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Quantum block coherence with respect to projective measurements

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Quantum coherence serves as a defining characteristic of quantum mechanics, finding extensive applications in quantum computing and quantum communication processing. This study explores quantum block coherence in the context of projective measurements, focusing on the quantification of such coherence. Firstly, we define the correlation function between the two general projective measurements P and Q, and analyze the connection between sets of block incoherent states related to two compatible projective measurements P and Q. Secondly, we discuss the measure of quantum block coherence with respect to projective measurements. Based on a given measure of quantum block coherence, we characterize the existence of maximal block coherent states through projective measurements. This research integrates the compatibility of projective measurements with the framework of quantum block coherence, contributing to the advancement of block coherence measurement theory.

Keywords: quantum coherence, compatibility, projective measurement, quantum block coherence

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1. Introduction and preliminaries

The principle of superposition is the most fundamental feature at the heart of quantum physics. Quantum coherence,[1–4] which arises from the superposition principle in quantum mechanics, is one of the most unique features of quantum physics. It is a valuable resource in fields like quantum computation and quantum communication processing.[5–10] Additionally, it has important implications for quantum metrology, quantum biology, and quantum thermodynamics.[11–16] Quantum resource theory has drawn a great deal of attention towards the quantization of quantum coherence, leading to the proposal of various coherence measurements. Quantum measurement, particularly projective measurement, is a vital tool for observing quantum systems and gathering quantum information, making it a fundamental aspect of quantum theory research.

In recent years, there has been a growing interest in the axiomatic and quantitative study of coherence, inspired by the seminal work of Baumgartz *et al.*[1] For general projective measurements, Åberg[17] proposed a theoretical framework for quantum block coherence based on projective measurements. Bischof *et al.*[18] further extended the quantum block coherence based on projective measurement to the case of positive-operator-valued measures (POVMs) and established a coherent resource theory based on POVMs. Moreover, Bischof *et al.*[19] gave the quantum block coherence measurement axiom based on POVMs and established a quantitative quantum coherence framework based on POVMs. In a striking study, Guilloramy introduced K-coherence an intuitive quantitative index of coherence based on Wigner–Yanase skew information.[20] The theory of block coherence is particularly important when the experimenter does not have the ability to measure a full set of observables and to prepare a complete basis of states. However, there is a lack of clarity in the study of effect of the relationship between projected measurements on block coherence, and thus our study focuses on this topic.

For a quantum system X described by a d-dimensional Hilbert space H, let I denote the identity operator on H. We use $\mathcal{B}(H)$ and $\mathcal{D}(H)$ to denote the sets of all linear operators and all density operators on H, respectively. For the fixed orthonormal basis $e=\{|e_i\rangle\}_{i=1}^{d}$ for H, $\mathcal{I}(e)$ denotes the set of incoherent states[1] on H with respect to e, i.e.,

$$\mathcal{I}(e)=\Big\{\rho=\sum_i \lambda_i|e_i\rangle\langle e_i| : \lambda_i\geq 0,\ \sum_i \lambda_i=1\Big\},$$

which have diagonal matrix representations under the basis e.

In quantum information theory, a POVM is a set $M=\{M_i\}_{i=1}^{m}$ of operators on H with $0\leq M_i\leq I$ for all $i=1,2,\ldots,m$ and $\sum_{i=1}^{m}M_i=I$. In particular, if $M_i^2=M_i$ for all i, then the POVM becomes a projective measurement (PM). For a rank-one PM $P=\{P_i\}_{i=1}^{d}$, there exists an orthonormal basis $e=\{|e_i\rangle\}_{i=1}^{d}$ such that $P_i=|e_i\rangle\langle e_i|$ for $i\in\{1,2,\ldots d\}$.

If $P=\{P_i\}_{i=1}^{n}$ is a PM on H, then P_i is the orthogonal projection of $H=\bigoplus_{i=1}^{n}H_i$ onto H_i. A state ρ is called block incoherent[18] with respect to P if it satisfies

$$P_i\rho P_j=0\ (\forall i\neq j),$$

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i.e.,

$$\Pi_P(\rho) := \sum_{i=1}^{n} P_i \rho P_i = \rho.$$

We use $\mathcal{I}(P)$ to denote the set of all block incoherent states with respect to P on H.

Observing quantum systems and obtaining information are carried out through quantum measurement, especially projective measurement, which is a basic tool for studying quantum theory. Measurement incompatibility refers to a phenomenon that occurs in quantum theory and general probability theory: some observations do not share a common refinement that can be obtained by coarse-graining the associated probability distribution. For these measurements, the exchangeability of the associated Hermitian operators is a necessary and sufficient condition for their compatibility.[22]

Since the coherence of a quantum state depends on the choice of reference basis, it is natural to study the relationship between the coherence in different bases. On this issue, Cheng *et al.*[21] proposed a complementary relationship between the two coherence measures under any complete set of mutually unbiased bases (MUBs). Inspired by this, we are interested in selecting two compatible projective measurements as reference bases to study measure of block coherence in block coherence theory.

This paper explores the compatibility of projective measurements and the relationship between the quantum block coherence based on projective measurements and the compatibility of projective measurements. In Section 2, we define the compatibility measure function of two general projection measurements. Then, we discuss the block coherence of quantum states based on two compatible projective measurements P and Q. In Section 3, we introduce a coherence measure of block coherence regarding PMs and discuss a block coherence measure $C(\rho|P)$ using the Wigner–Yanase skew information. Focusing on the effect of the choice of reference system on coherence measurements, our work combines the compatibility of projective measurements with the theory of block coherence and further refines the theory of block coherence measurements.

