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An integrable generalization of the Fokas–Lenells equation: Darboux transformation, reduction and explicit soliton solutions

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Under investigation is an integrable generalization of the Fokas–Lenells equation, which can be derived from the negative power flow of a 2×2 matrix spectral problem with three potentials. Based on the gauge transformation of the matrix spectral problem, one kind of Darboux transformation with multi-parameters for the three-component coupled Fokas–Lenells system is constructed. As a reduction, the N-fold Darboux transformation for the generalized Fokas–Lenells equation is obtained, from which the N-soliton solution in a compact Vandermonde-like determinant form is given. Particularly, the explicit one- and two-soliton solutions are presented and their dynamical behaviors are shown graphically.

Keywords: Darboux transformation, soliton solutions, generalized Fokas–Lenells equation

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1. Introduction

The nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t+\nu u_{xx}+\sigma|u|^2u=0, \tag{1}$$

in which ν is a real parameter and $\sigma=\pm1$, plays a significant role to model the propagations of light pulses in nonlinear optical fibers.[1] The complete integrability as well as the explicit soliton, breather and rogue wave solutions of Eq. (1) have been a topic of general interest in multidisciplinary research fields over the past half century.[2–5] Considering the diverse physical environments, some generalized NLS equations which involve higher-order perturbation terms such as the higher-order dispersion, self-steepening and Raman scattering effects should be taken into account.[6,7]

The Fokas–Lenells (FL) equation

$$\mathrm{i}u_t-\iota u_{xt}+\nu u_{xx}+\sigma|u|^2(u+\mathrm{i}\iota u_x)=0, \tag{2}$$

as a generalization of the NLS equation, where ι and ν are real parameters, $u(x,t)$ represents a complex-valued function, was initially proposed by Fokas with the aid of the bi-Hamiltonian formulation.[8] It is apparently seen that when $\iota\to0$, namely, the mixed derivative term u_{xt} and the self-steepening effect $|u|^2u_x$ disappear, equation (2) reduces to the standard NLS equation (1). As was pointed out in Ref. [9], equation (2) has a close relationship with the derivative NLS equation

$$\mathrm{i}u_t+u_{xx}+\mathrm{i}\sigma(|u|^2u)_x=0. \tag{3}$$

Actually, through the gauge transformation

$$u\to\sqrt{\frac{\nu}{\iota}}\frac{1}{\iota}\exp\left[\mathrm{i}\left(\frac{1}{\iota}x+2\frac{\nu}{\iota^2}t\right)\right]u,\quad \frac{\nu}{\iota}>0, \tag{4}$$

equation (2) becomes

$$u_{xt}-\frac{\nu}{\iota}u_{xx}=\frac{\nu}{\iota^3}(-u+\mathrm{i}\sigma|u|^2u_x). \tag{5}$$

By introducing the new variables

$$\xi=x+\frac{\nu}{\iota}t,\quad \tau=-\frac{\nu}{\iota^3}t, \tag{6}$$

one has

$$u_{\tau\xi}=u-\mathrm{i}\sigma|u|^2u_\xi, \tag{7}$$

which is nothing but the first negative power flow of derivative NLS hierarchy. For $u\to\sqrt{2}u$, $\xi=t$, $\tau=-x$, $\sigma=1$, equation (7) can be rewritten as a more common form

$$u_{xt}=2\mathrm{i}|u|^2u_t-u. \tag{8}$$

The study of the FL equation (2) or its equivalent form Eq. (8) has attracted increasing attention during the past decade. The Lax pair formulation, conservation laws and inverse scattering transform for solving the initial value problem were presented in Ref. [10]. The solution on the half-line for the initial–boundary value problem was expressed in terms of a matrix-valued Riemann–Hilbert (RH) problem.[11] The explicit formulas of soliton, breather and rogue wave solutions were obtained by the dressing method,[12] the Hirota's bilinear method[13–16] and the Darboux transformation (DT).[17,18] Recently, the long-time asymptotics for the FL equation with decaying initial value problem without solitons and the global well-posedness for the FL equation on the line with solitons were established by the RH approach.[19,20] Additionally, some generalizations of the FL equation such as the nonholonomic deformed FL equation and the multi-component coupled FL equation were also investigated by many authors.[21–25]

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It is well known that searching for new integrable equations that possess N-solution solutions is one of the most significant subjects in the theory of integrable system.[26–31] In this paper, we propose a novel generalized FL equation

$$\begin{aligned} &u_{xt} = \mathrm{i}(2|u|^2 u_t + \mu u_x + 2u_t w) - u, \\ &\mathrm{i}w_t = \mu(u_x u_t^* - u_t u_x^*), \end{aligned} \tag{9}$$

where μ is a real parameter and $w(x,t)$ is an additional real-valued function compared with the original FL equation (8). One can readily find that when $w=0$ and $\mu=0$, the linear term μu_x and the nonlinear term $2u_t w$ in the first equation of Eq. (9) as well as the second equation of Eq. (9) all vanish, then Eq. (9) degenerates to the standard FL equation (8).

In this paper, we demonstrate that equation (9) is a special reduction of a three-component coupled FL system, which is the first negative member of a 2×2 matrix spectral problem with three potentials.

Among several systematic methods that have been developed to construct explicit solutions of integrable equations such as the inverse scattering transformation,[2] Hirota's bilinear method,[32] DT,[33–37] algebra-geometric method[38,39] and RH approach,[40–42] the DT method possesses many advantages compared with others. Firstly, it doesn't require to do the inverse spectral analysis and provides a direct way to solve the corresponding Lax pair algebraically. Furthermore, by starting with a trivial seed solution, DT can be performed again to produce another set of new solutions. This process can be repeated and will yield a series of new explicit solutions.

