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# Strip method to construct a two-dimensional quasilattice with eight-fold symmetry

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Based on the substitution rule and symmetry, we propose a method to generate an octagonal quasilattice consisting of square and rhombus tiles. Local configurations and Ammann lines are used to guide the growth of the tiles in a quasiperiodic order. The structure obtained is a perfect eight-fold symmetric quasilattice, which is confirmed by the radial distribution function and the diffraction pattern.

**Keywords:** quasicrystals, eight-fold symmetry, Ammann–Beenker tiling

**PACS:** 61.44.Br, 61.50.Ah, 02.60.Cb **DOI:** 10.1088/1674-1056/ad51f2

## 1. Introduction

The geometric and structural properties of quasiperiodic tiling have been of general interest to mathematicians and physicists, especially since the first experimental discovery of quasicrystals.[1] A quasiperiodic tiling with certain decorations can serve as a quasicrystal model, which is known as a quasilattice. Compared to traditional crystal lattices, quasilattices may possess five or more than six degrees of rotational symmetry, which makes it possible to understand the detailed information about the structural and physical properties. The theoretical models provide important information about the formation mechanism and physical properties of quasicrystals.[2–6]

There are several ways to generate a quasilattice, including the inflation–deflation transformation, the general dual method and the cut-and-projection method.[7–10] However, the formation of a real quasicrystal is different from the above processes, and the growth mechanisms of a perfect quasicrystal have not been thoroughly understood. In general, prototiles plus matching rules is an intuitive picture to explain the formation of a quasicrystal. For example, a patch of Penrose tiling is produced through edge-to-edge matching of fat and thin rhombuses. However, mismatch in the process of tiling will inevitably occur because the matching rules only determine the local order. Therefore, some growth methods have been proposed by considering other constraints. Pioneer work was done by Onoda *et al.*[11] who demonstrated that a perfect Penrose tiling can be formed based on vertex configurations.

In this paper, we study the formation of a quasilattice with eight-fold rotational symmetry, which is known as Ammann–Beenker tiling.[12,13] In an earlier work, Cai *et al.* developed a computer algorithm to produce an Ammann–Beenker tiling.[14] In their growth process, the local vertex configurations are first identified according to the matching rules and suitable tiles are selected to form allowed configurations. If multiple choices exist, Ammann lines are also used to guide the growth rules. The complexity of this method is that every time a layer is added, the whole boundary must be traversed, resulting in much lower efficiency. To overcome this disadvantage, we propose a strip method to generate an Ammann–Beenker tiling that is more efficient on the one hand, and provides implications for the growth mechanism of quasicrystals on the other.

## 2. Geometric properties of the Ammann–Beenker tiling

The most popular eight-fold symmetric quasilattice is the Ammann–Beenker tiling. It consists of two types of tiles, a square and a rhombus with an acute angle of 45°. A straightforward way to obtain an Ammann–Beenker tiling is the so-called deflation–inflation or self-similar transformation, which transforms a square and a rhombus to several smaller ones and then the pattern is rescaled by a factor of $\sqrt{2}+1$ so that the size of the squares and rhombuses are equal to the original ones.[15,16] As shown in Fig. 1(a), a square is cut into nine parts consisting of one small square, four rhombuses and four half squares. The small half square will combine with the adjacent half, which resulted from the transformation outside the original large square, and thus form a full square. Equivalently, a square will produce three smaller squares and four smaller rhombuses after one transformation. Similarly, a rhombus will produce two smaller squares and three smaller rhombuses after one transformation, as shown in Fig. 1(b). Starting from an eight-fold symmetric patch of tiling, as shown in Fig. 2(a), one can get successive patterns during the transformation. Figure 2(b) is the resulting pattern after a self-similar transformation. The arrows on the sides indicate the direction of a tile

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and are a guide to match adjacent tiles.

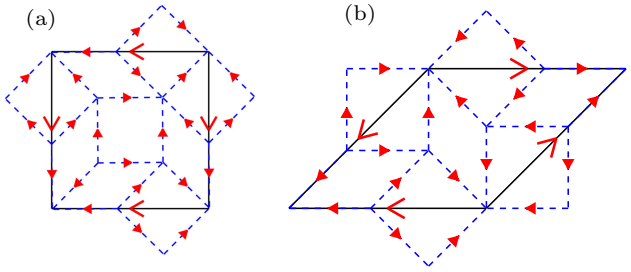



**Fig. 1.** The self-similar transformation for (a) a square and (b) a rhombus. The solid lines show the original tiles and the dashed lines represent the pattern after a transformation.