2. Compatibility of projective measurements

In the realm of quantum theory, observables are used to define the compatibility of measurements. An observable is a Hermitian operator that operates on the Hilbert space of a quantum system. It is associated with a projective measurement through its spectral decomposition. Two observables A and B are said to be compatible if they commute.[23] The same definition can be extended to POVMs as follows.

Two POVMs $M = \{M_i\}_{i=1}^m$ and $N = \{N_j\}_{j=1}^n$ are said to be compatible[22] if there exists a common refinement, i.e., a POVM $G = \{G_{ij}\}$ such that

$$M_i = \sum_j G_{ij}, N_j = \sum_i G_{ij}.$$

Otherwise, they are said to be incompatible.

In particular, Ravi in Ref. [24] proved that if $M = \{M_i\}_{i=1}^m$ and $N = \{N_j\}_{j=1}^n$ are projective measurements, then M and N are compatible if and only if $[M_i, N_j] = M_i N_j - N_j M_i = 0$ for all i, j. In Ref. [25] Yu *et al.* proved that the compatibility of two POVM measurements is equivalent to the joint measurability. The compatibility (or joint measurability) of a set of projective measurements is a binary property, characterized entirely by their pairwise commutativity.

Since rank-one projection measurements can be generated from orthonormal bases, we inscribe the compatibility of projective measurements by considering the generalization of the following correlation functions to projective measurements.

For two orthonormal bases $e = \{|e_i\rangle\}_{i=1}^d$ and $f = \{|f_j\rangle\}_{j=1}^d$ for H, the correlation function between e and f is defined in Ref. [26] as

$$m(e,f) = \sum_{i,j=1}^{d} |\langle e_i|f_j\rangle| - d,$$

which satisfies

(a1) $0 \le m(e,f) \le d^{\frac{3}{2}} - d$;

(a2) $m(e,f) = 0$ if and only if $\{|e_i\rangle\langle e_i|\}_{i=1}^d = \{|f_j\rangle\langle f_j|\}_{j=1}^d$;

(a3) $m(e,f) = d^{\frac{3}{2}} - d$ if and only if e and f are mutually unbiased bases, that is, $|\langle e_i|f_j\rangle| = 1/\sqrt{d}$ for any i, j.

Let $P = \{P_i\}_{i=1}^m$ be a projective measurement, we know that there exists an orthonormal basis

$$e = \{|e_{is}\rangle : s \in [m_i],\ i \in [m]\},$$

such that $P_i = \sum_{s=1}^{m_i} |e_{is}\rangle\langle e_{is}| (i \in [m])$, where $[m] = \{1, 2, \ldots, m\}$. In this sense, we say that P is generated by e. Let G_P denote the set of orthonormal bases which could generate P.

Let $P = \{P_i\}_{i=1}^m$ and $Q = \{Q_j\}_{j=1}^n$ be two projective measurements on H. We define the correlation function between P and Q as

$$m(P,Q) = \min_{e \in G_P, f \in G_Q} m(e,f).$$

We now generalize the results (a1)–(a3) in Ref. [26].

Theorem 1 (1) $0 \le m(P,Q) \le d^{\frac{3}{2}} - d$;

(2) $m(P,Q) = 0$ if and only if P and Q are compatible;

(3) $m(P,Q) = d^{\frac{3}{2}} - d$ if and only if P and Q are two von Neumann measurements generated by two mutually unbiased bases, respectively.

Proof (1) By definition and (a1), it is clear.

(2) We know that P and Q are compatible if and only if $[P_i,Q_j]=0(\forall i,j)$ if and only if P_i and Q_j can be diagonalized simultaneously under an orthonormal basis $e=\{|e_i\rangle\}_{i=1}^d$, i.e.,

$$P_i=\sum_{k\in\Lambda_i}|e_k\rangle\langle e_k|, Q_j=\sum_{s\in\Gamma_j}|e_s\rangle\langle e_s|$$

if and only if $e\in G_P\bigcap G_Q$ if and only if $m(P,Q)=0$, where Λ_i, Γ_j are the sets of indexes and G_P, G_Q are the sets of orthonormal bases which could generate PMs P, Q.

(3) It is clear by (a3).

The correlation function $m(P,Q)$ reflects the compatibility between P and Q. Now we naturally present the theorem about the relationship between the bases generating P,Q and their compatibility.

Theorem 2 Let $\{e=\{|e_i\rangle\}_{i=1}^d, f=\{|f_j\rangle\}_{j=1}^d\}$ be a pair of bases for H. If $P=\{P_i\}_{i=1}^m$ and $Q=\{Q_j\}_{j=1}^n$ $(2\le m,n\le d)$ are the projective measurements generated by e,f, respectively, i.e.,

$$P_i=\sum_{s\in\Lambda_i}|e_s\rangle\langle e_s|,\ \ Q_j=\sum_{t\in\Gamma_j}|f_t\rangle\langle f_t|, \tag{1}$$

then:

(i) P and Q are compatible if and only if

$$\forall i\in[m],\ j\in[n],\ \sum_{t\in\Gamma_j}\alpha_{st}\overline{\alpha_{rt}}=0,\ \forall s\in\Lambda_i,\ \forall r\notin\Lambda_i, \tag{2}$$

where $\alpha_{rt}=\langle e_r|f_t\rangle$.

(ii) When $|\langle e_s|f_t\rangle|>0$ for all s,t, if $|\Gamma_j|<m$ for some j or $|\Lambda_i|<n$ for some i, then P and Q are incompatible.