The aim of this paper is to construct the N-fold DT for Eq. (9), and then present the N-soliton solution in a compact Vandermonde-like determinant form. The outline of our paper is as follows. In Section 2, we introduce a gauge transformation, which is rigorously proved to be a DT with multi-parameters for the three-component coupled FL system. In Section 3, through the reduction technique, the N-fold DT for Eq. (9) is derived. The N-soliton solution under trivial seed solution in a compact Vandermonde-like determinant representation is put forward. In particular, the one- and two-soliton solutions of Eq. (9) are explicitly given, and the corresponding dynamical behaviors are shown graphically. Finally, we give the conclusion.

2. Darboux transformation

In this section, we shall construct the DT with multi-parameters for the three-component coupled FL system

$$\begin{aligned} &u_{xt} = 2\mathrm{i}uvu_t + 2\mathrm{i}u_t w + \mathrm{i}\mu u_x - u, \\ &v_{xt} = -2\mathrm{i}uvv_t - 2\mathrm{i}v_t w - \mathrm{i}\mu v_x - v, \\ &\mathrm{i}w_t = \mu(u_x v_t - u_t v_x), \end{aligned} \tag{10}$$

which can be reproduced from the compatibility condition $U_t - V_x + [U,V] = 0$ of the matrix spectral problems

$$\begin{aligned} &y_x = Uy, \\ &U = \frac{1}{1-\mu\lambda}\begin{pmatrix} \dfrac{\mathrm{i}\lambda}{2} + \mathrm{i}uv + \mathrm{i}w & -\mathrm{i}u - \mu u_x \\ \lambda(\mathrm{i}v - \mu v_x) & -\dfrac{\mathrm{i}\lambda}{2} - \mathrm{i}uv - \mathrm{i}w \end{pmatrix}, \end{aligned} \tag{11}$$

$$y_t = Vy,\ \ V = \begin{pmatrix} \dfrac{\mathrm{i}\lambda^{-1}}{2} & \lambda^{-1}u_t \\ v_t & -\dfrac{\mathrm{i}\lambda^{-1}}{2} \end{pmatrix}, \tag{12}$$

where $y=(y_1,y_2)^{\mathrm{T}}$, $u(x,t)$ and $v(x,t)$ are two complex-valued potentials, $w(x,t)$ is a real-valued potential and λ is a constant spectral parameter. When $v=u^*$, equation (10) is reduced to the generalized FL equation (9). Now, we introduce a gauge transformation of the matrix spectral problems (11) and (12)

$$\widehat{y} = Ty,\ \ T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned} &A = \lambda^N + \sum_{k=0}^{N-1} A_k \lambda^k,\ \ B = \sum_{k=0}^{N-1} B_k \lambda^k, \\ &C = \sum_{k=1}^{N} C_k \lambda^k,\ \ D = \lambda^N + \sum_{k=0}^{N-1} D_k \lambda^k. \end{aligned} \tag{14}$$

In order to construct a DT for Eq. (10), we assume that $\phi = (\phi_1,\phi_2)^{\mathrm{T}}$, $\psi = (\psi_1,\psi_2)^{\mathrm{T}}$ are two basic solutions of Eqs. (11) and (12). Then we introduce the following linear algebraic system:

$$\begin{aligned} &\sum_{k=0}^{N-1}(A_k + \alpha_j B_k)\lambda_j^k = -\lambda_j^N,\ \ j=1,\ldots,2N, \\ &\sum_{k=1}^{N} C_k \lambda_j^k + \sum_{k=0}^{N-1} \alpha_j D_k \lambda_j^k = -\alpha_j \lambda_j^N,\ \ j=1,\ldots,2N, \end{aligned} \tag{15}$$

where

$$\alpha_j = \frac{\phi_2(\lambda_j) - \gamma_j \psi_2(\lambda_j)}{\phi_1(\lambda_j) - \gamma_j \psi_1(\lambda_j)},\ \ j=1,\ldots,2N, \tag{16}$$

in which the constants λ_j and γ_j ($\lambda_j \neq 0$, $\lambda_j \neq 1/\mu$, $\lambda_j \neq \lambda_k$, $\gamma_j \neq \gamma_k$, $j \neq k$) are suitably chosen such that the coefficients in system (15) are nonzero. Hence, A_k, B_k, D_k ($k = 0,\ldots,N-1$) and C_k ($k=1,\ldots,N$) are uniquely determined by the linear algebraic system (15). At this point, it is not difficult to verify that $\det T(\lambda)$ is a $(2N)$th-order polynomial and

$$\begin{aligned} \det T(\lambda_j) = &\left(\lambda_j^N + \sum_{k=0}^{N-1} A_k \lambda_j^k\right)\left(\lambda_j^N + \sum_{k=0}^{N-1} D_k \lambda_j^k\right) \\ &- \sum_{k=0}^{N-1}\sum_{l=1}^{N} B_k C_l \lambda_j^{k+l} = 0, \end{aligned} \tag{17}$$

which implies that λ_j $(j=1,\ldots,2N)$ are $2N$ roots of the $(2N)$th-order polynomial $\det T$. Therefore, $\det T$ can be rewritten in the form of the finite products as

$$\det T=\prod_{j=1}^{2N}(\lambda-\lambda_j). \tag{18}$$

Under the gauge transformation (13), the spectral problem (11) is transformed into the following new matrix spectral problem with respect to $\widehat{y}$:

$$\widehat{y}_x=\widehat{U}\widehat{y}, \tag{19}$$

where

$$\widehat{U}=(T_x+TU)T^{-1}. \tag{20}$$

It turns out that $\lambda=\lambda_j$ $(j=1,\ldots,2N)$ are removable isolated singularities of $\widehat{U}$ (see Eq. (21) below). Thus, we can define $\widehat{U}$ for all λ by analytic continuation.