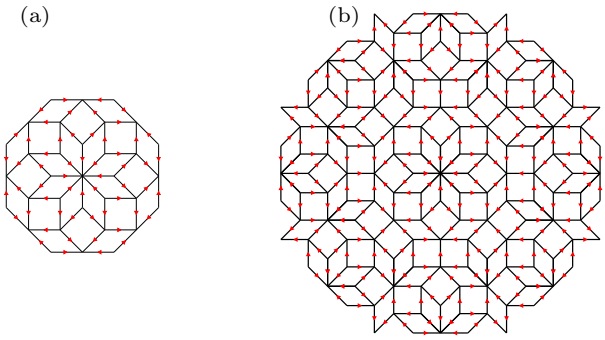



**Fig. 2.** The self-similar transformation is performed on a patch of tiling with eight-fold rotational symmetry (a) and a larger pattern is obtained (b).

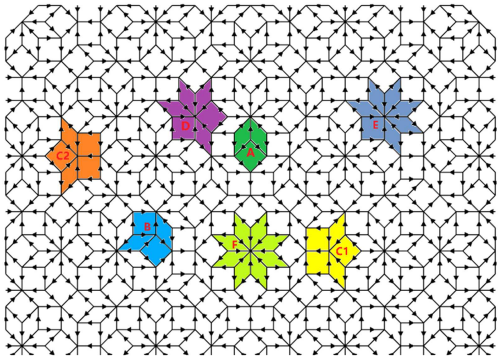



**Fig. 3.** Seven types of vertex configuration in the Ammann–Beenker tiling.

The nearest neighbors surrounding the center of the squares and rhombuses in a quasiperiodic pattern are specified by the vertex configurations. The vertex configurations constrain the arrangement of the nearest tiles that share a common vertex. Even though there are many possible combinations of tiles obeying the matching rules between squares and rhombuses, only certain vertex configurations are allowed in an ideal Ammann–Beenker tiling.[17,18] If we combine the squares and rhombuses in a way that the sum of the angles at the vertex is 360° and the arrows on the edges match one another, there will be 19 cases. However, some of them never occur in the Ammann–Beenker tiling due to the long-range order requirement. The number of allowed vertex configurations is only seven, and they are labeled A, B, C1, C2, D, E and F, as shown in Fig. 3. The vertex configurations of type C1 and type C2 have the same tile arrangement with different arrow directions on the common edge of the two squares, one to the left and the other to the right.

## 3. The strip method

The self-similar transformation is a mathematical approach to obtaining a quasilattice, but the practical growth of a quasicrystal never follows this path. In the tiling model, tiles aggregate one by one or cluster by cluster. When a tiling is formed, the basic requirement is that the matching rules that constrain adjacent tiles to take the same arrow directions along their shared sides must be satisfied. However, the addition of tiles at the boundary of a cluster governed by the matching rules alone is not sufficient to keep the growth process going. Deviation from a perfect quasiperiodic order inevitably occurs after only less than a hundred tiles are added. It is generally believed that the alignment of distant tiles must be considered to produce a perfect quasiperiodic structure, and long-range rules are required to make sure that this alignment is correct.