(iii) When $|\langle e_s|f_t\rangle|=\frac{1}{\sqrt{d}}$ for all s,t, if $mn>d$ or $|\Lambda_i|\cdot|\Gamma_j|<d$ for some i,j, then P and Q are incompatible.

Proof (i) Since $\langle e_r|f_t\rangle=\alpha_{rt}$, $|f_t\rangle=\sum_{r=1}^d\alpha_{rt}|e_r\rangle$, we can reach

$$\begin{aligned}P_iQ_j&=\sum_{s\in\Lambda_i}\sum_{r=1}^d\sum_{t\in\Gamma_j}\alpha_{st}\overline{\alpha_{rt}}|e_s\rangle\langle e_r|\\&=\sum_{s,r\in\Lambda_i}\Big(\sum_{t\in\Gamma_j}\alpha_{st}\overline{\alpha_{rt}}\Big)|e_s\rangle\langle e_r|\\&\quad+\sum_{s\in\Lambda_i}\sum_{r\notin\Lambda_i}\Big(\sum_{t\in\Gamma_j}\alpha_{st}\overline{\alpha_{rt}}\Big)|e_s\rangle\langle e_r|.\end{aligned}$$

Then $P_iQ_j=P_jQ_i$ if and only if P_iQ_j is self-adjoint if and only if for any

$$\sum_{t\in\Gamma_j}\alpha_{st}\overline{\alpha_{rt}}=0,\ \forall s\in\Lambda_i,\ \forall r\notin\Lambda_i. \tag{3}$$

(ii) Let $|\langle e_s|f_t\rangle|>0$ for all s,t and suppose that P and Q are compatible. Then we obtain from Eq. (2) that for every $j\in[n]$, there are m mutually orthogonal nonzero vectors

$$x_i=(\alpha_{s_ir_1},\alpha_{s_ir_2},\ldots,\alpha_{s_ir_{n_j}})\ \ (i=1,2,\ldots,m),$$

where $s_i\in\Lambda_i, r_k\in\Gamma_j$. Thus,

$$\min\{|\Gamma_j|:j=1,2,\ldots,n\}\ge m. \tag{4}$$

Similarly, we get

$$\min\{|\Lambda_i|:i=1,2,\ldots,m\}\ge n. \tag{5}$$

(iii) When $|\langle e_s|f_t\rangle|=\frac{1}{\sqrt{d}}$ for all s,t, $\{e,f\}$ is a pair of mutually unbiased bases[27] for H. For any $t_0\in\Gamma_j$, we get $Q_j|f_{t_0}=|f_{t_0}$ and thus

$$P_iQ_j|f_{t_0}\rangle=P_i|f_{t_0}\rangle=\sum_{s\in\Lambda_i}\langle e_s|f_{t_0}\rangle|e_s\rangle, \tag{6}$$

$$Q_jP_i|f_{t_0}\rangle=\sum_{t\in\Gamma_j}\sum_{s\in\Lambda_i}\langle f_t|e_s\rangle\langle e_s|f_{t_0}\rangle|f_t\rangle. \tag{7}$$

Assuming that P and Q are compatible, then for all i,j, we get $P_iQ_j=Q_jP_i$. Using Eqs. (6) and (7) imply that

$$\Big\|\sum_{s\in\Lambda_i}\langle e_s|f_{t_0}\rangle|e_s\rangle\Big\|^2=\Big\|\sum_{t\in\Gamma_j}\sum_{s\in\Lambda_i}\langle f_t|e_s\rangle\langle e_s|f_{t_0}\rangle|f_t\rangle\Big\|^2. \tag{8}$$

Since the left-hand side of Eq. (7) equals

$$\sum_{s\in\Lambda_i}|\langle e_s|f_{t_0}\rangle|^2=\frac{|\Lambda_i|}{d},$$

where $|\Lambda_i|$ denotes the cardinality of the index set Λ_i and the right-hand side of Eq. (7) equals

$$\begin{aligned}&\sum_{t\in\Gamma_j}\Big|\sum_{s\in\Lambda_i}\langle f_t|e_s\rangle\langle e_s|f_{t_0}\rangle\Big|^2\\&\le\sum_{t\in\Gamma_j}\sum_{s\in\Lambda_i}|\langle f_t|e_s\rangle|^2\sum_{s\in\Lambda_i}|\langle e_s|f_{t_0}\rangle|^2=\frac{|\Lambda_i|^2|\Gamma_j|}{d^2},\end{aligned}$$

we have $|\Lambda_i|\cdot|\Gamma_j|\ge d(\forall i,j)$ and so $mn\le d$. Thus, when $mn>d$ or $|\Lambda_i|\cdot|\Gamma_j|<d$ for some i,j, P and Q are incompatible.

The theorem presents a way of combining the compatibility of projective measurements with the mutual unbiasedness between orthogonal bases by generating PMs. However, projection measurements generated by mutually unbiased bases may be compatible or incompatible, for example: For $d=4$, we need to consider the case of $m=n=2$. The following example shows that even if P and Q are rank-two projective measurements generated by a pair of mutually unbiased bases e,f, respectively, then they may be compatible or incompatible.

Example 1 Let $e=\{|e_i\rangle\}_{i=1}^4$ be an orthonormal basis for $\mathbb{C}^4$ and

$$U=[U_{ij}]=\frac{1}{2}\begin{pmatrix}1&1&-1&1\\1&1&1&-1\\1&-1&1&1\\1&-1&-1&-1\end{pmatrix}.$$

Then we can get an orthonormal basis $f=\{|f_j\rangle\}_{j=1}^4$ with $|f_j\rangle=\sum_i U_{ij}|e_i\rangle$ and calculate $|\langle e_i|f_j\rangle|=\frac{1}{2}(\forall i,\ j\in[4])$.