Proposition 1 The matrix $\widehat{U}$ determined by Eq. (20) has the same form as U

$$\widehat{U}=\frac{1}{1-\mu\lambda}\begin{pmatrix}\dfrac{\mathrm{i}\lambda}{2}+\mathrm{i}\widehat{u}\widehat{v}+\mathrm{i}\widehat{w} & -\mathrm{i}\widehat{u}-\mu\widehat{u}_x\\ \lambda(\mathrm{i}\widehat{v}-\mu\widehat{v}_x) & -\dfrac{\mathrm{i}\lambda}{2}-\mathrm{i}\widehat{u}\widehat{v}-\mathrm{i}\widehat{w}\end{pmatrix}, \tag{21}$$

where the transformation formulae from the old potentials u, v and w into the new ones are given by

$$\begin{aligned}&\widehat{u}=u+B_{N-1},\ \ \widehat{v}=v+C_N,\\ &\widehat{w}=w+\mathrm{i}\mu(A_{N-1,x}+v_xB_{N-1}-u_xC_N-B_{N-1,x}C_N),\end{aligned} \tag{22}$$

where

$$A_{N-1}=\frac{\Delta_1^{(1)}}{\Delta_1},\ \ B_{N-1}=\frac{\Delta_1^{(2)}}{\Delta_1},\ \ C_N=\frac{\Delta_2^{(1)}}{\Delta_2}, \tag{23}$$

with

$$\Delta_1=\begin{vmatrix}1&\lambda_1&\cdots&\lambda_1^{N-2}&\lambda_1^{N-1}&\alpha_1&\alpha_1\lambda_1&\cdots&\alpha_1\lambda_1^{N-2}&\alpha_1\lambda_1^{N-1}\\ 1&\lambda_2&\cdots&\lambda_2^{N-2}&\lambda_2^{N-1}&\alpha_2&\alpha_2\lambda_2&\cdots&\alpha_2\lambda_2^{N-2}&\alpha_2\lambda_2^{N-1}\\ \vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 1&\lambda_{2N-1}&\cdots&\lambda_{2N-1}^{N-2}&\lambda_{2N-1}^{N-1}&\alpha_{2N-1}&\alpha_{2N-1}\lambda_{2N-1}&\cdots&\alpha_{2N-1}\lambda_{2N-1}^{N-2}&\alpha_{2N-1}\lambda_{2N-1}^{N-1}\\ 1&\lambda_{2N}&\cdots&\lambda_{2N}^{N-2}&\lambda_{2N}^{N-1}&\alpha_{2N}&\alpha_{2N}\lambda_{2N}&\cdots&\alpha_{2N}\lambda_{2N}^{N-2}&\alpha_{2N}\lambda_{2N}^{N-1}\end{vmatrix}, \tag{24}$$

$$\Delta_2=\begin{vmatrix}\lambda_1&\lambda_1^2&\cdots&\lambda_1^{N-1}&\lambda_1^N&\alpha_1&\alpha_1\lambda_1&\cdots&\alpha_1\lambda_1^{N-2}&\alpha_1\lambda_1^{N-1}\\ \lambda_2&\lambda_2^2&\cdots&\lambda_2^{N-1}&\lambda_2^N&\alpha_2&\alpha_2\lambda_2&\cdots&\alpha_2\lambda_2^{N-2}&\alpha_2\lambda_2^{N-1}\\ \vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ \lambda_{2N-1}&\lambda_{2N-1}^2&\cdots&\lambda_{2N-1}^{N-1}&\lambda_{2N-1}^N&\alpha_{2N-1}&\alpha_{2N-1}\lambda_{2N-1}&\cdots&\alpha_{2N-1}\lambda_{2N-1}^{N-2}&\alpha_{2N-1}\lambda_{2N-1}^{N-1}\\ \lambda_{2N}&\lambda_{2N}^2&\cdots&\lambda_{2N}^{N-1}&\lambda_{2N}^N&\alpha_{2N}&\alpha_{2N}\lambda_{2N}&\cdots&\alpha_{2N}\lambda_{2N}^{N-2}&\alpha_{2N}\lambda_{2N}^{N-1}\end{vmatrix}, \tag{25}$$

$$\Delta_1^{(1)}=\begin{vmatrix}1&\lambda_1&\cdots&\lambda_1^{N-2}&-\lambda_1^N&\alpha_1&\alpha_1\lambda_1&\cdots&\alpha_1\lambda_1^{N-2}&\alpha_1\lambda_1^{N-1}\\ 1&\lambda_2&\cdots&\lambda_2^{N-2}&-\lambda_2^N&\alpha_2&\alpha_2\lambda_2&\cdots&\alpha_2\lambda_2^{N-2}&\alpha_2\lambda_2^{N-1}\\ \vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 1&\lambda_{2N-1}&\cdots&\lambda_{2N-1}^{N-2}&-\lambda_{2N-1}^N&\alpha_{2N-1}&\alpha_{2N-1}\lambda_{2N-1}&\cdots&\alpha_{2N-1}\lambda_{2N-1}^{N-2}&\alpha_{2N-1}\lambda_{2N-1}^{N-1}\\ 1&\lambda_{2N}&\cdots&\lambda_{2N}^{N-2}&-\lambda_{2N}^N&\alpha_{2N}&\alpha_{2N}\lambda_{2N}&\cdots&\alpha_{2N}\lambda_{2N}^{N-2}&\alpha_{2N}\lambda_{2N}^{N-1}\end{vmatrix}, \tag{26}$$

$$\Delta_1^{(2)}=\begin{vmatrix}1&\lambda_1&\cdots&\lambda_1^{N-2}&\lambda_1^{N-1}&\alpha_1&\alpha_1\lambda_1&\cdots&\alpha_1\lambda_1^{N-2}&-\lambda_1^N\\ 1&\lambda_2&\cdots&\lambda_2^{N-2}&\lambda_2^{N-1}&\alpha_2&\alpha_2\lambda_2&\cdots&\alpha_2\lambda_2^{N-2}&-\lambda_2^N\\ \vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 1&\lambda_{2N-1}&\cdots&\lambda_{2N-1}^{N-2}&\lambda_{2N-1}^{N-1}&\alpha_{2N-1}&\alpha_{2N-1}\lambda_{2N-1}&\cdots&\alpha_{2N-1}\lambda_{2N-1}^{N-2}&-\lambda_{2N-1}^N\\ 1&\lambda_{2N}&\cdots&\lambda_{2N}^{N-2}&\lambda_{2N}^{N-1}&\alpha_{2N}&\alpha_{2N}\lambda_{2N}&\cdots&\alpha_{2N}\lambda_{2N}^{N-2}&-\lambda_{2N}^N\end{vmatrix}, \tag{27}$$