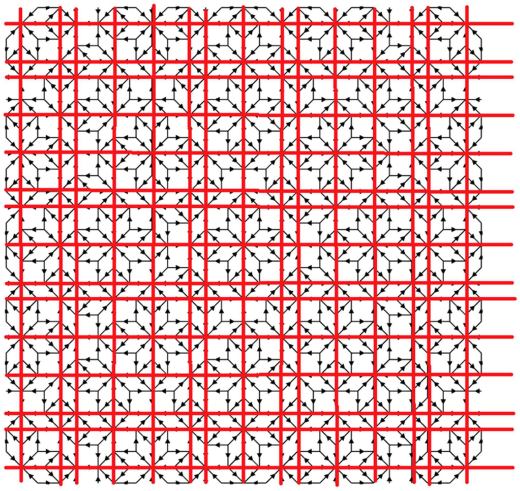


**Fig. 4.** Two sets of Ammann lines in the Ammann–Beenker tiling.

One of the characteristics of a quasilattice is that the forbidden rotational symmetries in a periodic lattice are allowed to exist. The other characteristic is quasiperiodicity. If we observe the structure along a specific direction in an Ammann–Beenker tiling, it is found that the sides or diagonals of the square or rhombus follow a one-dimensional quasiperiodic order known as the generalized Fibonacci sequence.[19,20] Let us do a simple decoration in the tiling by drawing horizontal and vertical lines through the diagonal of the squares, as shown in Fig. 4. The two sets of lines each have the same quasiperiodic order, $LLSLLSLLLSLLSL\cdots$, which is denoted as the GF(2,1) sequence. Here, $L$ and $S$ stand for the long and short spacings between adjacent lines, respectively, with $L/S=\sqrt{2}+1$. The GF(2,1) sequence is generated by the substitution rule $L\to LLS$ and $S\to L$. The idea of specifying a tiling in this way was proposed by Ammann and the resulting

lines are called Ammann lines.[21,22] The Ammann lines used here are different from the original ones and are simpler and more intuitive.

The distribution of tiles along an Ammann line follows the GF(2,1) sequence. This reminds us that it is possible to construct a ribbon by certain rules in a specific Ammann line. Vertex type A is the simplest of the seven types of vertices and its self-similar transformation has a simple correspondence to the substitution of the GF(2,1) sequence. Let us start from a vertex of type A and add tiles along its symmetric line. There are eight vertices of type A in Fig. 2(a), so the growth from this configuration will lead to a pattern with eight-fold symmetry. Different from a GF(2,1) sequence that has only two elements $L$ and $S$, there are three segments in an Ammann–Beenker tiling along an Ammann line, which are a side of a square, a side of a rhombus and a diagonal of a square. When the arrow or the orientation is also considered, the number of segments becomes six. Therefore, we must study all the combinations of the six types of segments. Regarding the vertex configurations in Fig. 3, there are five cases to add tiles, as shown in Fig. 5. In Figs. 5(a)–5(c), tiles are grown on the right-hand side, with vertices of type E, types C1+C2 and types C2+C1 added, respectively. In Figs. 5(d) and 5(e), tiles are grown on the left-hand side with vertices of types F and E added, respectively. According to the above growth rules, a quasiperiodic strip is constructed by considering squares along an Ammann line, as shown in Fig. 6. There are two sets of squares, the interior and exterior ones. The interior squares lie at the center of the strip and are determined by the Ammann line. The exterior squares are not crossed by the Ammann lines and are uniquely determined by one of the seven types of vertex configuration.

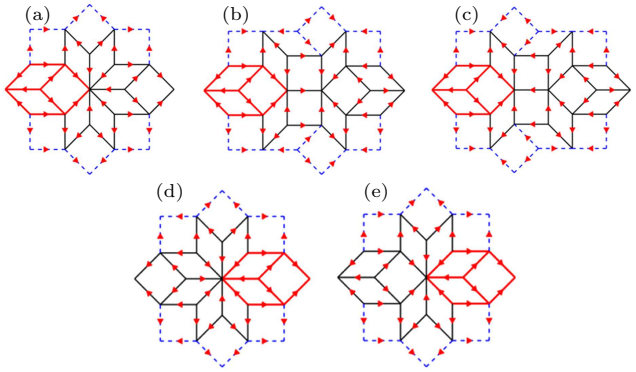



**Fig. 5.** Growth rules of tiles along an Ammann line starting from a vertex of type A. The interior squares (solid lines) are determined by the Ammann line and the exterior squares (dashed lines) are determined by the vertex types.

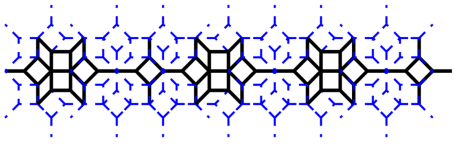


**Fig. 6.** The strip method to grow a quasiperiodic sequence of squares along an Ammann line. The space between the squares is filled by rhombuses according to the vertex configurations.

Using rotational symmetry, we obtain two strips perpendicular to each other, as shown in Fig. 7(a). There is no difference between the two strips except for their orientations.

Now that the framework has been constructed, we can extend the pattern along other Ammann lines using the growth rules by adding squares with two orientations. Thus, a two-dimensional pattern consisting of squares is formed in the first quadrant, as shown in Fig. 7(a). Regarding the seven vertex configurations and the orientations of the squares relative to the Ammann line, the gaps between the squares are filled with additional tiles, leading to a complete tiling, as shown in Fig. 7(b). Using eight-fold symmetry, the pattern generated in the first quadrant can be extended to the other three quadrants and a finite Ammann–Beenker tiling is produced.

