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New construction of mutually unbiased bases for odd-dimensional state space

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We study the construction of mutually unbiased bases in Hilbert space for composite dimensions d which are not prime powers. We explore the results for composite dimensions which are true for prime power dimensions. We then provide a method for selecting mutually unbiased vectors from the eigenvectors of generalized Pauli matrices to construct mutually unbiased bases. In particular, we present four mutually unbiased bases in $\mathbb{C}^{15}$.

Keywords: mutually unbiased bases, Hilbert space, Pauli matrix, quantum measurement

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1. Introduction

Mutually unbiased bases (MUBs), first introduced by Schwinger,[1] have many important applications in the field of quantum information theory; for instance, Bell inequalities,[2] quantum key distribution protocols,[3] quantum state tomography,[4] and non-Gaussianity of quantum states in quantum optics communities.[5] A complete set of MUBs can be used to construct an optimal scheme of orthogonal quantum measurement for the quantum system.[6] Mathematically, this notion arose independently in combinatorial design theory.[7] Recent research has shown that MUBs reveal a profound connection between quantum mechanics theory and many fields in mathematics, such as polynomial optimization problem over reals, Galois rings, group theory, combinatorics, and number theory.[8–10]

Two orthonormal bases in $\mathbb{C}^d$ are called mutually unbiased if the square of the magnitude of the inner products between their vectors is $1/d$. A set of orthonormal basis is called mutually unbiased if every pair of bases in the set is mutually unbiased. Constructing an extremal set of MUBs is still a fundamental problem in quantum information theory. Let $N(d)$ be the maximum number of any set which contains pairwise MUBs of $\mathbb{C}^d$. It is shown that $N(d) = d+1$ holds for prime power dimensions d by explicit construction and $N(d) \leq d+1$ holds for any dimension d.[11–15] It is known that if there exists d MUBs in $\mathbb{C}^d$, then the $(d+1)$-th unbiased basis can be also constructed.[16] If d is not a power of a prime number, then it remains an unsolved question of finding maximal sets of MUBs in $\mathbb{C}^d$. In particular, for the simplest case, $d = 6$, Zauner has given a conjecture that one cannot construct a set of 4 MUBs.[17]

By developing a close relation between maximal commuting bases of orthogonal unitary matrices and extremal sets of MUBs, Bandyopadhyay *et al.* gave a completely different constructional proof of $N(d) = d+1$ for prime power dimensions d.[12] However, many of the theorems in their paper do not hold for non-prime-power d. In this paper, we mainly explore some similar results in the case of non-prime-power d based on Ref. [12] and provide a different construction of MUBs, i.e., we obtain $p_1 + 1$ MUBs for $d = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$. As can be seen from our construction, the maximal number of MUBs is closely associated with the number of prime divisors of the dimension d. It is also implied that two or more prime divisors are of the same difficulty for the construction of MUBs.

The rest of the paper is arranged as follows. In Section 2, we review some notations and previous results. We give a simple result on congruence in elementary number theory, which will be used in the subsequent construction. In Section 3, we provide a new construction of MUBs in composite dimensional space. In Section 4, we consider a specific example and present four MUBs in $\mathbb{C}^{15}$. Finally, we conclude our paper and put forward some relevant questions.

2. Preliminaries

In this section, we recall some basic definitions and notations about quantum computation (i.e., MUBs).

Definition 1[11] Let $\mathcal{B}_1 = \{|\phi_i\rangle\}_{i=0}^{d-1}$ and $\mathcal{B}_2 = \{|\psi_j\rangle\}_{j=0}^{d-1}$ be two orthonormal bases in the state space $\mathbb{C}^d$. $\mathcal{B}_1$ and $\mathcal{B}_2$ are said to be MUBs if

$$\left|\langle\phi_i|\psi_j\rangle\right| = \frac{1}{\sqrt{d}}, \quad \text{for any} \quad i,j = 0,1,\ldots,d.$$

A set of orthonormal bases $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m\}$ in $\mathbb{C}^d$ is called mutually unbiased if $\mathcal{B}_s$ and $\mathcal{B}_t$ are MUBs for any $s \neq t, 1 \leq s \leq m, 1 \leq t \leq m$. For example, in the state space