Thus $\{e,f\}$ is a pair of mutually unbiased bases. Put $P=\{P_1,P_2\}$ and $Q=\{Q_1,Q_2\}$, where

$$P_1=|e_1\rangle\langle e_1|+|e_2\rangle\langle e_2|,\ \ P_2=|e_3\rangle\langle e_3|+|e_4\rangle\langle e_4|,$$
$$Q_1=|f_1\rangle\langle f_1|+|f_2\rangle\langle f_2|,\ \ Q_2=|f_3\rangle\langle f_3|+|f_4\rangle\langle f_4|,$$

then by Theorem 2 (i), we obtain that P and Q are compatible.

If we give $e=\{|e_i\rangle\}_{i=1}^4$ and

$$V=[V_{ij}]=\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&\mathrm{i}&-1&-\mathrm{i}\\1&-1&1&-1\\1&-\mathrm{i}&-1&\mathrm{i}\end{pmatrix}.$$

Then we can get an othonormal basis $g=\{|g_j\rangle\}_{j=1}^4$ with $|g_j\rangle=\sum_i V_{ij}|e_i\rangle$ and calculate $|\langle e_i|g_j\rangle|=\frac{1}{2}(\forall i,\ j\in[4])$.

Thus $\{e,g\}$ is a pair of mutually unbiased bases. Put $P=\{P_1,P_2\}$ and $L=\{L_1,L_2\}$, where

$$P_1=|e_1\rangle\langle e_1|+|e_2\rangle\langle e_2|,\ \ P_2=|e_3\rangle\langle e_3|+|e_4\rangle\langle e_4|,$$
$$L_1=|g_1\rangle\langle g_1|+|g_2\rangle\langle g_2|,\ \ L_2=|g_3\rangle\langle g_3|+|g_4\rangle\langle g_4|,$$

then by Theorem 2 (i), we obtain that P and L are incompatible.

In the resource theory of block coherence, the block incoherent states have a block diagonal structure which is determined by a projective (not necessarily rank one) measurement. When we select compatible projective measurements as the reference base, we notice a close relationship between corresponding sets of block incoherent quantum states. According to Theorem 1, if P and Q are compatible, then there exists an orthonormal basis e such that

$$\mathcal{I}(e)\subset\mathcal{I}(P)\bigcap\mathcal{I}(Q).$$

We say that $Q=\{Q_j\}_{j=1}^n$ is a coarsening of $P=\{P_i\}_{i=1}^m$ if for all $i\in[m]$, there exists a $j_i\in[n]$ such that $\mathrm{Ran}(P_i)\subset\mathrm{Ran}(Q_{j_i})$. By definition, the coarsening of projective measurements is a special kind of compatibility.

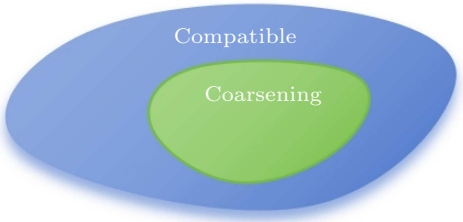


Fig. 1. Coarsening is included in compatibility.

Based on the definition of coarsening, we consider the inclusion relation between $\mathcal{I}(P)$ and $\mathcal{I}(Q)$ when the reference measurements is chosen to satisfy that Q is a coarsening of P.

Theorem 3 Let $P=\{P_i\}_{i=1}^m$ and $Q=\{Q_j\}_{j=1}^n$ be two PMs. Then $\mathcal{I}(P)\subset\mathcal{I}(Q)$ if and only if Q is a coarsening of P.

Proof For sufficiency, suppose that Q is a coarsening of P, then $\mathrm{Ran}(P_i)\subset\mathrm{Ran}(Q_{j_i})$, i.e., $P_i\le Q_{j_i}$ for all $i=1,2,\ldots,m$. Since

$$\sum_{i=1}^m P_i=\sum_{j=1}^n Q_j=I,$$

we conclude that for each $Q_jQ_\ell=0(j\neq\ell)$ and for $i\in[m]$, the the index j_i such that $P_i\le Q_{j_i}$ must be unique. Thus, $P_iQ_{j_i}=Q_{j_i}P_i=P_i$ and $P_iQ_j=Q_jP_i=0$ for all $j\neq j_i$.

Suppose $\rho\in\mathcal{I}(P)$. According to the fact that

$$Q_j=\sum_{i=1}^m Q_jP_i=\sum_{i=1}^m P_iQ_j,\quad \sum_{i=1}^m P_i\rho P_i=\rho,$$

we obtain

$$\begin{aligned}\sum_{j=1}^n Q_j\rho Q_j&=\sum_{j=1}^n\sum_{i=1}^m\sum_{k=1}^m Q_jP_i\rho P_kQ_j=\sum_{j=1}^n\sum_{i=1}^m Q_jP_i\rho P_iQ_j\\&=\sum_{i=1}^m\sum_{j=1}^n Q_jP_i\rho P_iQ_j=\sum_{i=1}^m Q_{j_i}P_i\rho P_iQ_{j_i}\\&=\sum_{i=1}^m P_i\rho P_i=\rho.\end{aligned}$$

Therefore, $\rho\in\mathcal{I}(Q)$. This shows that $\mathcal{I}(P)\subset\mathcal{I}(Q)$.