$$\Delta_2^{(1)}=\begin{vmatrix}\lambda_1&\lambda_1^2&\cdots&\lambda_1^{N-1}&-\alpha_1\lambda_1^N&\alpha_1&\alpha_1\lambda_1&\cdots&\alpha_1\lambda_1^{N-2}&\alpha_1\lambda_1^{N-1}\\ \lambda_2&\lambda_2^2&\cdots&\lambda_2^{N-1}&-\alpha_2\lambda_2^N&\alpha_2&\alpha_2\lambda_2&\cdots&\alpha_2\lambda_2^{N-2}&\alpha_2\lambda_2^{N-1}\\ \vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ \lambda_{2N-1}&\lambda_{2N-1}^2&\cdots&\lambda_{2N-1}^{N-1}&-\alpha_{2N-1}\lambda_{2N-1}^N&\alpha_{2N-1}&\alpha_{2N-1}\lambda_{2N-1}&\cdots&\alpha_{2N-1}\lambda_{2N-1}^{N-2}&\alpha_{2N-1}\lambda_{2N-1}^{N-1}\\ \lambda_{2N}&\lambda_{2N}^2&\cdots&\lambda_{2N}^{N-1}&-\alpha_{2N}\lambda_{2N}^N&\alpha_{2N}&\alpha_{2N}\lambda_{2N}&\cdots&\alpha_{2N}\lambda_{2N}^{N-2}&\alpha_{2N}\lambda_{2N}^{N-1}\end{vmatrix}. \tag{28}$$

Proof Let $T^{-1}=T^*/\det T$ and

$$(1-\mu\lambda)(T_x+TU)T^*=\begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda)\\ f_{21}(\lambda) & f_{22}(\lambda)\end{pmatrix}. \tag{29}$$

Here T^* stands for the adjoint matrix of T. A direct calculation shows that $f_{11}(\lambda)$, $f_{21}(\lambda)$, $f_{22}(\lambda)$ are $(2N+1)$th-order polynomials in λ and $f_{12}(\lambda)$ is a $(2N)$th-order polynomial in λ. With the aid of Eqs. (11), (15), and (16), we have

$$\begin{aligned}&(1-\mu\lambda_j)\alpha_{jx}=\lambda_j(\mathrm{i}v-\mu v_x)-(\mathrm{i}\lambda_j+2\mathrm{i}uv+2\mathrm{i}w)\alpha_j\\&\qquad+(\mathrm{i}u+\mu u_x)\alpha_j^2,\end{aligned} \tag{30}$$

$$\begin{aligned}&A_x(\lambda_j)=-\alpha_{jx}B(\lambda_j)-\alpha_jB_x(\lambda_j),\\&C_x(\lambda_j)=-\alpha_{jx}D(\lambda_j)-\alpha_jD_x(\lambda_j),\end{aligned} \tag{31}$$

from which it is not difficult to verify that λ_j $(j=1,\ldots,2N)$ are roots of f_{kl} $(k,l=1,2)$ in Eq. (29). Therefore, according to Eq. (18) we obtain

$$(1-\mu\lambda)(T_x+TU)T^*=(\det T)P(\lambda), \tag{32}$$

where

$$P(\lambda)=\begin{pmatrix} p_{11}^{(1)}\lambda+p_{11}^{(0)} & p_{12}^{(0)}\\ p_{21}^{(1)}\lambda+p_{21}^{(0)} & p_{22}^{(1)}\lambda+p_{22}^{(0)}\end{pmatrix}, \tag{33}$$

with $p_{kl}^{(0)}$ and $p_{kl}^{(1)}$ being independent of λ. Equation (32) can be rewritten as

$$(1-\mu\lambda)(T_x+TU)=P(\lambda)T. \tag{34}$$

By comparing the coefficients of λ^{N+1},λ^N and λ^0 in Eq. (32), we have

$$\begin{aligned}&p_{11}^{(1)}=\frac{\mathrm{i}}{2},\ \ p_{22}^{(1)}=-\frac{\mathrm{i}}{2},\ \ p_{21}^{(0)}=0,\\&p_{21}^{(1)}=\mathrm{i}(v+C_N)-\mu(v_x+C_{N,x}),\\&p_{12}^{(0)}=-\mathrm{i}(u+B_{N-1})-\mu(u_x+B_{N-1,x}),\\&p_{11}^{(0)}=\mathrm{i}(u+B_{N-1})(v+C_N)+\mathrm{i}w\\&\qquad-\mu(A_{N-1,x}+v_xB_{N-1}-u_xC_N-B_{N-1,x}C_N),\\&p_{22}^{(0)}=-\mathrm{i}(u+B_{N-1})(v+C_N)-\mathrm{i}w\\&\qquad-\mu(D_{N-1,x}+u_xC_N-v_xB_{N-1}-B_{N-1}C_{N,x}).\end{aligned} \tag{35}$$

Besides, comparing the coefficients of λ^{2N-1} of $\det T(\lambda)$ in Eqs. (13) and (18), one obtains

$$A_{N-1}+D_{N-1}-B_{N-1}C_N=-\sum_{j=1}^{2N}\lambda_j=\text{const}. \tag{36}$$

We thus have

$$(A_{N-1}+D_{N-1}-B_{N-1}C_N)_x=0. \tag{37}$$

Substituting Eqs. (22) and (37) into Eq. (35) leads to

$$\begin{aligned}&p_{21}^{(1)}=\mathrm{i}\widehat{v}-\mu\widehat{v}_x,\ \ p_{12}^{(0)}=-\mathrm{i}\widehat{u}-\mu\widehat{u}_x,\\&p_{11}^{(0)}=-p_{22}^{(0)}=\mathrm{i}\widehat{u}\widehat{v}+\mathrm{i}\widehat{w}.\end{aligned} \tag{38}$$

Moreover, by solving the linear algebraic system (15), one can obtain the explicit expressions of A_{N-1}, B_{N-1} and C_N given by Eq. (23). This completes the proof.