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$\mathbb{C}^2$, the following orthonormal bases form a set of MUBs:

$$\{|0\rangle,|1\rangle\},$$
$$\left\{\frac{|0\rangle+|1\rangle}{\sqrt{2}},\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right\},$$
$$\left\{\frac{|0\rangle+\mathrm{i}|1\rangle}{\sqrt{2}},\frac{|0\rangle-\mathrm{i}|1\rangle}{\sqrt{2}}\right\}.$$

Now let us consider a simple concept about elementary number theory. Two integers a and b are relatively prime if their positive greatest common divisor is 1 and we use the notation $(a,b)=1$. Denote the set of congruence classes modulo a positive integer d by $\mathbb{Z}/d\mathbb{Z}$.

Definition 2 Let $d\geq 2$ and $f:\mathbb{Z}/d\mathbb{Z}\to\mathbb{Z}/d\mathbb{Z}$ be a map. f is called a complete cyclic shift modulo d if for any $\bar{t}\in\mathbb{Z}/d\mathbb{Z}$, the number of elements of the set $\{\bar{t},f(\bar{t}),f^2(\bar{t}),\ldots,f^{d-1}(\bar{t})\}$ is exactly d, i.e., this set is a complete set of residues modulo d.

According to the theory of congruence classes in elementary number theory, the following lemma is easily proved.

Lemma 1 Let $d>1$ be an integer, and $k\in\mathbb{Z}$. If $(a,b)=1$, then the map $f_k:\mathbb{Z}/d\mathbb{Z}\to\mathbb{Z}/d\mathbb{Z}$ defined by $f_k(\overline{n})=\overline{n+k}$ is a complete cyclic shift modulo d.

In order to derive our main result, we need the core theorem about constructing MUBs for prime dimensions in Ref. [12]. They gave a new constructive proof for the existence of MUBs for which the dimension of state space is prime or power of a prime. We write it down here for the convenience of readers. Many theorems in that paper do not hold in the case that the dimension of quantum system is a product of different primes. However, it is not difficult to check that the following theorem is true for any dimensional quantum systems.

Theorem 1[12] Let $\mathcal{B}_1=\{|\phi_1\rangle,|\phi_2\rangle,\ldots,|\phi_d\rangle\}$ be an orthonormal basis in $\mathbb{C}^d$. Suppose that there is a unitary operator V such that $V|\phi_i\rangle=\beta_i|\phi_{i+1}\rangle$, where $|\beta_i|=1$ and $|\phi_{d+1}\rangle=|\phi_1\rangle$, i.e., V applies a complete cyclic shift modulo d on the vectors of the basis $\mathcal{B}_1$. Assume that the orthonormal basis $\mathcal{B}_2=\{|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_d\rangle\}$ consists of eigenvectors of V. Then, $\mathcal{B}_1$ and $\mathcal{B}_2$ are MUBs.

3. Construction of MUBs for odd dimensions

In this section, we will investigate the construction of MUBs. We mainly study the property of product of generalized Pauli operators. For prime or prime power case, Bandyopadhyay *et al*. have developed a complete theory to construct corresponding MUBs. We shall prove the partial results of other situations, i.e., the composite dimensional system.

Let $\{|0\rangle,|1\rangle,\ldots,|d-1\rangle\}$ be the computational basis of $\mathbb{C}^d$. The generalizations X_d and Z_d of Pauli operators over $\mathbb{C}^d$ are defined as follows:[18]

$$X_d|i\rangle=|i+1\rangle, \tag{1}$$

$$Z_d|i\rangle=\omega^i|i\rangle, \tag{2}$$

where the algebraic operation is modulo d and ω is a primitive d-th root of unity, specifically $\omega=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}}$. For $0\leq n\leq d-1$, denote the unitary operator $X_dZ_d^n$ by V_n. Obviously, $V_n|i\rangle=\omega^{ni}|i+1\rangle$.

Theorem 2 Suppose that $d\geq 3$ is an odd number. Then, the following d vectors:

$$|v_{n,k}\rangle=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}\omega^{\frac{n[(i^2-i)-(d^2-d)]}{2}-ki}|i\rangle\quad(k=0,\ldots,d-1) \tag{3}$$

are eigenvectors of V_n. Furthermore, these vectors form an orthonormal basis for $\mathbb{C}^d$.

