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Verifying hierarchical network nonlocality in general quantum networks

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Recently, a class of innovative notions on quantum network nonlocality (QNN), called full quantum network nonlocality (FQNN), have been proposed in *Phys. Rev. Lett.* **128** 010403 (2022). As the generalization of full network nonlocality (FNN), l-level quantum network nonlocality (l-QNN) was defined in arxiv. 2306.15717 quant-ph (2024). FQNN is a NN that can be generated only from a network with all sources being non-classical. This is beyond the existing standard network nonlocality, which may be generated from a network with only a non-classical source. One of the challenging tasks is to establish corresponding Bell-like inequalities to demonstrate the FQNN or l-QNN. Up to now, the inequality criteria for FQNN and l-QNN have only been established for star and chain networks. In this paper, we devote ourselves to establishing Bell-like inequalities for networks with more complex structures. Note that star and chain networks are special kinds of tree-shaped networks. We first establish the Bell-like inequalities for verifying l-QNN in k-forked tree-shaped networks. Such results generalize the existing inequalities for star and chain networks. Furthermore, we find the Bell-like inequality criteria for l-QNN for general acyclic and cyclic networks. Finally, we discuss the demonstration of l-QNN in the well-known butterfly networks.

Keywords: full network nonlocality, hierarchical network nonlocality, tree network

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1. Introduction

By taking into account complex causal structures beyond the standard local hidden variable model, the definition of nonlocality on bipartite systems has been generalized to the cases of quantum networks, such as entanglement-swapping networks,[1,2] chain networks,[3,4] star networks,[5,6] polygon networks,[7,8] tree-shaped networks[9–11] and arbitrary non-cyclic networks,[12–14] and even arbitrary networks.[15] Because of the complexity of network structures and the independence of sources, the set of local correlations in a network cannot generally be convex. This implies that linear Bell-like inequalities are not sufficient to characterize network nonlocality (NN). Thus, nonlinear Bell-like inequalities for networks were proposed.[1,2,5,16] In addition, Denis *et al.* put forward an iterative procedure for constructing Bell-like inequalities by adding independent sources.[13]

However, the violation of one of these aforementioned inequalities guarantees only the existence of one non-classical source in this network. This network nonlocality may not be genuinely beyond Bell bipartite nonlocality, and also cannot display the superiority of network communications.[17] Currently, Alejandro *et al.* displayed the notion of full network nonlocality (FNN) to describe the class of NN correlations formed by requiring all sources to be nonclassical.[17] So FNN correlations can genuinely be applied in quantum information communications. In particular, the inequality criteria for FNN in the star network were built by finding a new upper bound of the existing Bell-like inequalities. The upper bound lies in the interval $(1,\sqrt{2})$. FNN is also observed in experiments for several networks.[18–21] Sequentially, in Ref. [22], Luo *et al.* generalized the notion of FNN to hierarchical NN, which is called l-level network nonlocality (l-NN). To define the notations of FNN and l-NN, researchers consider the different kinds of sources, including no-signaling and quantum sources. Consider a behavior as the joint probability $p(a,b|x,y)$ with output vectors a,b and input vectors x, y respectively. The behavior is no-signaling if[23]

$$\sum_b p(a,b|x,y)=\sum_b p(a,b|x,y') \quad \text{for all} \quad a,x,y,y',$$
$$\sum_a p(a,b|x,y)=\sum_a p(a,b|x',y) \quad \text{for all} \quad b,y,x,x'.$$

The behavior is quantum if $p(a,b|x,y)=\mathrm{tr}(\rho_{AB}M_{a|x}\otimes M_{b|y})$ for some bipartite quantum states ρ_{AB} and $M_{a|x},M_{b|y}$ constitute the POVMs of Alice and Bob, respectively. Denote by $\mathcal{Q}$ and $\mathcal{NS}$ the sets of quantum correlations and no-signaling correlations respectively, then $\mathcal{Q}\subset\mathcal{NS}$. The aforementioned correlations can play different roles in NN theory. For instance, roughly speaking, for the binary CHSH inequality, the CHSH value is bounded by 1 if all sources are classical, the maximal quantum violation is $\sqrt{2}$, and the CHSH value for no-signaling sources can arrive at 2 maximally. Based on the above, Luo *et al.*[22] provided a notion of the hierarchical network nonlocality between NN and FNN: l-level network nonlocality. This means its corresponding probability distribution cannot

