 3.25$	0.142497	0.141703	0.140067	0.139782

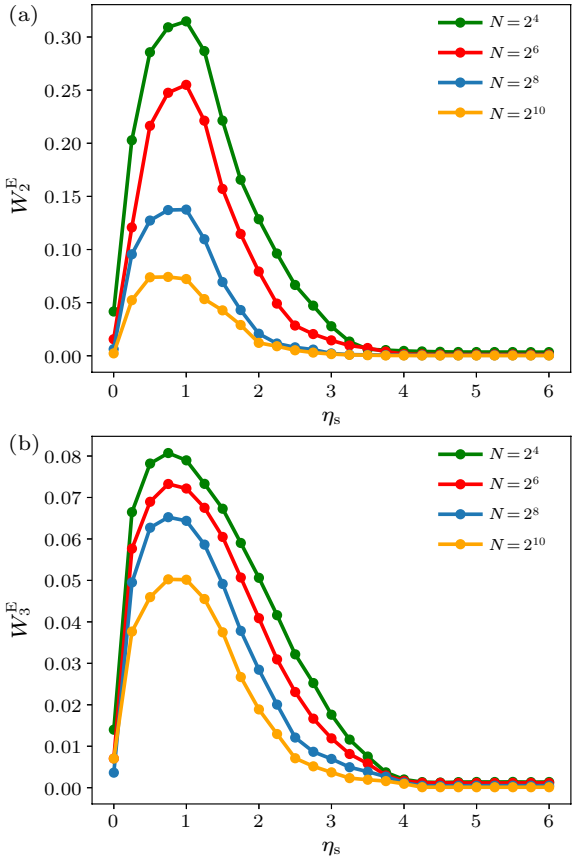


Fig. 4. Phase transition of eigen microstates W_2^E and W_3^E in the standard Vicsek model under scalar noise when $\rho = 1$: (a) W_2^E versus scalar noise η_s under different population sizes, (b) W_3^E versus scalar noise η_s under different population sizes.

Figure 4 provides enhanced insights into the analysis of phase transition in the standard Vicsek model subject to scalar noise at a density ρ of 1, utilizing eigen microstates W_2^E and W_3^E. As illustrated in Fig. 4, both W_2^E and W_3^E decrease as the population size increases. This indicates that for W_2^E and W_3^E, there is no critical noise strength η_s that makes β [as described in Eq. (19)] equal to zero. Therefore, W_2^E and W_3^E exhibit continuous phase transitions.

Compared to the previous scenarios, the phase transition in the standard Vicsek model at a lower population density ($\rho = 0.5$) and a higher population density ($\rho = 2$) also needs to be discussed further, for the sake of estimate of the criticality of the system's density to its phase transitions.

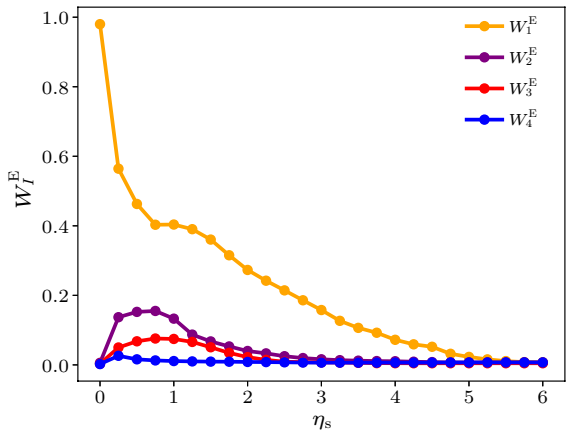


Fig. 5. Top four eigen microstates W_1^E, W_2^E, W_3^E, and W_4^E under scalar noise when $\rho = 0.5$ and $N = 128$ ($L = 2^4$).

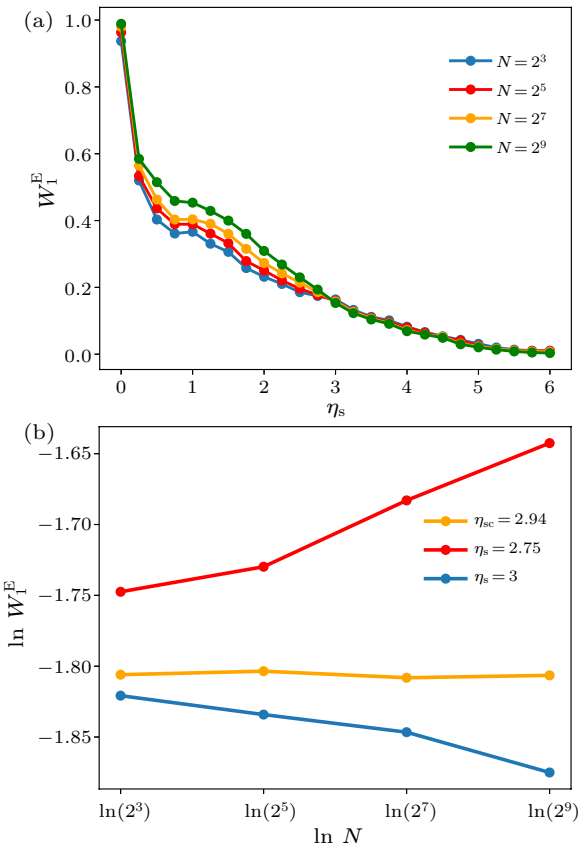


Fig. 6. The phase transition in the standard Vicsek model under scalar noise when $\rho = 0.5$: (a) W_1^E versus scalar noise η_s under different population sizes, (b) a logarithmic plot of W_1^E against N to highlight the phase transition point $\eta_{sc} = 2.94$.