Proof Since V_n is a unitary operator, we only need to prove that $|v_{n,k}\rangle$ are eigenvectors of V_n belonging to different eigenvalues.

By definitions of X_d and Z_d, we have

$$\begin{aligned}
V_n|v_{n,k}\rangle &= \frac{1}{\sqrt{d}}X_dZ_d^n\sum_{i=0}^{d-1}\omega^{\frac{n[(i^2-i)-(d^2-d)]}{2}-ki}|i\rangle\\
&= \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}\omega^{\frac{n[(i^2-i)-(d^2-d)]}{2}-ki+ni}|i+1\rangle\\
&= \frac{1}{\sqrt{d}}|1\rangle+\frac{1}{\sqrt{d}}\sum_{i=1}^{d-2}\omega^{\frac{n[(i^2-i)-(d^2-d)]}{2}-ki+ni}|i+1\rangle\\
&\quad+\frac{1}{\sqrt{d}}\omega^k|0\rangle\\
&= \frac{1}{\sqrt{d}}\omega^k|0\rangle+\frac{1}{\sqrt{d}}|1\rangle\\
&\quad+\frac{1}{\sqrt{d}}\omega^k\sum_{i=1}^{d-2}\omega^{\frac{n[((i+1)^2-(i+1))-(d^2-d)]}{2}-k(i+1)}|i+1\rangle\\
&= \frac{1}{\sqrt{d}}\omega^k|0\rangle+\frac{1}{\sqrt{d}}\omega^k\omega^{\frac{n[-(d^2-d)]}{2}-k}|1\rangle\\
&\quad+\frac{1}{\sqrt{d}}\omega^k\sum_{i=2}^{d-1}\omega^{\frac{n[(i^2-i)-(d^2-d)]}{2}-ki}|i\rangle\\
&= \frac{1}{\sqrt{d}}\omega^k|0\rangle+\frac{1}{\sqrt{d}}\omega^k\sum_{i=1}^{d-1}\omega^{\frac{n[(i^2-i)-(d^2-d)]}{2}-ki}|i\rangle\\
&= \frac{1}{\sqrt{d}}\omega^k\sum_{i=0}^{d-1}\omega^{\frac{n[(i^2-i)-(d^2-d)]}{2}-ki}|i\rangle\\
&= \omega^k|v_{n,k}\rangle,
\end{aligned}$$

where we used the condition that d is an odd number for the fourth to fifth equal signs. Therefore, we have proved this theorem.

Similarly, we can prove the following conclusion.

Theorem 3 Suppose that $d\geq 2$ is an even number and $0\leq n\leq d-1$ is also even. Then, the following d vectors:

$$|v_{n,k}\rangle=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}\omega^{\frac{n[(i^2-i)-(d^2-d)]}{2}-ki}|i\rangle\quad(k=0,\ldots,d-1) \tag{4}$$

are eigenvectors of V_n. Furthermore, these vectors form an orthonormal basis for $\mathbb{C}^d$.

Next, let us keep the notation as above. We have the following theorem.

Theorem 4 Suppose that $d \geq 3$ is an odd number and $d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ $(p_1 < p_2 < \cdots < p_r)$ is the prime decomposition. Assume that $0 \leq n, m \leq p_1 - 1$ and $m \neq n$. Then, the eigenvectors $\{|v_{m,k}\rangle\}_{k=0}^{d-1}$ of V_m are cyclically shifted under the action of operator V_n.