On the other hand, differentiating $\widehat{y}=Ty$ with respect to t, the auxiliary spectral problem (12) is transformed into the following auxiliary problem:

$$\widehat{y}_t=\widehat{V}\widehat{y},\quad \widehat{V}=(T_t+TV)T^{-1}. \tag{39}$$

In a similar way to proposition 1, we can prove the following assertion.

Proposition 2 The matrix $\widehat{V}$ defined by the second expression of Eq. (39) has the same form as V, in which the old potentials u and v are mapped into $\widehat{u}$ and $\widehat{v}$ according to the same DT (22).

Proof Denote

$$(T_t+TV)T^*=\begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda)\\ g_{21}(\lambda) & g_{22}(\lambda)\end{pmatrix}. \tag{40}$$

Then a direct calculation shows that $\lambda g_{11}(\lambda)$, $\lambda g_{12}(\lambda)$, $\lambda g_{22}(\lambda)$ are $(2N)$th-order polynomials in λ and $g_{12}(\lambda)$ is a $(2N)$th-order polynomial in λ. Resorting to Eqs. (12), (15), and (16), one has

$$\begin{aligned}&\alpha_{jt}=v_t-\mathrm{i}\lambda_j^{-1}\alpha_j-\lambda_j^{-1}u_t\alpha_j^2,\\&A_t(\lambda_j)=-\alpha_{jt}B(\lambda_j)-\alpha_jB_t(\lambda_j),\\&C_t(\lambda_j)=-\alpha_{jt}D(\lambda_j)-\alpha_jD_t(\lambda_j).\end{aligned} \tag{41}$$

It can be verified that λ_j $(j=1,\ldots,2N)$ are roots of g_{kl} $(k,l=1,2)$ in Eq. (40). Hence we arrive at

$$(T_t+TV)T^{-1}=(\det T)Q(\lambda), \tag{42}$$

where

$$Q(\lambda)=\begin{pmatrix} q_{11}\lambda^{-1} & q_{12}\lambda^{-1}\\ q_{21} & q_{22}\lambda^{-1}\end{pmatrix}, \tag{43}$$

with q_{kl} being independent of λ. Equation (42) can be rewritten as

$$T_t+TV=Q(\lambda)T. \tag{44}$$

Equating the coefficients of λ^{-1}, λ^{N-1}, λ^N in Eq. (44) leads to

$$q_{11}=\frac{\mathrm{i}}{2},\ q_{22}=-\frac{\mathrm{i}}{2},\ \ q_{12}=u_t+B_{N-1,t},\ q_{21}=v_t+C_{N,t}. \tag{45}$$

Then, by using Eq. (22), we obtain

$$q_{12}=\widehat{u}_t,\ \ q_{21}=\widehat{v}_t. \tag{46}$$

The proof is thus completed.

According to the propositions 1 and 2, the transformations (13) and (22) convert the matrix spectral problems (11) and (12) into another one with the same form (20) and (39). Therefore, both of the two matrix spectral problems lead to the same integrable equation. The transformation (22) is also called a DT with multi-parameters for the three-component coupled FL system (10). We immediately get the following fact.

Proposition 3 Every solution (u,v,w) for the three-component coupled FL system (10) is mapped into a new solution $(\widehat{u},\widehat{v},\widehat{w})$ under the DT (22), where A_{N-1}, B_{N-1} and C_N are given by Eq. (23).

3. The reduction of Darboux transformation and soliton solutions

In this section, we will derive the N-fold DT for the generalized FL equation (9) and construct its N-soliton solution in a compact Vandermonde-like determinant form. Under the reduction condition $v=u^*$ in Eq. (10), the matrix spectral problems (11) and (12) turn into

$$y_x = Uy,$$
$$U = \frac{1}{1-\mu\lambda}\begin{pmatrix} \frac{\mathrm{i}\lambda}{2}+\mathrm{i}|u|^2+\mathrm{i}w & -\mathrm{i}u-\mu u_x \\ \lambda(\mathrm{i}u^*-\mu u_x^*) & -\frac{\mathrm{i}\lambda}{2}-\mathrm{i}|u|^2-\mathrm{i}w \end{pmatrix}, \quad (47)$$
$$y_t = Vy,\ V = \begin{pmatrix} \frac{\mathrm{i}\lambda^{-1}}{2} & \lambda^{-1}u_t \\ u_t^* & -\frac{\mathrm{i}\lambda^{-1}}{2} \end{pmatrix}. \quad (48)$$

In this case, we choose two solutions of Eqs. (47) and (48)

$$\phi(\lambda)=(\phi_1(\lambda),\phi_2(\lambda))^{\mathrm{T}},\ \ \psi(\lambda)=(\phi_2^*(\lambda^*),\lambda\phi_1^*(\lambda^*))^{\mathrm{T}}, \quad (49)$$

and set the parameters

$$\lambda_{2j}=\lambda_{2j-1}^*,\ \ \gamma_{2j}=\lambda_{2j}^{-1}\gamma_{2j-1}^{*-1},\ \ j=1,\ldots,N. \quad (50)$$