Figure 5 shows the values of the first four largest eigen microstates in the case of $\rho = 0.5$ with $N = 128$ ($L = 2^4$). Similar to the case of $\rho = 1$, Fig. 5 indicates the existence of three phase transitions (corresponding to W_1^{E}, W_2^{E}, and W_3^{E}).

Figure 6(a) presents the largest eigenvalue W_1^{E} for different population sizes at $\rho = 0.5$. According to the data in Fig. 6(a), the critical value η_{sc} for discontinuous phase transition lies within the range [2.75, 3]. Table 3 displays the results of all the experiments conducted to approximate the critical threshold of scalar noise η_{s} using the dichotomy method in the standard Vicsek model when $\rho = 0.5$. According to Fig. 6(b), $\beta = 0$ when $\eta_{\mathrm{s}} = 2.94$. Consequently, under scalar noise, the standard Vicsek model exhibits a discontinuous phase transition, occurring at the critical value $\eta_{\mathrm{sc}} = 2.94$ when $\rho = 0.5$.

Table 3. Relationship between W_1^{E} and N, facilitating the determination of phase transition point η_{sc} in the standard Vicsek model under scalar noise η_{s} when $\rho = 0.5$.

W_1^E	$N = 8$	$N = 32$	$N = 128$	$N = 512$
$\eta_{\mathrm{s}} = 2.75$	0.174215	0.177319	0.185817	0.193485
$\eta_{\mathrm{s}} = 2.88$	0.170624	0.174652	0.180366	0.185619
$\eta_{\mathrm{s}} = 2.92$	0.166239	0.168417	0.17037	0.174624
$\eta_{\mathrm{s}} = 2.94$	0.16431	0.164717	0.163956	0.164235
$\eta_{\mathrm{s}} = 3$	0.163896	0.159748	0.157777	0.15335

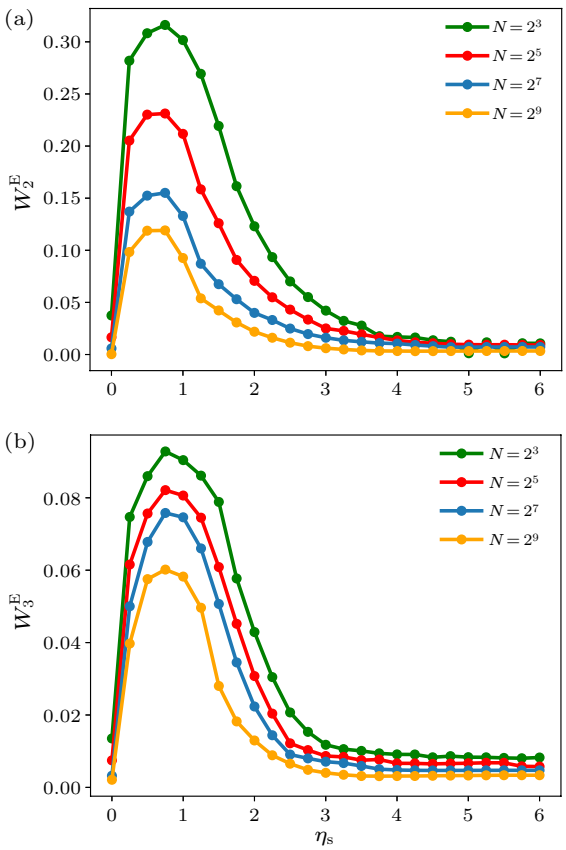


Fig. 7. Variations in the eigenvalues W_2^{E} and W_3^{E} under scalar noise conditions, at a density of $\rho = 0.5$, across different population sizes: (a) W_2^{E} versus scalar noise η_{s} under different population sizes, (b) W_3^{E} versus scalar noise η_{s} under different population sizes.

Figure 7 shows the relationship of eigen microstates W_2^{E} and W_3^{E} versus varying scalar noises when system's density is $\rho = 0.5$. Similar to the situation with a density of $\rho = 1$, there is no critical value for W_2^{E} and W_3^{E} that makes β equal to zero. Therefore, it can be deduced that the type of phase transition reflected by these order parameters is continuous.

Figure 8 shows the values of the first four largest eigen microstates in the case of $\rho = 2$ with $N = 512$ ($L = 2^4$). Similarly, Fig. 8 indicates the existence of three phase transitions (corresponding to W_1^{E}, W_2^{E}, and W_3^{E}).