Proof At first, we have

$$\begin{aligned}
V_n|v_{m,k}\rangle &= \frac{1}{\sqrt{d}} X_d Z_d^n \sum_{i=0}^{d-1} \omega^{\frac{m[(i^2-i)-(d^2-d)]}{2} - ki} |i\rangle \\
&= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \omega^{\frac{m[(i^2-i)-(d^2-d)]}{2} - ki + ni} |i+1\rangle \\
&= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-2} \omega^{\frac{m[(i^2-i)-(d^2-d)]}{2} + mi - mi - ki + ni} |i+1\rangle \\
&\quad + \frac{1}{\sqrt{d}} \omega^{k+(n-m)(d-1)} |0\rangle \\
&= \frac{1}{\sqrt{d}} \omega^{k+m-n} |0\rangle \\
&\quad + \frac{1}{\sqrt{d}} \sum_{i=0}^{d-2} \omega^{\frac{m[((i+1)^2-(i+1))-(d^2-d)]}{2} - (k+m-n)i} |i+1\rangle \\
&= \frac{1}{\sqrt{d}} \omega^{k+m-n} |0\rangle \\
&\quad + \frac{1}{\sqrt{d}} \omega^{k+m-n} \sum_{i=1}^{d-1} \omega^{\frac{m[(i^2-i)-(d^2-d)]}{2} - (k+m-n)i} |i\rangle \\
&= \frac{1}{\sqrt{d}} \omega^{k+m-n} |0\rangle \\
&\quad + \frac{1}{\sqrt{d}} \omega^{k+m-n} \sum_{i=1}^{d-1} \omega^{\frac{m[(i^2-i)-(d^2-d)]}{2} - (k+m-n)i} |i\rangle \\
&= \frac{1}{\sqrt{d}} \omega^{k+m-n} \sum_{i=0}^{d-1} \omega^{\frac{m[(i^2-i)-(d^2-d)]}{2} - (k+m-n)i} |i\rangle \\
&= \omega^{k+m-n} |v_{m,k+m-n}\rangle,
\end{aligned}$$

where we used the condition that d is an odd number for the fourth to fifth equal signs. Notice that $m-n$ is relatively prime to d. According to Lemma 1, we obtain that V_n applies a cyclic shift on the eigenvectors $\{|v_{m,k}\rangle\}_{k=0}^{d-1}$ of V_m. □

The above theorem is not true when d is an even integer. For example, let $d = 6$. Then, V_0 acts on the eigenvectors $\{|v_{2,k}\rangle\}_{k=0}^{5}$ of V_2 with two loops. That is,

$$v_{2,0} \xrightarrow{V_0} v_{2,2} \xrightarrow{V_0} v_{2,4} \xrightarrow{V_0} v_{2,0}, \tag{5}$$

$$v_{2,1} \xrightarrow{V_0} v_{2,3} \xrightarrow{V_0} v_{2,5} \xrightarrow{V_0} v_{2,1}, \tag{6}$$

where we omit the scalars of the 6th root of unity.

For even integer d, we can just derive the following result.

Theorem 5 Suppose that $d \geq 2$ is an even number. Assume that $0 \leq n, m \leq d-1$, $m \neq n$, and n, m are even. Then for any $0 \leq k \leq d-1$, we have $V_n|v_{m,k}\rangle = \omega^{k+m-n}|v_{m,k+m-n}\rangle$.

Next, it is obvious to obtain the MUBs by the above results when the dimension d is odd. We conclude it in the following theorem. Denote $\mathcal{B}_n = \{|v_{n,0}\rangle, |v_{n,1}\rangle, \ldots, |v_{n,d-1}\rangle\}$.

Theorem 6 Suppose that $d \geq 3$ is an odd number and $d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ $(p_1 < p_2 < \cdots < p_r)$ is the prime decomposition. Then, the p_1 orthonormal bases $\{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{p_1-1}\}$ are mutually unbiased. Together with the standard basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$, these bases form a set of $p_1 + 1$ MUBs.

If $d = p_1$ is a prime number, then by Theorem 6 there exists a set of $d+1$ MUBs. However, according to our construction we cannot obtain better results when the dimension d has at least two different prime divisors.

4. MUBs in $\mathbb{C}^{15}$

The concept of MUBs was first introduced by Schwinger in 1960. It is used to construct measurement bases for quantum information and quantum tomography. The explicit construction of maximal sets of MUBs is obtained only for prime power dimensions. However, the problem of finding (extremal) sets of MUBs in any composite dimensional state space is still open. In this section, we will construct four MUBs in $\mathbb{C}^{15}$ by Section 3.