By means of Eqs. (15) and (16), it is easy to prove that $\alpha_{2j}=\lambda_{2j}\alpha_{2j-1}^{*-1}$, $D_k=A_k^*$, $C_{k+1}=B_k^*$, $(j=1,\ldots,N,\ k=0,\ldots,N-1)$. Here, α_{2j-1} can be expressed as

$$\alpha_{2j-1}=\frac{\phi_2(\lambda_{2j-1})-\lambda_{2j-1}\gamma_{2j-1}\phi_1^*(\lambda_{2j-1}^*)}{\phi_1(\lambda_{2j-1})-\gamma_{2j-1}\phi_2^*(\lambda_{2j-1}^*)}. \quad (51)$$

Then, the functions A_k and B_k are determined by the following linear algebraic system:

$$\begin{aligned}&\sum_{k=0}^{N-1}(A_k+\alpha_{2j-1}B_k)\lambda_{2j-1}^k=-\lambda_{2j-1}^N,\ \ j=1,\ldots,N,\\ &\sum_{k=0}^{N-1}\alpha_{2j-1}^*A_k\lambda_{2j-1}^{*k}+\sum_{k=1}^{N}B_{k-1}\lambda_{2j-1}^{*k}=-\alpha_{2j-1}^*\lambda_{2j-1}^{*N},\\ &j=1,\ldots,N. \end{aligned} \quad (52)$$

According to Eq. (22) and $v=u^*$, we arrive at $\widehat{v}=v+C_N=u^*+B_{N-1}^*=(u+B_{N-1})^*=\widehat{u}^*$, $\widehat{w}=w+\mathrm{i}\mu(A_{N-1,x}+u_x^*B_{N-1}-u_xB_{N-1}^*-B_{N-1,x}B_{N-1}^*)$. Thus we have the following assertion.

Proposition 4 Every solution (u,w) for the generalized FL equation (9) is mapped into a new solution $(\widehat{u},\widehat{w})$ under the DT

$$\begin{aligned}&\widehat{u}=u+B_{N-1},\\ &\widehat{w}=w+\mathrm{i}\mu(A_{N-1,x}+u_x^*B_{N-1}-u_xB_{N-1}^*-B_{N-1,x}B_{N-1}^*),\end{aligned} \quad (53)$$

where A_{N-1} and B_{N-1} are determined by the linear algebraic system (52) and have the form

$$A_{N-1}=\frac{\widetilde{\Delta}^{(1)}}{\widetilde{\Delta}},\ \ B_{N-1}=\frac{\widetilde{\Delta}^{(2)}}{\widetilde{\Delta}}, \quad (54)$$

with

$$\widetilde{\Delta}=\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-2} & \lambda_1^{N-1} & \alpha_1 & \alpha_1\lambda_1 & \cdots & \alpha_1\lambda_1^{N-2} & \alpha_1\lambda_1^{N-1} \\ \alpha_1^* & \alpha_1^*\lambda_1^* & \cdots & \alpha_1^*\lambda_1^{*N-2} & \alpha_1^*\lambda_1^{*N-1} & \lambda_1^* & \lambda_1^{*2} & \cdots & \lambda_1^{*N-1} & \lambda_1^{*N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \lambda_{2N-1} & \cdots & \lambda_{2N-1}^{N-2} & \lambda_{2N-1}^{N-1} & \alpha_{2N-1} & \alpha_{2N-1}\lambda_{2N-1} & \cdots & \alpha_{2N-1}\lambda_{2N-1}^{N-2} & \alpha_{2N-1}\lambda_{2N-1}^{N-1} \\ \alpha_{2N-1}^* & \alpha_{2N-1}^*\lambda_{2N-1}^* & \cdots & \alpha_{2N-1}^*\lambda_{2N-1}^{*N-2} & \alpha_{2N-1}^*\lambda_{2N-1}^{*N-1} & \lambda_{2N-1}^* & \lambda_{2N-1}^{*2} & \cdots & \lambda_{2N-1}^{*N-1} & \lambda_{2N-1}^{*N} \end{vmatrix}, \quad (55)$$

$$\widetilde{\Delta}^{(1)}=\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-2} & -\lambda_1^N & \alpha_1 & \alpha_1\lambda_1 & \cdots & \alpha_1\lambda_1^{N-2} & \alpha_1\lambda_1^{N-1} \\ \alpha_1^* & \alpha_1^*\lambda_1^* & \cdots & \alpha_1^*\lambda_1^{*N-2} & -\alpha_1^*\lambda_1^{*N} & \lambda_1^* & \lambda_1^{*2} & \cdots & \lambda_1^{*N-1} & \lambda_1^{*N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \lambda_{2N-1} & \cdots & \lambda_{2N-1}^{N-2} & -\lambda_{2N-1}^N & \alpha_{2N-1} & \alpha_{2N-1}\lambda_{2N-1} & \cdots & \alpha_{2N-1}\lambda_{2N-1}^{N-2} & \alpha_{2N-1}\lambda_{2N-1}^{N-1} \\ \alpha_{2N-1}^* & \alpha_{2N-1}^*\lambda_{2N-1}^* & \cdots & \alpha_{2N-1}^*\lambda_{2N-1}^{*N-2} & -\alpha_{2N-1}^*\lambda_{2N-1}^{*N} & \lambda_{2N-1}^* & \lambda_{2N-1}^{*2} & \cdots & \lambda_{2N-1}^{*N-1} & \lambda_{2N-1}^{*N} \end{vmatrix}, \quad (56)$$

$$\widetilde{\Delta}^{(2)}=\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-2} & \lambda_1^{N-1} & \alpha_1 & \alpha_1\lambda_1 & \cdots & \alpha_1\lambda_1^{N-2} & -\lambda_1^N \\ \alpha_1^* & \alpha_1^*\lambda_1^* & \cdots & \alpha_1^*\lambda_1^{*N-2} & \alpha_1^*\lambda_1^{*N-1} & \lambda_1^* & \lambda_1^{*2} & \cdots & \lambda_1^{*N-1} & -\alpha_1^*\lambda_1^{*N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \lambda_{2N-1} & \cdots & \lambda_{2N-1}^{N-2} & \lambda_{2N-1}^{N-1} & \alpha_{2N-1} & \alpha_{2N-1}\lambda_{2N-1} & \cdots & \alpha_{2N-1}\lambda_{2N-1}^{N-2} & -\lambda_{2N-1}^N \\ \alpha_{2N-1}^* & \alpha_{2N-1}^*\lambda_{2N-1}^* & \cdots & \alpha_{2N-1}^*\lambda_{2N-1}^{*N-2} & \alpha_{2N-1}^*\lambda_{2N-1}^{*N-1} & \lambda_{2N-1}^* & \lambda_{2N-1}^{*2} & \cdots & \lambda_{2N-1}^{*N-1} & -\alpha_{2N-1}^*\lambda_{2N-1}^{*N} \end{vmatrix}. \tag{57}$$