For *necessity*, we set $\mathcal{I}(P)\subset\mathcal{I}(Q)$. Suppose that there exists an i_0 such that $\mathrm{Ran}(P_{i_0})\nsubseteq\mathrm{Ran}(Q_j)$, i.e., $\Lambda_{i_0}\nsubseteq\Gamma_j$ for $P_{i_0}=\sum_{k\in\Lambda_{i_0}}|e_k\rangle\langle e_k|$ and $Q_j=\sum_{s\in\Gamma_j}|e_s\rangle\langle e_s|(\forall j\in[n])$, where $\{|e_i\rangle\}_{i=1}^d$ is an orthonormal basis for H. Then the set Λ_{i_0} has at least two different elements. Since $\cup_{j=1}^n\Gamma_j=[d]$, Λ_{i_0} contains two different elements k_1,k_2 such that $k_1\in\Gamma_{j_1}$ and $k_2\in\Gamma_{j_2}$ for some different indexes $j_1,j_2\in[n]$. Thus,

$$P_{i_0}=|e_{k_1}\rangle\langle e_{k_1}|+|e_{k_2}\rangle\langle e_{k_2}|+\sum_{k\in\Lambda_{i_0}\setminus\{k_1,k_2\}}|e_k\rangle\langle e_k|,$$
$$Q_{j_1}=|e_{k_1}\rangle\langle e_{k_1}|+\sum_{s\in\Gamma_{j_1}\setminus\{k_1\}}|e_s\rangle\langle e_s|,$$
$$Q_{j_2}=|e_{k_2}\rangle\langle e_{k_2}|+\sum_{t\in\Gamma_{j_2}\setminus\{k_2\}}|e_t\rangle\langle e_t|.$$

Consider the state

$$\rho=\rho_{11}|e_{k_1}\rangle\langle e_{k_1}|+\rho_{12}|e_{k_1}\rangle\langle e_{k_2}|+\overline{\rho_{12}}|e_{k_2}\rangle\langle e_{k_1}|+\rho_{22}|e_{k_2}\rangle\langle e_{k_2}|$$

with $\rho_{12}\neq0$, when $i,s\in[m]$ with $i\neq s$, we have $i\neq i_0$ or $s\neq i_0$. because $P_i\rho=0$ or $\rho P_s=0$, $P_i\rho P_s=0$.

Note that $k_1\notin\Gamma_{j_2}$ and $k_2\notin\Gamma_{j_1}$, we obtain

$$\begin{aligned}Q_{j_1}\rho Q_{j_2}&=(\rho_{11}|e_{k_1}\rangle\langle e_{k_1}|+\rho_{12}|e_{k_1}\rangle\langle e_{k_2}|)Q_{j_2}\\&=\rho_{12}|e_{k_1}\rangle\langle e_{k_2}|\neq0.\end{aligned}$$

Thus, $\rho\in\mathcal{I}(P)\backslash\mathcal{I}(Q)$. This contradicts the assumption.

However, for two compatible projective measurements P and Q (without coarsening relation), it is possible that $\mathcal{I}(P)$

and $\mathcal{I}(Q)$ do not contain each other. We present an example in the following.

Example 2 Let $e=\{|e_i\rangle\}_{i=1}^4$ be an orthonormal basis for $\mathbb{C}^4$,

$$P=\{P_1=|e_1\rangle\langle e_1|+|e_2\rangle\langle e_2|,\ P_2=|e_3\rangle\langle e_3|+|e_4\rangle\langle e_4|\},$$
$$Q=\{Q_1=|e_1\rangle\langle e_1|+|e_3\rangle\langle e_3|,\ Q_2=|e_2\rangle\langle e_2|+|e_4\rangle\langle e_4|\},$$

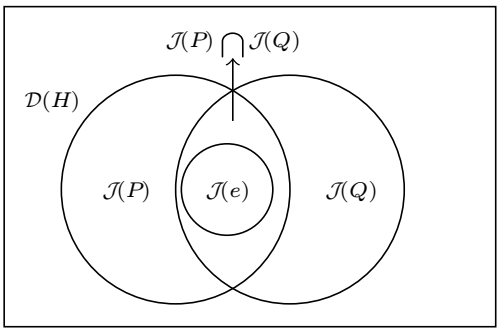
then we get that P and Q are two compatible projective measurements. Further calculations give

$$\rho\in\mathcal{I}(P)\Leftrightarrow\rho=\begin{pmatrix}\rho_{00}&\rho_{01}&0&0\\ \rho_{10}&\rho_{11}&0&0\\ 0&0&\rho_{22}&\rho_{23}\\ 0&0&\rho_{32}&\rho_{33}\end{pmatrix},$$
$$\rho'\in\mathcal{I}(Q)\Leftrightarrow\rho=\begin{pmatrix}\rho'_{00}&0&\rho'_{02}&0\\ 0&\rho'_{11}&0&\rho'_{13}\\ \rho'_{20}&0&\rho'_{22}&0\\ 0&\rho'_{31}&0&\rho'_{33}\end{pmatrix},$$

where ρ and ρ' are two quantum states with respect to e. Clearly, $\mathcal{I}(P)$ and $\mathcal{I}(Q)$ could not contain each other.

According to Theorem 3, we use the following figure to show the relationship between $\mathcal{I}(P)$ and $\mathcal{I}(Q)$, where P and Q are compatible.

(a) The first scenario: P and Q are compatible.



(b) The second scenario: Q is a coarsening of P.

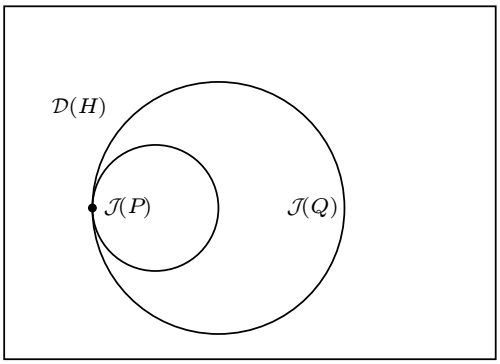


Fig. 2. Relationship between $\mathcal{I}(P)$ and $\mathcal{I}(Q)$ when P and Q are compatible.