Let $d = 15$, and let ω be a primitive d-th root of unity, specifically $\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{d}}$. Assume that $\{|i\rangle | i = 0, 1, \ldots, 14\}$ is the standard basis. Next, we shall write down the eigenvectors $|v_{m,k}\rangle_{k=0}^{14}$ of the operators V_m $(m = 0, 1, 2)$ in the form of column vectors

$$|v_{0,0}\rangle = \frac{1}{\sqrt{15}}\big(1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1\big)^{\mathrm{T}},$$

$$|v_{0,1}\rangle = \frac{1}{\sqrt{15}}\big(1,\ \omega^{14},\ \omega^{13},\ \omega^{12},\ \omega^{11},\ \omega^{10},\ \omega^{9},\ \omega^{8},\ \omega^{7},\ \omega^{6},\ \omega^{5},\ \omega^{4},\ \omega^{3},\ \omega^{2},\ \omega\big)^{\mathrm{T}},$$

$$|v_{0,2}\rangle = \frac{1}{\sqrt{15}}\big(1,\ \omega^{13},\ \omega^{11},\ \omega^{9},\ \omega^{7},\ \omega^{5},\ \omega^{3},\ \omega,\ \omega^{14},\ \omega^{12},\ \omega^{10},\ \omega^{8},\ \omega^{6},\ \omega^{4},\ \omega^{2}\big)^{\mathrm{T}},$$

$$|v_{0,3}\rangle = \frac{1}{\sqrt{15}}\big(1,\ \omega^{12},\ \omega^{9},\ \omega^{6},\ \omega^{3},\ 1,\ \omega^{12},\ \omega^{9},\ \omega^{6},\ \omega^{3},\ 1,\ \omega^{12},\ \omega^{9},\ \omega^{6},\ \omega^{3}\big)^{\mathrm{T}},$$

$$|v_{0,4}\rangle = \frac{1}{\sqrt{15}}\big(1,\ \omega^{11},\ \omega^{7},\ \omega^{3},\ \omega^{14},\ \omega^{10},\ \omega^{6},\ \omega^{2},\ \omega^{13},\ \omega^{9},\ \omega^{5},\ \omega,\ \omega^{12},\ \omega^{8},\ \omega^{4}\big)^{\mathrm{T}},$$

$$|v_{0,5}\rangle = \frac{1}{\sqrt{15}}\big(1,\ \omega^{10},\ \omega^{5},\ 1,\ \omega^{10},\ \omega^{5},\ 1,\ \omega^{10},\ \omega^{5},\ 1,\ \omega^{10},\ \omega^{5},\ 1,\ \omega^{10},\ \omega^{5}\big)^{\mathrm{T}},$$

$$|v_{0,6}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{9},\ \omega^{3},\ \omega^{12},\ \omega^{6},\ 1,\ \omega^{9},\ \omega^{3},\ \omega^{12},\ \omega^{6},\ 1,\ \omega^{9},\ \omega^{3},\ \omega^{12},\ \omega^{6}\right)^{\mathrm{T}},$$

$$|v_{0,7}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{8},\ \omega,\ \omega^{9},\ \omega^{2},\ \omega^{10},\ \omega^{3},\ \omega^{11},\ \omega^{4},\ \omega^{12},\ \omega^{5},\ \omega^{13},\ \omega^{6},\ \omega^{14},\ \omega^{7}\right)^{\mathrm{T}},$$

$$|v_{0,8}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{7},\ \omega^{14},\ \omega^{6},\ \omega^{13},\ \omega^{5},\ \omega^{12},\ \omega^{4},\ \omega^{11},\ \omega^{3},\ \omega^{10},\ \omega^{2},\ \omega^{9},\ \omega,\ \omega^{8}\right)^{\mathrm{T}},$$

$$|v_{0,9}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{6},\ \omega^{12},\ \omega^{3},\ \omega^{9},\ 1,\ \omega^{6},\ \omega^{12},\ \omega^{3},\ \omega^{9},\ 1,\ \omega^{6},\ \omega^{12},\ \omega^{3},\ \omega^{9}\right)^{\mathrm{T}},$$

$$|v_{0,10}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{5},\ \omega^{10},\ 1,\ \omega^{5},\ \omega^{10},\ 1,\ \omega^{5},\ \omega^{10},\ 1,\ \omega^{5},\ \omega^{10},\ 1,\ \omega^{5},\ \omega^{10}\right)^{\mathrm{T}},$$

$$|v_{0,11}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{4},\ \omega^{8},\ \omega^{12},\ \omega,\ \omega^{5},\ \omega^{9},\ \omega^{13},\ \omega^{2},\ \omega^{6},\ \omega^{10},\ \omega^{14},\ \omega^{3},\ \omega^{7},\ \omega^{11}\right)^{\mathrm{T}},$$