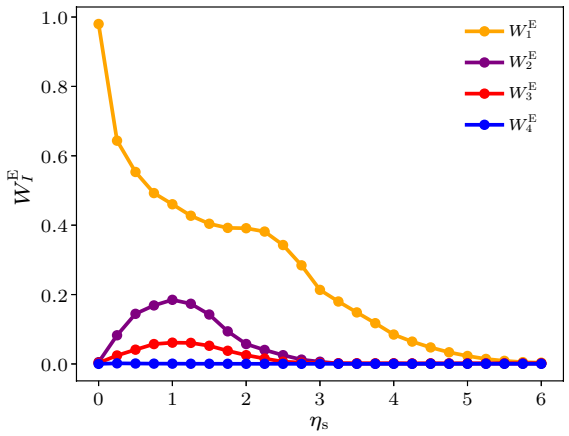


Fig. 8. Top four eigen microstates W_1^{E}, W_2^{E}, W_3^{E}, and W_4^{E} under scalar noise when $\rho = 2$ and $N = 512$ ($L = 2^4$).

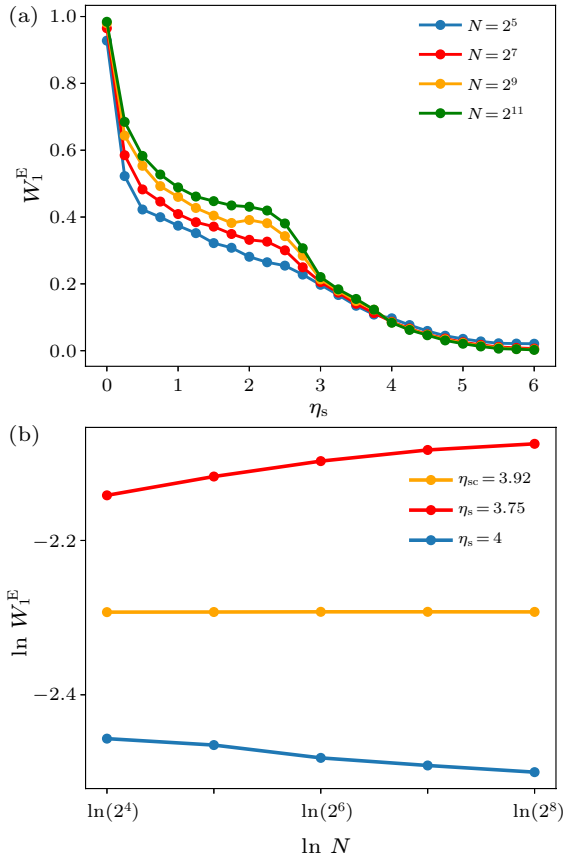


Fig. 9. Phase transition in the standard Vicsek model under scalar noise when $\rho = 2$: (a) W_1^{E} versus scalar noise η_{s} under different population sizes, (b) a logarithmic plot of W_1^{E} against N to highlight the phase transition point $\eta_{\mathrm{sc}} = 3.92$.

Table 4. The relationship between W_1^{E} and N facilitates the determination of phase transition point η_{sc} in the standard Vicsek model under scalar noise η_{s} when $\rho = 2$.

W_1^{E}	$N = 32$	$N = 128$	$N = 512$	$N = 2048$
$\eta_{\mathrm{s}} = 3.75$	0.104478	0.112505	0.117468	0.122805
$\eta_{\mathrm{s}} = 3.88$	0.102546	0.107428	0.112818	0.117605
$\eta_{\mathrm{s}} = 3.92$	0.100984	0.100935	0.100972	0.101016
$\eta_{\mathrm{s}} = 3.94$	0.098310	0.092231	0.088505	0.086762
$\eta_{\mathrm{s}} = 4$	0.094988	0.089557	0.084720	0.086762