3. Measure of block coherence with respect to PMs

In this section we discuss the measure of block coherence with respect to PMs. The effect of compatibility between reference PMs on the existence of their common maximum block coherent state is discussed under a selected block coherence measure.

Before presenting our main results, we first briefly recall the framework for quantifying the block coherence has been established in Ref. [19]: a measure of block coherence with respect to PM P is a map $C:\mathcal{D}(H)\mapsto\mathbb{R}$ satisfying the following four conditions.

(1) Faithfulness: $C(\rho|P)\geq 0$ for all $\rho\in\mathcal{D}(H)$; $C(\rho|P)=0$ if and only if $\rho\in\mathcal{I}_B(P)$.

(2) Monotonicity: $C(\Phi(\rho)|P)\leq C(\rho|P)$ for any $\Phi\in\mathcal{IO}_B(P)$ ($\mathcal{IO}_B$ is the set of block incoherent operations).

(3) Strong monotonicity: $\sum_{i=1}^N p_iC(\rho_i|P)\leq C(\rho|P)$, for any set of Kraus operators $\{K_i\}_{i=1}^N$ of $\Phi\in\mathcal{IO}_B(P)$ defining $p_i=\mathrm{tr}(K_i\rho K_i^\dagger)\neq 0$ and $\rho_i=p_i^{-1}\left(K_i\rho K_i^\dagger\right)$.

(4) Convexity: $C(\sum_{i=1}^n p_i\rho_i|P)\leq\sum_{i=1}^n p_iC(\rho_i|P)$ for any states ρ_i $(i=1,2,\ldots,n)$ and any probability distribution $\{p_i\}_{i=1}^n$.

Various coherence measures, such as the relative entropy of PM-based coherence measure, l_1-norm of PM-based coherence measure,[19] and Tsallis relative entropy of PM-based coherence measure,[28] have been formulated and tested to meet the aforementioned criteria. Ghirolami defined a K-coherence measure based on the observable K using the Wigner–Yenners skew information

$$I(\rho,K)=-\frac{1}{2}\mathrm{tr}[\sqrt{\rho},K]^2$$

in Ref. [20]. However, the given measure does not satisfy conditions (2) and (3). In Ref. [29], Luo *et al.* discussed and addressed this issue. Specifically, firstly they supposed that the observable K has the spectral decomposition $K=\sum_i a_iP_i$ with P_i orthogonal projections satisfying $\sum_i P_i=I$, and then defined a block coherence measure with respect to the Lüders measurement $P=\{P_i\}$ as

$$C(\rho|P)=\sum_{i=1}^m I(\rho,P_i). \tag{9}$$

Finally, they proved that this block coherence measure indeed satisfies the requirements of block coherence measure, i.e., Eqs. (1)–(4).

We therefore choose the block coherence measure (9) for our subsequent study. In this context, we consider the maximal block coherence of the quantum state with respect to the projective measurements.

Theorem 4 For any PM $P=\{P_i\}_{i=1}^m(1\leq m\leq d)$, we get

$$C(\rho|P)\leq 1-\frac{1}{m}(\forall\rho\in\mathcal{D}(H)),$$

moreover $C(\rho|P)=1-\frac{1}{m}$ if and only if $\mathrm{tr}\rho P_i=\frac{1}{m},\forall i\in[m]$.

Proof Let $P=\{P_i\}_{i=1}^m(1\leq m\leq d)$ be a PM. Then for any quantum state ρ, we have

$$C(\rho|P)=\sum_{i=1}^m I(\rho,P_i)=\sum_{i=1}^m(\mathrm{tr}\rho P_i-(\mathrm{tr}\rho P_i)^2)$$

$$= 1 - \sum_{i=1}^{m} (\mathrm{tr}\rho P_i)^2 \leq 1 - \frac{1}{m},$$

where $C(\rho|P) = 1 - \frac{1}{m}$ if and only if $\mathrm{tr}\rho P_i = \frac{1}{m}$, $i \in [m]$.

The above theorem provides an upper bound on the block coherence measure of Wigner–Yanase skew information and the conditions to be satisfied for a quantum state to be the maximal block coherent state under this measure. Naturally, by Theorem 4 we get $C(\rho|P) + C(\rho|Q) \leq 2 - (\frac{1}{m} + \frac{1}{n})(\forall \rho \in \mathcal{D}(H))$ for any two PMs $P = \{P_i\}_{i=1}^m$ and $Q = \{Q_j\}_{j=1}^n$. Now we consider whether there exists a quantum state that is maximally block coherent under two PMs.

Theorem 5 Let $P = \{P_i\}_{i=1}^m$ and $Q = \{Q_j\}_{j=1}^n$ be two compatible PMs. There exists a quantum pure state ρ such that

$$C(\rho|P) + C(\rho|Q) = 2 - \left(\frac{1}{m} + \frac{1}{n}\right)$$

if and only if $P_i \neq Q_j (\forall i \in [m], j \in [n])$.