$$|v_{0,12}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{3},\ \omega^{6},\ \omega^{9},\ \omega^{12},\ 1,\ \omega^{3},\ \omega^{6},\ \omega^{9},\ \omega^{12},\ 1,\ \omega^{3},\ \omega^{6},\ \omega^{9},\ \omega^{12}\right)^{\mathrm{T}},$$

$$|v_{0,13}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{2},\ \omega^{4},\ \omega^{6},\ \omega^{8},\ \omega^{10},\ \omega^{12},\ \omega^{14},\ \omega,\ \omega^{3},\ \omega^{5},\ \omega^{7},\ \omega^{9},\ \omega^{11},\ \omega^{13}\right)^{\mathrm{T}},$$

$$|v_{0,14}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega,\ \omega^{2},\ \omega^{3},\ \omega^{4},\ \omega^{5},\ \omega^{6},\ \omega^{7},\ \omega^{8},\ \omega^{9},\ \omega^{10},\ \omega^{11},\ \omega^{12},\ \omega^{13},\ \omega^{14}\right)^{\mathrm{T}};$$

$$|v_{1,0}\rangle = \frac{1}{\sqrt{15}}\left(1,\ 1,\ \omega,\ \omega^{3},\ \omega^{6},\ \omega^{10},\ 1,\ \omega^{6},\ \omega^{13},\ \omega^{6},\ 1,\ \omega^{10},\ \omega^{6},\ \omega^{3},\ \omega\right)^{\mathrm{T}},$$

$$|v_{1,1}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{14},\ \omega^{14},\ 1,\ \omega^{2},\ \omega^{5},\ \omega^{9},\ \omega^{14},\ \omega^{5},\ \omega^{12},\ \omega^{5},\ \omega^{14},\ \omega^{9},\ \omega^{5},\ \omega^{2}\right)^{\mathrm{T}},$$

$$|v_{1,2}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{13},\ \omega^{12},\ \omega^{12},\ \omega^{13},\ 1,\ \omega^{3},\ \omega^{7},\ \omega^{12},\ \omega^{3},\ \omega^{10},\ \omega^{3},\ \omega^{12},\ \omega^{7},\ \omega^{3}\right)^{\mathrm{T}},$$

$$|v_{1,3}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{12},\ \omega^{10},\ \omega^{9},\ \omega^{9},\ \omega^{10},\ \omega^{12},\ 1,\ \omega^{4},\ \omega^{9},\ 1,\ \omega^{7},\ 1,\ \omega^{9},\ \omega^{4}\right)^{\mathrm{T}},$$

$$|v_{1,4}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{11},\ \omega^{8},\ \omega^{6},\ \omega^{5},\ \omega^{5},\ \omega^{6},\ \omega^{8},\ \omega^{11},\ 1,\ \omega^{5},\ \omega^{11},\ \omega^{3},\ \omega^{11},\ \omega^{5}\right)^{\mathrm{T}},$$

$$|v_{1,5}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{10},\ \omega^{6},\ \omega^{3},\ \omega,\ 1,\ 1,\ \omega,\ \omega^{3},\ \omega^{6},\ \omega^{10},\ 1,\ \omega^{6},\ \omega^{13},\ \omega^{6}\right)^{\mathrm{T}},$$

$$|v_{1,6}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{9},\ \omega^{4},\ 1,\ \omega^{12},\ \omega^{10},\ \omega^{9},\ \omega^{9},\ \omega^{10},\ \omega^{12},\ 1,\ \omega^{4},\ \omega^{9},\ 1,\ \omega^{7}\right)^{\mathrm{T}},$$

$$|v_{1,7}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{8},\ \omega^{2},\ \omega^{12},\ \omega^{8},\ \omega^{5},\ \omega^{3},\ \omega^{2},\ \omega^{2},\ \omega^{3},\ \omega^{5},\ \omega^{8},\ \omega^{12},\ \omega^{2},\ \omega^{8}\right)^{\mathrm{T}},$$