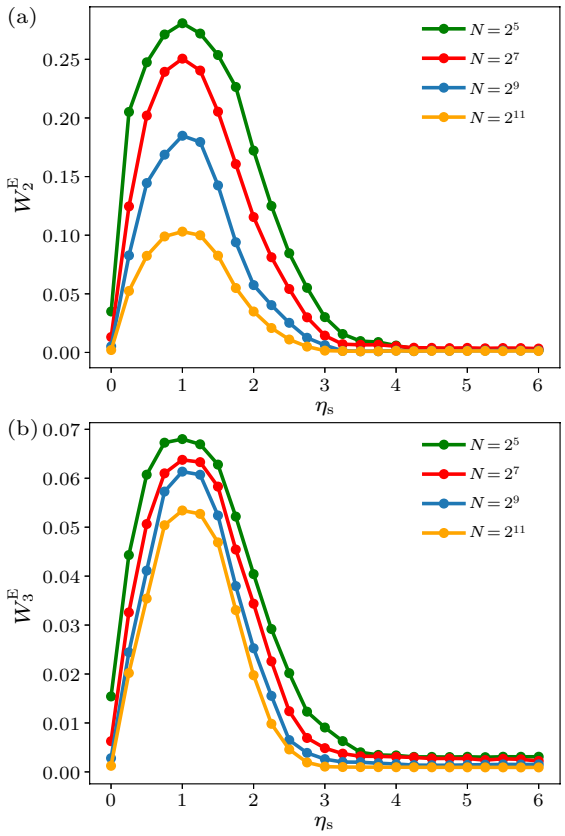


Fig. 10. Variations in the eigenvalues W_2^{E} and W_3^{E} under scalar noise conditions, at a density of $\rho = 2$, across different population sizes: (a) W_2^{E} versus scalar noise η_{s} under different population sizes, (b) W_3^{E} versus scalar noise η_{s} under different population sizes.

Figure 9 shows the simulation results when $\rho = 2$. Figure 9 and Table 4 give the details of how to find the critical value $\eta_{\mathrm{sc}} = 3.92$ for discontinuous phase transition lies within the range [3.75, 4] when $\rho = 2$. According to Fig. 10, there is no critical value for W_2^{E} and W_3^{E} that makes $\beta = 0$ when $\rho = 2$, meaning that the type of phase transition reflected by these order parameters is continuous.

It has been established that $\eta_{\mathrm{sc}} = 2.94$ at $\rho = 1$, $\eta_{\mathrm{sc}} = 3.19$ at $\rho = 1$, and $\eta_{\mathrm{sc}} = 3.92$ at $\rho = 2$. Therefore, to some extent, it can be inferred that in the Vicsek model under scalar noise, a discontinuous phase transition occurs, and the critical noise value for this transition increases with the increase in population density.

In summary, under scalar noise, the Vicsek model exhibits a first-order discontinuous phase transition, with the critical noise value for this transition increasing as the cluster density increases. Additionally, the Vicsek model also undergoes two continuous phase transitions.

5. Phase transition in the standard Vicsek model with vector noise

The phase transition problem in the standard Vicsek model based on vector noise involves studying the changes in the collective behavior of particles within the model under the influence of vector noise. This model is a key area of research in understanding collective dynamics and phase transitions in systems influenced by directional noise, as opposed to scalar noise. The standard Vicsek model with vector noise explores how the alignment of particles and the resultant order in the system are affected by the directional randomness introduced by the noise.

Under vector noise, the velocity $v_i(t+1)$ of a particle in the next time step can be described by Eq. (3). The same simulation parameters are employed as used in the scalar noise case, including the value of velocity $v_0 = 0.3$, density ρ, periodic boundary length L, cluster size N, number of microstates M, and interaction radius r. This consistent approach allows for a direct comparison between the effects of scalar and vector noise on the system's dynamics, providing valuable insights into the influence of different types of noise on the phase transition in the standard Vicsek model. The parameter settings of this experiment are the same as those of the scalar noise experiment, and the specific data is shown in Table 1.

5.1. Order parameter *P*

Firstly, the impact of vector noise on the collective behavior of the system at different densities and population scales is elucidated by using order parameter P. Figure 11 depicts the variation of the order parameter P with the vector noise under different densities. Figure 11 shows that the order parameter gradually decreases as noise increases and also decreases with an increase in population size. According to Eq. (19), there is no noise value that makes $\beta = 0$, indicating the absence of a discontinuous phase transition point. Therefore, the eigen microstate method is utilized to study the phase transition in the standard Vicsek model under vector noise further.

5.2. Eigen microstates and phase transition

Then, the phase transition in the standard Vicsek model under vector noise at a density of $\rho = 1$ is investigated. Figure 12 illustrates the changes of the first four largest eigen microstates with varying vector noise when $N = 256$, corresponding to $L = 2^4$. This figure helps in understanding the influence of vector noise on the standard Vicsek model's dynamics, particularly how these vector noises affect system's eigen microstates at a given population size.

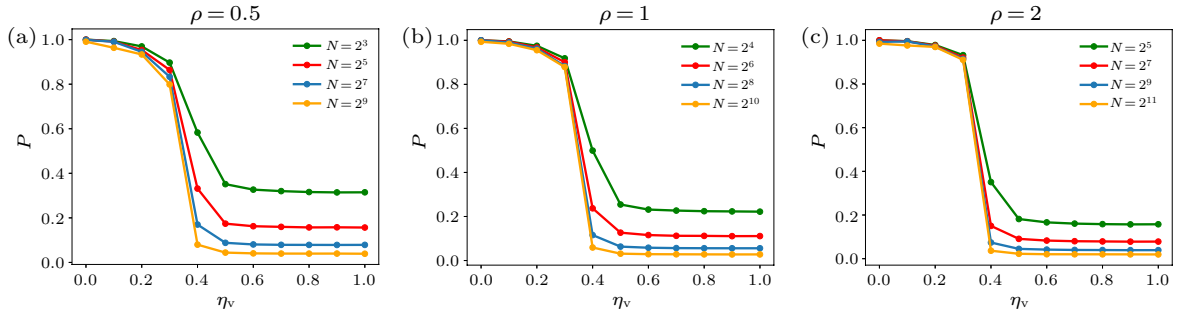


Fig. 11. Order parameter P versus vector noise η_v and population size N in the standard Vicsek model: (a) $\rho = 0.5$, (b) $\rho = 1$, (c) $\rho = 2$.