In the following, we shall apply the DT to construct explicit soliton solutions of Eq. (9). Substituting the trivial solution $u=0, w=0$ into Eqs. (47) and (48), we have the following two basic solutions:

$$\phi(\lambda)=\begin{pmatrix} \exp\left[\frac{\mathrm{i}}{2}\left(\frac{\lambda}{1-\mu\lambda}x+\frac{1}{\lambda}t\right)\right] \\ 0 \end{pmatrix},$$
$$\psi(\lambda)=\begin{pmatrix} 0 \\ \lambda\exp\left[-\frac{\mathrm{i}}{2}\left(\frac{\lambda}{1-\mu\lambda}x+\frac{1}{\lambda}t\right)\right] \end{pmatrix}. \tag{58}$$

Then we can calculate

$$\alpha_{2j-1}=-\lambda_{2j-1}\gamma_{2j-1}\exp\left[-\mathrm{i}\left(\frac{\lambda_{2j-1}}{1-\mu\lambda_{2j-1}}x+\frac{1}{\lambda_{2j-1}}t\right)\right],$$
$$j=1,\ldots,N. \tag{59}$$

By inserting Eq. (59) into Eqs. (53) and (54), we can obtain the N-soliton solution in a compact Vandermonde-like determinant form for Eq. (9).

For simplicity, we only discuss two special cases of $N=1$ and $N=2$.

(i) For $N=1$, we have from Eq. (54) that

$$A_0=\frac{\lambda_1^*(\alpha_1\alpha_1^*-\lambda_1)}{\lambda_1^*-\alpha_1\alpha_1^*},\quad B_0=\frac{\alpha_1^*(\lambda_1-\lambda_1^*)}{\lambda_1^*-\alpha_1\alpha_1^*}. \tag{60}$$

According to Eq. (53), the one-soliton solution of Eq. (9) reads

$$\widehat{u}[1]=B_0,\quad \widehat{w}[1]=\mathrm{i}\mu(A_{0,x}-B_{0,x}B_0^*), \tag{61}$$

with

$$\alpha_1=-\lambda_1\gamma_1\exp\left[-\mathrm{i}\left(\frac{\lambda_1}{1-\mu\lambda_1}x+\frac{1}{\lambda_1}t\right)\right]. \tag{62}$$

Through direct calculation, one has

$$|\widehat{u}[1]|^2=\frac{4[\mathrm{Im}(\lambda_1)]^2|\alpha_1|^2}{|\lambda_1|^2-2\mathrm{Re}(\lambda_1)|\alpha_1|^2+|\alpha_1|^4}. \tag{63}$$

At this very point, it can be found that, when setting $\mathrm{Re}(\lambda_1)\leq 0$, the above solution would be nonsingular. Here, we take $\lambda_1=\mathrm{i}\eta_1$, then it follows that

$$|\widehat{u}[1]|^2=2|\eta_1||\gamma_1|^2\mathrm{sech}\left[2\mathrm{Im}(\theta_1)+\ln|\eta_1|+2\ln|\gamma_1|\right], \tag{64}$$

where

$$\theta_1=-\frac{\mu\eta_1^2x}{\mu^2\eta_1^2+1}+\mathrm{i}\frac{\eta_1^2x-(\mu^2\eta_1^2+1)t}{\eta_1(\mu^2\eta_1^2+1)}. \tag{65}$$

The maximum amplitude of the one-soliton solution is $\sqrt{2|\eta_1|}|\gamma_1|$ and is localized at $x=(\mu^2+\frac{1}{\eta_1^2})t$. We display the dynamical evolution plots of the one-soliton solution in Fig. 1

(ii) For $N=2$, through a straightforward calculation, it holds from Eq. (54) that

$$A_1=\frac{\widetilde{\Delta}_1^{(1)}}{\widetilde{\Delta}_1},\quad B_1=\frac{\widetilde{\Delta}_1^{(2)}}{\widetilde{\Delta}_1}, \tag{66}$$

where

$$\widetilde{\Delta}_1=\begin{vmatrix} 1 & \lambda_1 & \alpha_1 & \alpha_1\lambda_1 \\ \alpha_1^* & \alpha_1^*\lambda_1^* & \lambda_1^* & \lambda_1^{*2} \\ 1 & \lambda_3 & \alpha_3 & \alpha_3\lambda_3 \\ \alpha_3^* & \alpha_3^*\lambda_3^* & \lambda_3^* & \lambda_3^{*2} \end{vmatrix},$$
$$\widetilde{\Delta}_1^{(1)}=\begin{vmatrix} 1 & -\lambda_1^2 & \alpha_1 & \alpha_1\lambda_1 \\ \alpha_1^* & -\alpha_1^*\lambda_1^{*2} & \lambda_1^* & \lambda_1^{*2} \\ 1 & -\lambda_3^2 & \alpha_3 & \alpha_3\lambda_3 \\ \alpha_3^* & -\alpha_3^*\lambda_3^{*2} & \lambda_3^* & \lambda_3^{*2} \end{vmatrix},$$
$$\widetilde{\Delta}_1^{(2)}=\begin{vmatrix} 1 & \lambda_1 & \alpha_1 & -\lambda_1^2 \\ \alpha_1^* & \alpha_1^*\lambda_1^* & \lambda_1^* & -\alpha_1^*\lambda_1^{*2} \\ 1 & \lambda_3 & \alpha_3 & -\lambda_3^2 \\ \alpha_3^* & \alpha_3^*\lambda_3^* & \lambda_3^* & -\alpha_3^*\lambda_3^{*2} \end{vmatrix}. \tag{67}$$

With the help of Eq. (53), the two-soliton solution of Eq. (9) can be presented as

$$\widehat{u}[2]=B_1,\quad \widehat{w}[2]=\mathrm{i}\mu(A_{1,x}-B_{1,x}B_1^*). \tag{68}$$

The corresponding dynamical evolution plots of the two-soliton solution under adequate parameters are exhibited in Figs. 2–4.