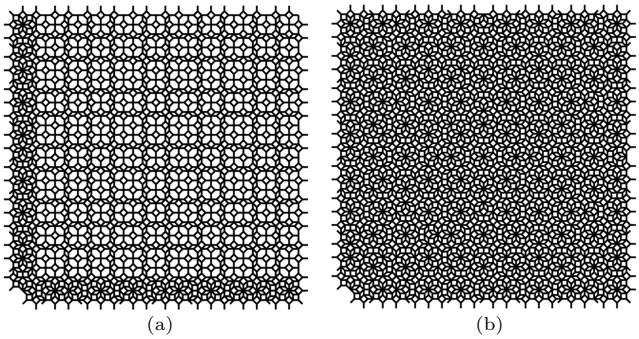



**Fig. 7.** (a) Two strips perpendicular to each other are generated according to the growth rule and then the space is extended in the first quadrant. (b) The gaps between the squares are filled with other tiles and a perfect Ammann–Beenker tiling is formed.

## 4. Diffraction pattern and radial distribution function

Using the strip method, we produced an Ammann–Beenker tiling with eight-fold rotational symmetry about the center, consisting of 3716 tiles. We double checked the vertex configurations in the tiling and no defects were found. To further confirm that the quasilattice generated by the above method is a perfect Ammann–Beenker tiling, the diffraction pattern and radial distribution function of the structure are generally used.[23,24] Suppose there is one atom at the center of every tile, then the corresponding distribution of atoms will be as presented in Fig. 8(a). The diffraction pattern is calculated by Fourier transformation and presented in Fig. 8(b), where the eight-fold symmetry is clearly shown.

The radial distribution function calculated from the quasilattice is shown in Fig. 9(a). The most probable position is $r = 1$, equal to a side length of the square and rhombus. The first peak appears at $r = 0.8$, equal to the length of the short diagonal of the rhombus. As shown in Fig. 9(b), the coordination number ranges from 3 to 8, with concentrations of 0.438(3), 0.327(4), 0.144(5), 0.054(6), 0.011(7) and 0.027(8), respectively, and the average value is 3.96.

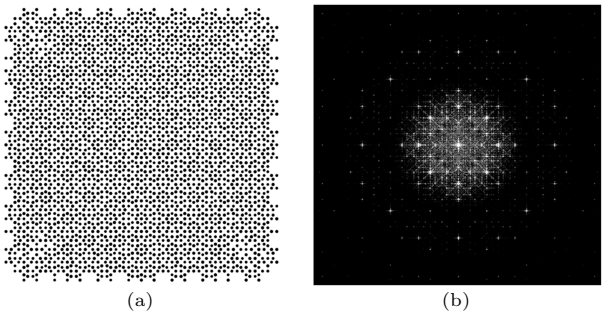



**Fig. 8.** (a) The structure of quasilattice with one atom at the center of each tile. (b) The diffraction pattern of the structure.

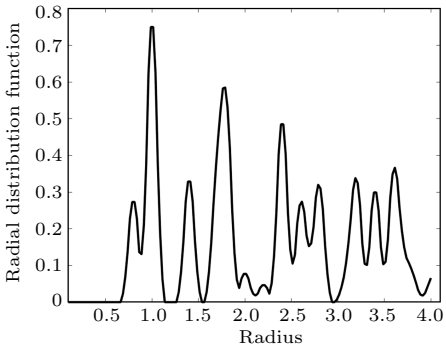



**Fig. 9.** The radial distribution function of the structure.

## 5. Discussion

While the strip method proposed in the present study is conducted from the matching rules, vertex types, Ammann lines and symmetry, the growth process is simpler and the algorithm is more efficient than the previous methods. The key idea is that a two-dimensional quasilattice is directly related to a one-dimensional quasiperiodic sequence. Combining the one-dimensional order and the vertex configurations, the arrangement of neighboring tiles has more restriction than the matching rules by arrows alone. This means that unforced vertices do not occur in a local cluster and tiles are grown in the correct long-range order. Since the matching rules, Ammann lines and symmetry are universal features of quasilattices, our method may be generalized to other two-dimensional quasicrystal models.

## Acknowledgement

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