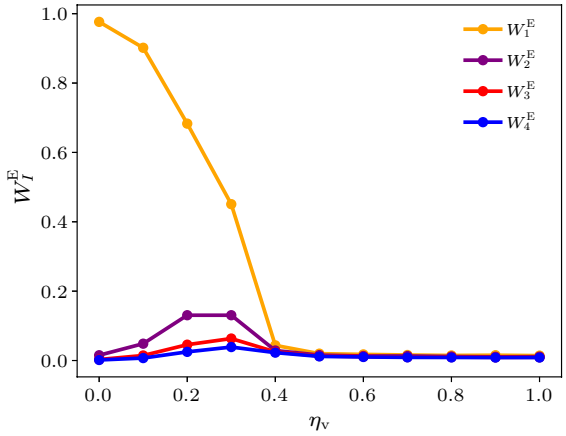


Fig. 12. The top four eigen microstates W_1^E, W_2^E, W_3^E, W_4^E under vector noise when $\rho = 1$ and $N = 256$ ($L = 2^4$).

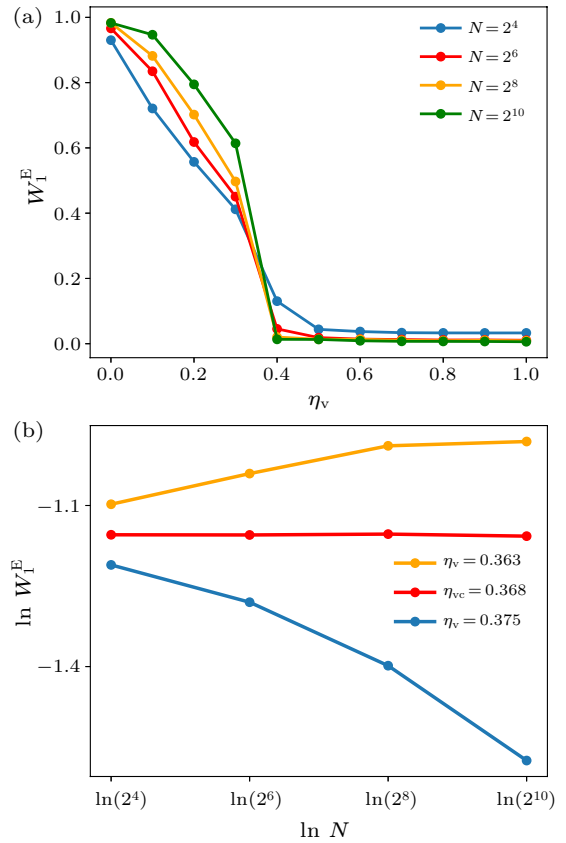


Fig. 13. Phase transition of eigen microstate W_1^E in the standard Vicsek model under vector noise when $\rho = 1$: (a) W_1^E versus vector noise η_v under different population sizes, (b) a logarithmic plot of W_1^E against N to highlight the phase transition point $\eta_{vc} = 0.368$.

Similarly to scalar noise, Fig. 13 indicates that W_1^E increases with population size when noise is below the critical level and decreases when noise is above it. Thus, a discontinuous phase transition exists in the standard Vicsek model under vector noise η_v and its critical value η_{vc} locates in the range [0.3, 0.4] according to Fig. 13(a). Table 5 presents results of all experiments conducted in the standard Vicsek model under scalar noise at $\rho = 1$, using the dichotomy method to approximate the critical noise value. Figure 13(b) shows that the critical value of a discontinuous phase transition in the standard Vicsek model under vector noise at $\rho = 1$ occurs when $\eta_{vc} = 0.368$, since β equals zero.

Table 5. Relationship between W_1^E and N, facilitating the determination of phase transition point η_{vc} in the standard Vicsek model under vector noise η_v when $\rho = 1$.

W_1^E	$N = 16$	$N = 64$	$N = 256$	$N = 1024$
$\eta_v = 0.3$	0.411792	0.450688	0.497063	0.614026
$\eta_v = 0.35$	0.352455	0.389213	0.395615	0.401254
$\eta_v = 0.363$	0.333647	0.353257	0.372089	0.375312
$\eta_v = 0.366$	0.315627	0.326455	0.330124	0.337451
$\eta_v = 0.368$	0.315355	0.315087	0.315398	0.315263
$\eta_v = 0.369$	0.315899	0.313462	0.313402	0.312401
$\eta = 0.375$	0.298281	0.278682	0.247113	0.207225
$\eta_v = 0.4$	0.130397	0.045455	0.020494	0.013397

Under the same vector noise conditions, Fig. 14 shows the relationship of eigen microstates W_2^E and W_3^E in the standard Vicsek model versus varying vector noises respectively when system's density is $\rho = 1$. It is clearly seen that the values of W_2^E and W_3^E decrease with increase in population size. There is no specific noise value at which β equals zero. Therefore, there is no discontinuous phase transition for W_2^E and W_3^E in the standard Vicsek model under vector noise.