$$|v_{1,8}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{7},\ 1,\ \omega^{9},\ \omega^{4},\ 1,\ \omega^{12},\ \omega^{10},\ \omega^{9},\ \omega^{9},\ \omega^{10},\ \omega^{12},\ 1,\ \omega^{4},\ \omega^{9}\right)^{\mathrm{T}},$$

$$|v_{1,9}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{6},\ \omega^{13},\ \omega^{6},\ 1,\ \omega^{10},\ \omega^{6},\ \omega^{3},\ \omega,\ 1,\ 1,\ \omega,\ \omega^{3},\ \omega^{6},\ \omega^{10}\right)^{\mathrm{T}},$$

$$|v_{1,10}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{5},\ \omega^{11},\ \omega^{3},\ \omega^{11},\ \omega^{5},\ 1,\ \omega^{11},\ \omega^{8},\ \omega^{6},\ \omega^{5},\ \omega^{5},\ \omega^{6},\ \omega^{8},\ \omega^{11}\right)^{\mathrm{T}},$$

$$|v_{1,11}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{4},\ \omega^{9},\ 1,\ \omega^{7},\ 1,\ \omega^{9},\ \omega^{4},\ 1,\ \omega^{12},\ \omega^{10},\ \omega^{9},\ \omega^{9},\ \omega^{10},\ \omega^{12}\right)^{\mathrm{T}},$$

$$|v_{1,12}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{3},\ \omega^{7},\ \omega^{12},\ \omega^{3},\ \omega^{10},\ \omega^{3},\ \omega^{12},\ \omega^{7},\ \omega^{3},\ 1,\ \omega^{13},\ \omega^{12},\ \omega^{12},\ \omega^{13}\right)^{\mathrm{T}},$$

$$|v_{1,13}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{2},\ \omega^{5},\ \omega^{9},\ \omega^{14},\ \omega^{5},\ \omega^{12},\ \omega^{5},\ \omega 14,\ \omega^{9},\ \omega^{5},\ \omega^{2},\ 1,\ \omega^{14},\ \omega^{14}\right)^{\mathrm{T}},$$

$$|v_{1,14}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega,\ \omega^{3},\ \omega^{6},\ \omega^{10},\ 1,\ \omega^{6},\ \omega^{13},\ \omega^{6},\ 1,\ \omega^{10},\ \omega^{6},\ \omega^{3},\ \omega,\ 1\right)^{\mathrm{T}};$$

$$|v_{2,0}\rangle = \frac{1}{\sqrt{15}}\left(1,\ 1,\ \omega^{2},\ \omega^{6},\ \omega^{12},\ \omega^{5},\ 1,\ \omega^{12},\ \omega^{11},\ \omega^{12},\ 1,\ \omega^{5},\ \omega^{12},\ \omega^{6},\ \omega^{2}\right)^{\mathrm{T}},$$

$$|v_{2,1}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{14},\ 1,\ \omega^{3},\ \omega^{8},\ 1,\ \omega^{9},\ \omega^{5},\ \omega^{3},\ \omega^{3},\ \omega^{5},\ \omega^{9},\ 1,\ \omega^{8},\ \omega^{3}\right)^{\mathrm{T}},$$

$$|v_{2,2}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{13},\ \omega^{13},\ 1,\ \omega^{4},\ \omega^{10},\ \omega^{3},\ \omega^{13},\ \omega^{10},\ \omega^{9},\ \omega^{10},\ \omega^{13},\ \omega^{3},\ \omega^{10},\ \omega^{4}\right)^{\mathrm{T}},$$

$$|v_{2,3}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{12},\ \omega^{11},\ \omega^{12},\ 1,\ \omega^{5},\ \omega^{12},\ \omega^{6},\ \omega^{2},\ 1,\ 1,\ \omega^{2},\ \omega^{6},\ \omega^{12},\ \omega^{5}\right)^{\mathrm{T}},$$

$$|v_{2,4}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{11},\ \omega^{9},\ \omega^{9},\ \omega^{11},\ 1,\ \omega^{6},\ \omega^{14},\ \omega^{9},\ \omega^{6},\ \omega^{5},\ \omega^{6},\ \omega^{9},\ \omega^{14},\ \omega^{6}\right)^{\mathrm{T}},$$