Proof Since P and Q are compatible, P_i and Q_j can be diagonalized simultaneously under an orthonormal basis $e = \{|e_s\rangle\}_{s=1}^d$, i.e.,

$$P_i = \sum_{k \in \Lambda_i} |e_k\rangle\langle e_k|, Q_j = \sum_{s \in I_j} |e_s\rangle\langle e_s|.$$

There exists a pure state $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sum_{s=1}^d a_s |e_s\rangle$ and $\sum_{s=1}^d |a_s|^2 = 1$ is maximally block coherent with respect to P and Q, respectively, if and only if ρ satisfies

$$\mathrm{tr}\rho P_i = \frac{1}{m},\ i \in \{1,2,\ldots,m\},$$
$$\mathrm{tr}\rho Q_j = \frac{1}{n},\ j \in \{1,2,\ldots,n\}.$$

Computing the above system of equations we get that $\{a_s\}_{s=1}^d$ is a solution to the equations

$$\sum_{s \in \Lambda_{P_i}} |a_s|^2 = \frac{1}{m},\ i \in \{1,2,\ldots,m\},$$
$$\sum_{t \in \Lambda_{Q_j}} |a_t|^2 = \frac{1}{n},\ j \in \{1,2,\ldots,n\},$$

if and only if $\Lambda_{P_i} \neq \Lambda_{Q_j}$, i.e., $P_i \neq Q_j$ $(\forall i \in [m], j \in [n])$.

Example 3 Let $e = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ be an orthonormal basis in the $\mathbb{C}^4$ and P, Q be the compatible PMs and

$$P = \{P_1 = |0\rangle\langle 0| + |1\rangle\langle 1|,\ P_2 = |2\rangle\langle 2| + |3\rangle\langle 3|\},$$
$$Q = \{Q_1 = |0\rangle\langle 0| + |3\rangle\langle 3|,\ Q_2 = |1\rangle\langle 1| + |2\rangle\langle 2|\}.$$

According to Theorem 5, P, Q are compatible and $P_i \neq Q_j (i, j = 1, 2)$, there exists a quantum state that is the maximally coherent state with respect to P and Q, for example, $\rho = \frac{1}{4}\sum_{i,j=0}^3 |i\rangle\langle j|$.

The conditions for the existence of a maximal block coherent state with respect to P and Q when P and Q are compatible are given by the above theorem. The next theorem deals with the existence of maximal block coherent states with respect to P and Q when P and Q are incompatible.

To prove the theorem provided, we will firstly introduce Lemma 1.

Lemma 1[30,31] Let $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ be the set of polynomials consisting of a finite number of polynomials in $K[x_1, x_2, \ldots, x_n]$ and $G = g$ be another polynomial. The set of zeros of the system of equations $\mathcal{F}$ in any extended domain by K but such that $G \neq 0$ is denoted as $\mathrm{Zero}(\mathcal{F}/G) = \{(x_1, x_2, \ldots, x_n)$: s.t. $f_1 = 0, f_2 = 0, \ldots, f_m = 0, g \neq 0\}$. There is an algorithm that determines whether $\mathrm{Zero}(\mathcal{F}/G)$ is the empty set after a finite number of steps. If it is not empty, then it can be decomposed in the following form:

$$\mathrm{Zero}(\mathcal{F}/G) = \bigcup \mathrm{Zero}(A_i/R_i),$$

where A_i is an irreducible sequence and R_i is the residual of $J_i G_i$ over A_i, J_i is the product of the primitives of the polynomials in A_i and G_i is some nonzero polynomial.

We consider the case where P and Q are incompatible below: two fixed incompatible PMs $P = \{P_i\}_{i=1}^m$ and $Q = \{Q_j\}_{j=1}^n (2 \leq m, n \leq d)$ generated by two orthonormal bases $e = \{|e_s\rangle\}_{s=1}^d$ and $f = \{|f_t\rangle\}_{t=1}^d$ serving as reference for block coherence. We perform an ascending ordering for these PMs P and Q:

$$P = \{P_i | \mathrm{rank}(P_1) \leq \mathrm{rank}(P_2) \leq \cdots \leq \mathrm{rank}(P_m)\}_{i=1}^m,$$
$$Q = \{Q_j | \mathrm{rank}(Q_1) \leq \mathrm{rank}(Q_2) \leq \cdots \leq \mathrm{rank}(Q_n)\}_{j=1}^n.$$

Theorem 6 Let $P = \{P_i\}_{i=1}^m$ and $Q = \{Q_j\}_{j=1}^n$ be two incompatible PMs. There exists a quantum pure state ρ such that

$$C(\rho|P) + C(\rho|Q) = 2 - \left(\frac{1}{m} + \frac{1}{n}\right)$$

if and only if $m = n$ and $\mathrm{rank}(P_i) = \mathrm{rank}(Q_i)$ for any $i = 1, \ldots, m$.

Proof Let $P = \{P_i\}_{i=1}^m$ and $Q = \{Q_j\}_{j=1}^n$ be two incompatible PMs generated by two orthonormal bases $e = \{|e_s\rangle\}_{s=1}^d$ and $f = \{|f_t\rangle\}_{t=1}^d$. For any pure state $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sum_{s=1}^d a_s |e_s\rangle = \sum_{t=1}^d b_t |f_t\rangle$, $\sum_{s=1}^d |a_s|^2 = 1 = \sum_{t=1}^d |b_t|^2$ and $b_t = \sum_{s=1}^d a_s \langle f_t | e_s\rangle$.