Figures 15–17 display the eigenvalues of the standard Vicsek model under vector noise at a density of $\rho = 0.5$. Similar to the case of $\rho = 1$, Fig. 15 indicates the existence of three phase transitions (corresponding to W_1^E, W_2^E, and W_3^E). Figure 16(a) determines the critical value η_{vc} for discontinuous phase transition be within the range [0.3, 0.4]. The dichotomy method was employed to approximate this critical value, and the results of all such experiments are summarized in Table 6.

Figure 16(b) indicates that the critical value for a discontinuous phase transition under this noise condition is $\eta_{vc} = 0.341$.

Table 6. Relationship between W_1^E and N, facilitating the determination of phase transition point η_{vc} in the standard Vicsek model under vector noise η_v when $\rho = 0.5$.

W_1^E	$N = 8$	$N = 32$	$N = 128$	$N = 512$
$\eta_v = 0.3$	0.393672	0.405752	0.450769	0.496592
$\eta_v = 0.325$	0.361588	0.371658	0.392615	0.406514
$\eta_v = 0.338$	0.312815	0.321782	0.341245	0.351546
$\eta_v = 0.341$	0.301256	0.303489	0.300154	0.302517
$\eta_v = 0.344$	0.281695	0.256178	0.226148	0.203157
$\eta_v = 0.35$	0.261539	0.226158	0.20136	0.189563
$\eta_v = 0.4$	0.177292	0.078652	0.043691	0.029656

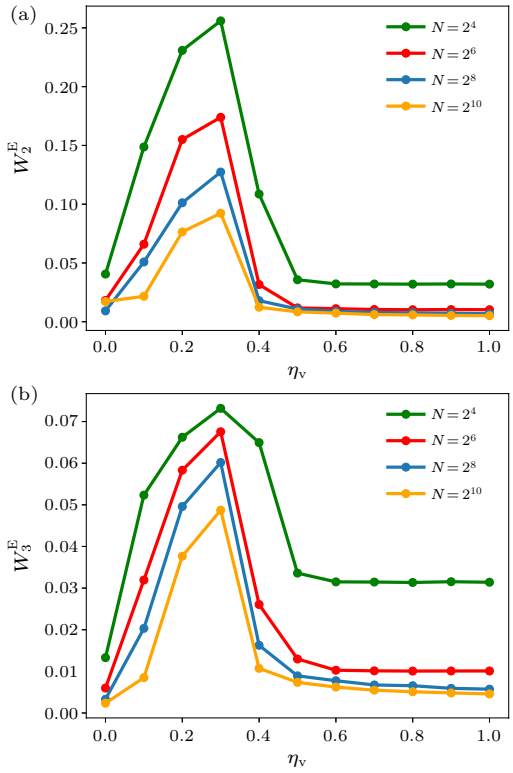


Fig. 14. Phase transition of eigen microstates W_2^E and W_3^E in the standard Vicsek model under vector noise when $\rho = 1$: (a) W_2^E versus vector noise η_v under different population sizes, (b) W_3^E versus vector noise η_v under different population sizes.

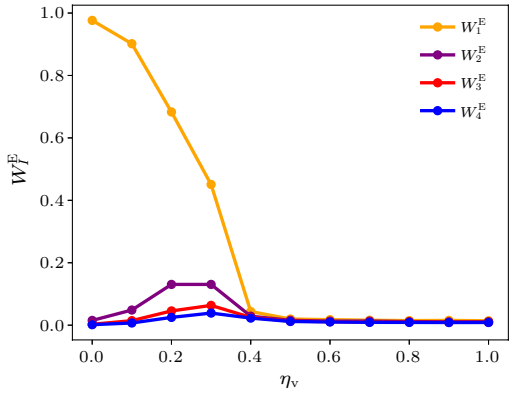


Fig. 15. Top four eigen microstates W_1^E, W_2^E, W_3^E, and W_4^E under vector noise when $\rho = 0.5$ and $N = 128$ ($L = 2^4$).