$$|v_{2,5}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{10},\ \omega^{7},\ \omega^{6},\ \omega^{7},\ \omega^{10},\ 1,\ \omega^{7},\ \omega,\ \omega^{12},\ \omega^{10},\ \omega^{10},\ \omega^{12},\ \omega,\ \omega^{7}\right)^{\mathrm{T}},$$

$$|v_{2,6}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{9},\ \omega^{5},\ \omega^{3},\ \omega^{3},\ \omega^{5},\ \omega^{9},\ 1,\ \omega^{8},\ \omega^{3},\ 1,\ \omega^{14},\ 1,\ \omega^{3},\ \omega^{8}\right)^{\mathrm{T}},$$

$$|v_{2,7}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{8},\ \omega^{3},\ 1,\ \omega^{14},\ 1,\ \omega^{3},\ \omega^{8},\ 1,\ \omega^{9},\ \omega^{5},\ \omega^{3},\ \omega^{3},\ \omega^{5},\ \omega^{9}\right)^{\mathrm{T}},$$

$$|v_{2,8}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{7},\ \omega,\ \omega^{12},\ \omega^{10},\ \omega^{10},\ \omega^{12},\ \omega,\ \omega^{7},\ 1,\ \omega^{10},\ \omega^{7},\ \omega^{6},\ \omega^{7},\ \omega^{10}\right)^{\mathrm{T}},$$

$$|v_{2,9}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{6},\ \omega^{14},\ \omega^{9},\ \omega^{6},\ \omega^{5},\ \omega^{6},\ \omega^{9},\ \omega^{14},\ \omega^{6},\ 1,\ \omega^{11},\ \omega^{9},\ \omega^{9},\ \omega^{11}\right)^{\mathrm{T}},$$

$$|v_{2,10}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{5},\ \omega^{12},\ \omega^{6},\ \omega^{2},\ 1,\ 1,\ \omega^{2},\ \omega^{6},\ \omega^{12},\ \omega^{5},\ 1,\ \omega^{12},\ \omega^{11},\ \omega^{12}\right)^{\mathrm{T}},$$

$$|v_{2,11}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{4},\ \omega^{10},\ \omega^{3},\ \omega^{13},\ \omega^{10},\ \omega^{9},\ \omega^{10},\ \omega^{13},\ \omega^{3},\ \omega^{10},\ \omega^{4},\ 1,\ \omega^{13},\ \omega^{13}\right)^{\mathrm{T}},$$

$$|v_{2,12}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{3},\ \omega^{8},\ 1,\ \omega^{9},\ \omega^{5},\ \omega^{3},\ \omega^{3},\ \omega^{5},\ \omega^{9},\ 1,\ \omega^{8},\ \omega^{3},\ 1,\ \omega^{14}\right)^{\mathrm{T}},$$

$$|v_{2,13}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega^{2},\ \omega^{6},\ \omega^{12},\ \omega^{5},\ 1,\ \omega^{12},\ \omega^{11},\ \omega^{12},\ 1,\ \omega^{5},\ \omega^{12},\ \omega^{6},\ \omega^{2},\ 1\right)^{\mathrm{T}},$$

$$|v_{2,14}\rangle = \frac{1}{\sqrt{15}}\left(1,\ \omega,\ \omega^{4},\ \omega^{9},\ \omega,\ \omega^{10},\ \omega^{6},\ \omega^{4},\ \omega^{4},\ \omega^{6},\ \omega^{10},\ \omega,\ \omega^{9},\ \omega^{4},\ \omega\right)^{\mathrm{T}}.$$

Now it is straightforward to check that $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2$ and the standard basis $\{|0\rangle, |1\rangle, \ldots, |14\rangle\}$ form a set of four MUBs for $\mathbb{C}^{15}$.

5. Conclusion

In this paper, we have mainly studied a new construction of MUBs in composite dimensional systems. For $d = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, we obtained $p_1 + 1$ new MUBs. For conclusions that hold for prime or prime power dimensions, we considered the similar results when the dimension d is a product of at least two different primes. We have proved conclusions that are still valid and explained which conclusions are invalid. Finally, we constructed a portion of MUBs in composite dimensions. Specifically, we provided an example of MUBs in $\mathbb{C}^{15}$. Combining the generalized Pauli matrices with tensor algebra will yield further results about MUBs in composite dimensions.

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