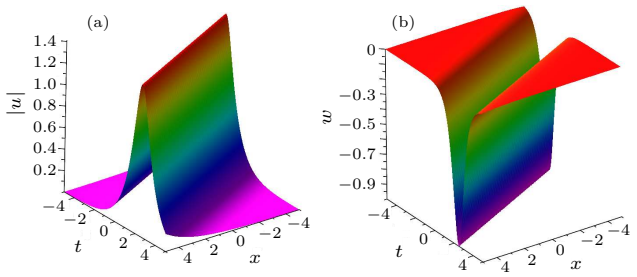


Fig. 1. (a), (b) Evolution plots of the one-soliton solution (61) with $\gamma_1=1$, $\lambda_1=-\mathrm{i}$, $\mu=1$.

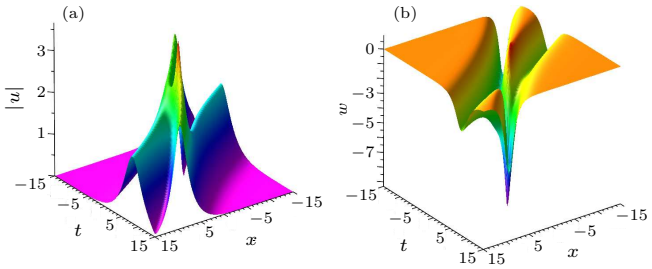


Fig. 2. (a), (b) Evolution plots of the two-soliton solution (68) with $\gamma_1=1$, $\gamma_3=-\mathrm{i}$, $\lambda_1=1+2\mathrm{i}$, $\lambda_3=-1+2\mathrm{i}$, $\mu=1$.

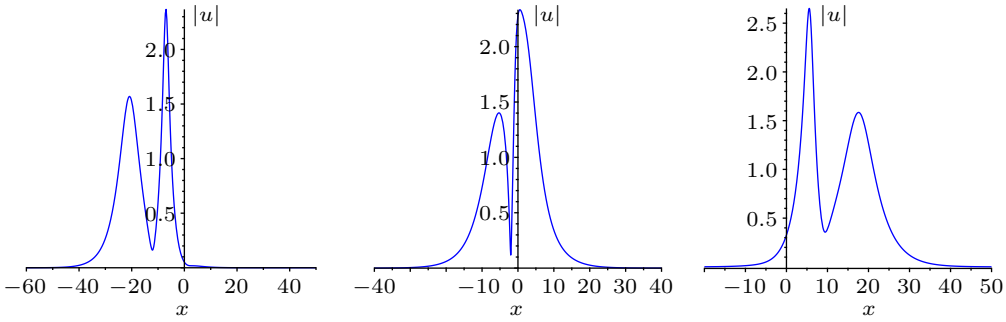


Fig. 3. (a), (b), (c) The two-soliton solution (68) for the u component at $t=-10$, $t=0$ and $t=10$, respectively. The parameters are the same as those in Fig. 2.

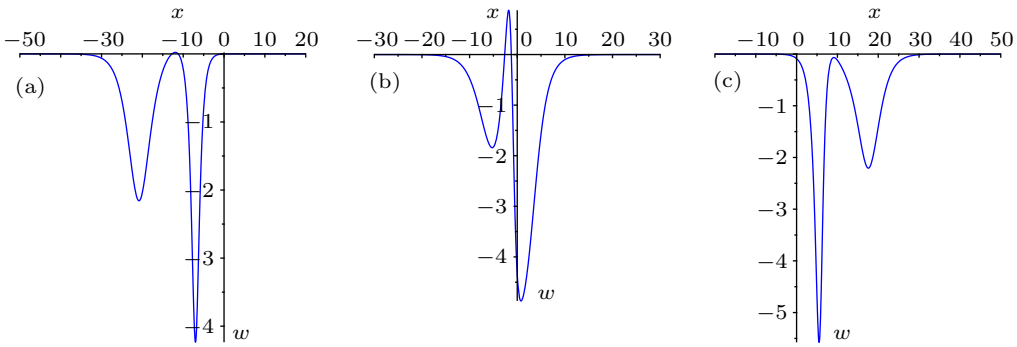


Fig. 4. (a), (b), (c) The two-soliton solution (68) for the w component at $t=-10$, $t=0$ and $t=10$, respectively. The parameters are the same as those in Fig. 2.

4. Conclusion

In summary, we investigated the integrable generalization of the FL equation, which can be viewed as the first negative member of a 2×2 matrix spectral problem with three potentials. Resorting to the matrix spectral problem, we introduced a gauge transformation. It was rigorously proved that the gauge transformation converted the matrix spectral problem into another matrix spectral problem with the same form except that the old potentials were replaced by the new ones, and hence justified the DT with multi-parameters for the three-component coupled FL system. By means of the reduction technique, the N-fold DT for the generalized FL equation was obtained. The N-soliton solution in a compact Vandermonde-like determinant form was presented. Explicitly, the one- and two-soliton solutions were given and the corresponding dynamical behaviors were graphically shown. We should emphasize that the DT presented in this paper can also be used to construct the breather and rogue wave solutions for the generalized FL equation under plane-wave seed solution. We will report these results in our future work.

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