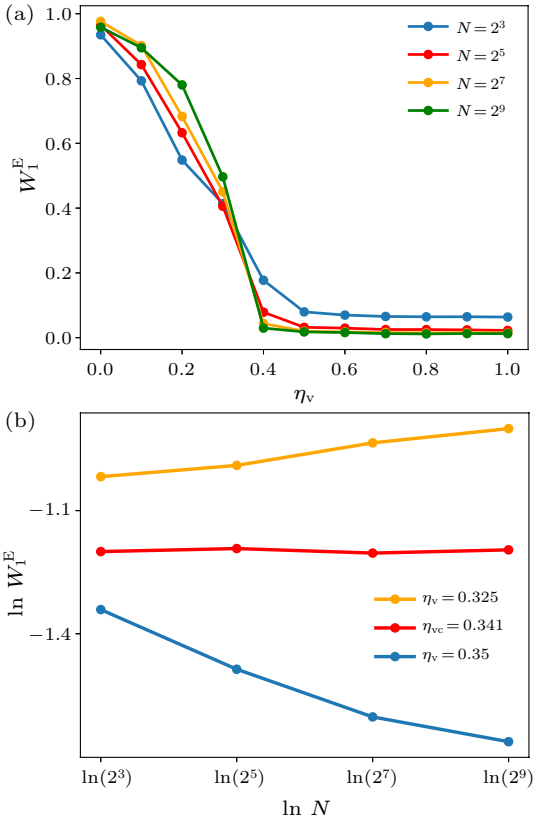


Fig. 16. Phase transition of eigen microstate W_1^E in the standard Vicsek model under vector noise when $\rho = 0.5$: (a) W_1^E versus vector noise η_v under different population sizes, (b) a logarithmic plot of W_1^E against N to highlight the phase transition point $\eta_{vc} = 0.341$.

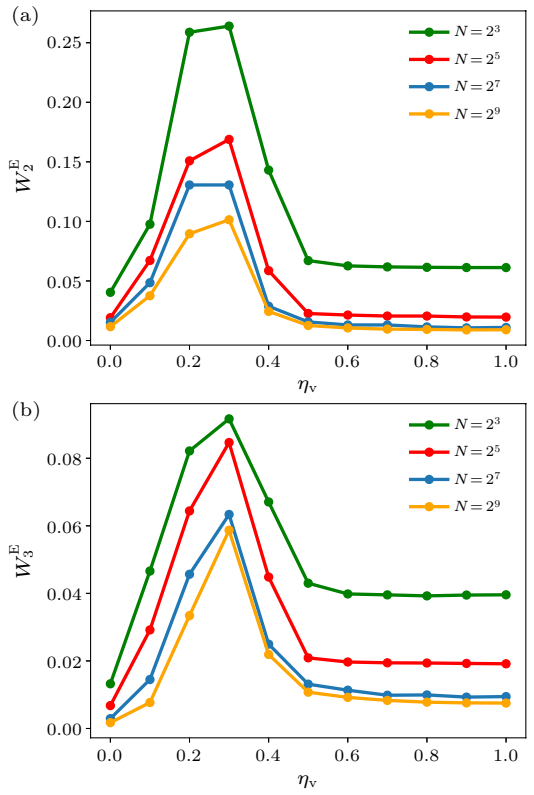


Fig. 17. Phase transition of eigen microstates W_2^E and W_3^E in the standard Vicsek model under vector noise when $\rho = 0.5$: (a) W_2^E versus vector noise η_v under different population sizes, (b) W_3^E versus vector noise η_v under different population sizes.

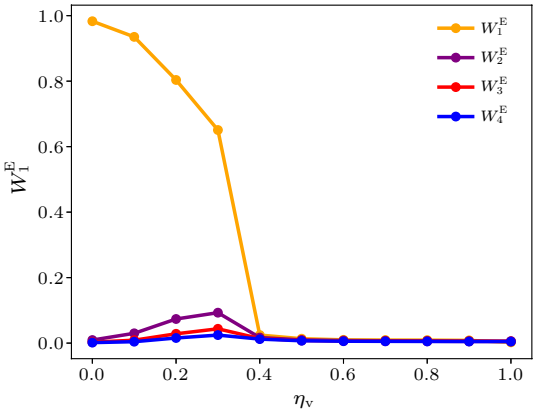


Fig. 18. Top four eigen microstates W_1^E, W_2^E, W_3^E, W_4^E under vector noise when $\rho = 2$ and $N = 512$ ($L = 2^4$).

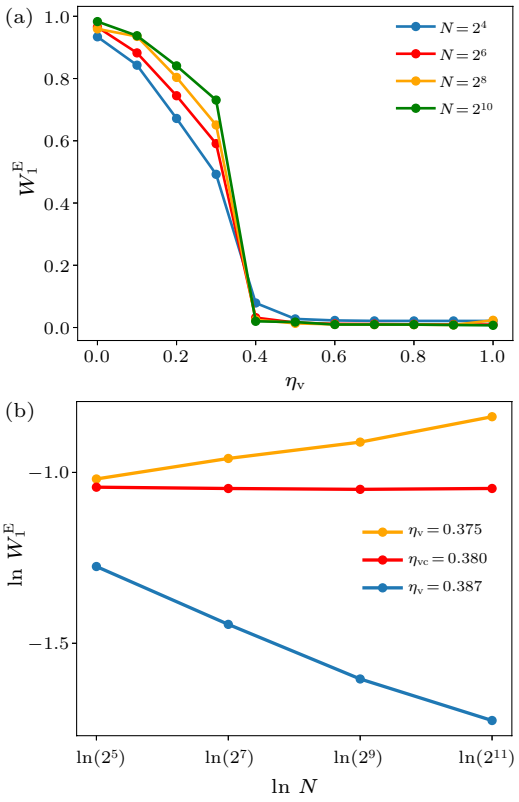


Fig. 19. Phase transition of eigen microstate W_1^E in the standard Vicsek model under vector noise when $\rho = 2$: (a) W_1^E versus vector noise η_v under different population sizes, (b) a logarithmic plot of W_1^E against N to highlight the phase transition point $\eta_{vc} = 0.380$.