For necessity, suppose that there exists a pure state ρ such that

$$C(\rho|P) + C(\rho|Q) = 2 - \left(\frac{1}{m} + \frac{1}{n}\right),$$

then there exists a solution $\{a_s\}$ to this system of equations:

$$\sum_{s \in \Lambda_{P_i}} |a_s|^2 = \frac{1}{m},\ i \in \{1,2,\ldots,m\},$$

$$\sum_{t\in\Lambda_{Q_j}}|b_t|^2=\frac{1}{n},\ j\in\{1,2,\ldots,n\},\ b_t=\sum_{s=1}^{d}a_s\langle f_t|e_s\rangle. \quad (10)$$

According to Lemma 1, we use the algorithm in Ref. [32] to calculate the above system of equations. Firstly we transform the system (10) into a polynomial set $\mathcal{F}=\{M_1,M_2,\ldots,M_m,N_1,\ldots,N_n\}$ that satisfy the conditions of the algorithm:

$$\begin{aligned}&M_i=\sum_{s\in\Lambda_{P_i}}|a_s|^2-\frac{1}{m}=0,\ i\in\{1,2,\ldots,m\},\\&N_j=\sum_{t\in\Lambda_{Q_j}}|\sum_{s=1}^{d}u_{ts}a_s|^2-\frac{1}{n}=0,\ j\in\{1,2,\ldots,n\},\end{aligned} \quad (11)$$

where $\{u_{ts}=\langle f_t|e_s\rangle,1\le s,t,\le d\}$ is the given set of parameters, Λ_{P_i} is the set of the indexes. Next we feed Eq. (11) directly into the algorithm.[32]

Assume $m<n$, there exists an $i\in[m]$ such that $|\Lambda_{P_i}|\neq|\Lambda_{Q_i}|$. Computing the above algorithm according to this case, we get $\mathrm{Zero}(\mathcal{F})=\emptyset$ and the algorithm terminates, i.e., Eq. (10) has no solution. Therefore, if Eq. (10) is solvable, then $\forall i\in[m]$ with $m\ge n$, it holds $|\Lambda_{P_i}|=|\Lambda_{Q_i}|$. And since $\sum_{i=1}^{m}|\Lambda_{P_i}|=d$, we get $\sum_{i=1}^{m}|\Lambda_{Q_i}|=d$ and $m=n$. We have tested this result by taking all possible values of m and n within the range of $2\le m,n\le d$, for each given d. This means that if Eq. (9) has solutions, then for any $i\in[m]$, $|\Lambda_{P_i}|=|\Lambda_{Q_i}|$, i.e., $\mathrm{rank}(P_i)=\mathrm{rank}(Q_i)(i=1,\ldots,m)$.

For *sufficiency*, let $m=n$ and $\mathrm{rank}(P_i)=\mathrm{rank}(Q_i)(i=1,\ldots,m)$. We use the *ND* algorithm to calculate the system of equations:

$$\begin{aligned}&M_i=\sum_{s\in\Lambda_{P_i}}|a_s|^2-\frac{1}{m}=0,\ i\in\{1,2,\ldots,m\},\\&N_j=\sum_{t\in\Lambda_{Q_j}}|\sum_{s=1}^{d}u_{ts}a_s|^2-\frac{1}{m}=0,\ j\in\{1,2,\ldots,m\}.\end{aligned}$$

The results show that the above system of equations has solutions, which are obtained by taking over all $2\le m\le d$ values for each given d. The form of the solutions to the above equations is more complex, so we will show the specific form of the solutions by means of examples in the following.

Thus, when P,Q are incompatible, there exists a quantum pure state ρ that is maximally block coherent concerning them if and only if $\mathrm{rank}(P_i)=\mathrm{rank}(Q_i)$ $(i=1,\ldots,m)$ if and only if $C(\rho|P)+C(\rho|Q)=2-(\frac{1}{m}+\frac{1}{n})$.

Example 4 Let $e=\{|0\rangle,|1\rangle,|2\rangle,|3\rangle\}$, $f=\{|f_t\rangle\}_{t=0}^{3}$ be two orthonormal bases in the $\mathbb{C}^4$ and P,Q be the incompatible PMs generated by e,f with

$$f=\Big\{|f_0\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle),|f_1\rangle=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle),$$
$$|f_2\rangle=\frac{1}{\sqrt{2}}(|2\rangle+\mathrm{i}|3\rangle),|f_3\rangle=\frac{1}{\sqrt{2}}(|2\rangle-\mathrm{i}|3\rangle)\Big\},$$

and

$$P=\Big\{P_1=|0\rangle\langle0|+|1\rangle\langle1|,\ P_2=|2\rangle\langle2|+|3\rangle\langle3|\Big\},$$
$$Q=\Big\{Q_1=|f_0\rangle\langle f_0|+|f_2\rangle\langle f_2|,\ Q_2=|f_1\rangle\langle f_1|+|f_3\rangle\langle f_3|\Big\}.$$

According to Theorem 6, P,Q are incompatible and $\mathrm{rank}(P_i)=\mathrm{rank}(Q_i)$ $(i=1,2)$, there exists a quantum pure state ρ that is the maximally coherent state with respect to P and Q, i.e., $C(\rho|P)+C(\rho|Q)=1$, for example, $\rho=\frac{1}{2}(|0\rangle\langle0|+|0\rangle\langle2|+|2\rangle\langle0|+|2\rangle\langle2|)$.

4. Conclusions

This paper explores the concept of quantum block coherence in relation to projective measurements. Projective measurements are a crucial tool for investigating quantum block coherence. Initially, we derive the correlation function between two general projective measurements and establish an equivalent theorem for two compatible quantum measurements. The compatibility of projective measurements is an important property that greatly impacts the quantum block coherence with respect to projective measurements. We obtain an inclusion relation between sets of incoherent quantum states with respect to two compatible PMs, and we categorize and discuss the maximal block coherence of the quantum state with respect to compatible and incompatible projective measurements. Our results can be used to determine the block coherence of quantum states based on compatible projective measurements and the existence of maximally block-coherent states based on projective measurements. The present results are a substantial addition to the theory of block coherence.

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