Figure 17 shows the relationship of eigen microstates W_2^E and W_3^E versus varying vector noises when system's density is $\rho = 0.5$. Similar to the situation with a density of $\rho = 1$, there is no critical value for W_2^E and W_3^E that makes β equal to zero. Therefore, it can be deduced that the type of phase transition reflected by these order parameters is continuous.

Figures 18–20 display the eigenvalues of the standard Vicsek model under vector noise when $\rho = 2$. Similar to the case of $\rho = 1$, Fig. 18 indicates the existence of three phase transitions (corresponding to W_1^E, W_2^E, and W_3^E), while Fig. 19 and Table 7 give the details of how to find out the critical value $\eta_{vc} = 0.380$ for discontinuous phase transition to be within the range [0.3, 0.4]. According to Fig. 20, there is no critical value for W_2^E and W_3^E that makes β equal to zero. Therefore, the type of phase transition reflected by these order parameters is continuous.

The conclusion drawn is that in the Vicsek model under vector noise, the critical noise value for a discontinuous phase transition increases with the density, following the same pattern observed in the model under scalar noise. This consistency in behavior across different noise types underscores the robustness of the model's dynamics in response to varying conditions.

Table 7. Relationship between W_1^E and N, facilitating the determination of phase transition point η_{vc} in the standard Vicsek model under vector noise η_v when $\rho = 2$.

W_1^E	$N = 32$	$N = 128$	$N = 512$	$N = 2048$
$\eta_v = 0.3$	0.492207	0.590839	0.650769	0.730814
$\eta_v = 0.35$	0.401719	0.441828	0.491274	0.564128
$\eta_v = 0.375$	0.360898	0.383215	0.401989	0.433071
$\eta_v = 0.378$	0.356142	0.358124	0.365133	0.371324
$\eta_v = 0.38$	0.352320	0.350966	0.350122	0.354214
$\eta_v = 0.382$	0.331548	0.326416	0.321568	0.319532
$\eta = 0.387$	0.279331	0.235820	0.201327	0.178178
$\eta_v = 0.4$	0.078226	0.031785	0.024570	0.019949

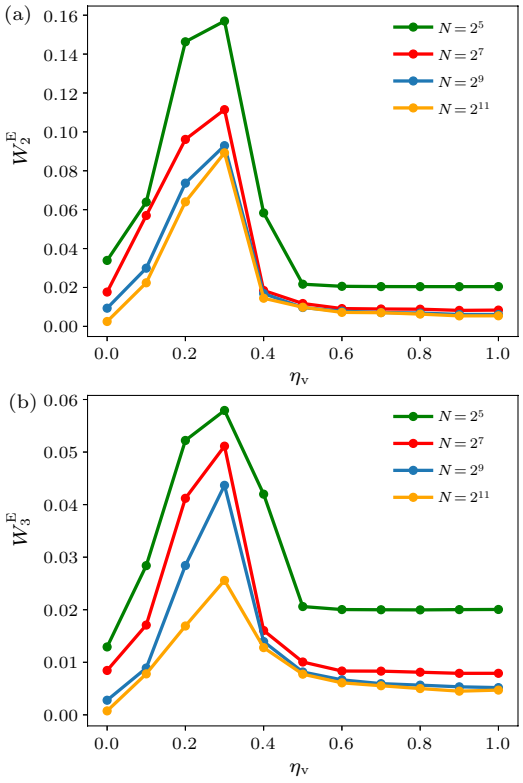


Fig. 20. Phase transition of eigen microstates W_2^E and W_3^E in the standard Vicsek model under vector noise when $\rho = 2$: (a) W_2^E versus vector noise η_v under different population sizes, (b) W_3^E versus vector noise η_v under different population sizes.

6. Conclusions

We have delineated a unified analytical framework leveraging the eigen microstates method to explore phase transi-

tions within the standard Vicsek model, subject to both scalar and vector noise. This methodology affords a nuanced understanding of the system's dynamic behavior, extending the analytical purview beyond mere particle velocity uniformity to encompass the intricacies of distribution and density fluctuations within the colony. The findings definitively indicate that the standard Vicsek model undergoes discontinuous phase transitions under both noise paradigms. This study meticulously quantifies the critical thresholds for these transitions, elucidating the interplay between these critical points and system density. It asserts that the critical thresholds for discontinuous phase transitions ascend with heightened noise levels, yet remain invariant to changes in cluster size.

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