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be generated when at least l sources in the network distribute classical physical systems, and the rest are no-signaling. Otherwise, the distribution is l-level network local (l-NL). When no-signaling sources are replaced by quantum sources, l-level quantum network nonlocality and locality (l-QNN and l-QNL) can also be defined similarly. If $l=1$, l-NN and l-QNN become FNN and FQNN, respectively. l-NN or l-QNN can be verified by violating nonlinear network local inequalities whose upper bounds are magnified generally and lie in the open interval $(1,\sqrt{2})$. Such hierarchical upper bounds (HUBs) should be explicit functions of the minimal classical resource number l, and show the clear change pattern dependent on the variation of l. However, complex network structures can hinder us from obtaining the aforementioned functions on HUBs. Indeed, up to now, only in star networks can satisfactory HUBs be obtained.[22] Besides, even in a slightly more complicated chain network, it is also not clear how HUBs vary as the minimal classical resource number l.[22] Star and chain networks are special kinds of tree-shaped networks, and general networks have more multi-level and indeterminate forked structures. Therefore, it is a more challenging task to establish the notions of l-QNN for general networks. In this paper, we devote ourselves to obtaining l-QNL (l-NL) inequalities, and so can verify the l-QNN (l-NN) in various kinds of more complicated networks, including acyclic networks, cyclic networks and even general networks.

2. Reviews and raising questions

Consider a quantum network $\Xi(n,m)$ consisting of n parties, say, $\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_n$, and m sources $S_1,S_2,\ldots,S_m$. All parties can perform measurements on their systems labeled by $x_1,x_2,\ldots,x_n$ for $\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_n$ and their outcomes are denoted by $a_1,a_2,\ldots,a_n$, $x_i,a_i\in\{0,1\}$. Here, denote the inputs and outcomes of parties by $\boldsymbol{x}$ and $\boldsymbol{a}$, respectively. The correlation is network local (NL) if its joint conditional probability distribution satisfies

$$p(\boldsymbol{a}|\boldsymbol{x})=\int \mathrm{d}\lambda_1\mu_1(\lambda_1)\cdots\int \mathrm{d}\lambda_m\mu_m(\lambda_m)\prod_{j=1}^{n}p(a_j|x_j,\Lambda_j), \quad (1)$$

where λ_i denotes the local variable distributed by source S_i, $\mu_i(\lambda_i)$ is the probability distribution for λ_i with $\int\mu_i(\lambda_i)\mathrm{d}\lambda_i=1$, Λ_j denotes the set of local variables associated with the sources that connect to party $\boldsymbol{A}_j$. Using the form of Eq. (1), Luo proposed nonlinear Bell-like inequalities based on geometric features of networks to characterize the multipartite correlations in general quantum networks. For a given quantum network $\Xi(n,m)$, it can be seen as a k-independence when there are k spacelike separated parties that do not share the same sources.[15] Each k local correlation in the network $\Xi(n,m)$ satisfies the inequality

$$|I(n,k)|^{\frac{1}{k}}+|J(n,k)|^{\frac{1}{k}}\leq 1, \quad (2)$$

where

$$I(n,k)=\frac{1}{2^k}\sum_{x_j,j\in\Gamma}\langle A_{x_1}A_{x_2}\cdots A_{x_n}\rangle,$$

$$J(n,k)=\frac{1}{2^k}\sum_{x_j,j\in\Gamma}(-1)^{\sum_{i\in\Gamma}x_i}\langle A_{x_1}A_{x_2}\cdots A_{x_n}\rangle,$$

$\Gamma=\{i_1,i_2,\ldots,i_k\}$ is the index set of k independent parties, $\bar{\Gamma}=\{1,2,\ldots,n\}\backslash\Gamma$ is the set of indexes for the remaining parties and $\langle A_{x_1}A_{x_2}\cdots A_{x_n}\rangle=\sum_{\boldsymbol{a}}(-1)^{\sum_{j=1}^{n}a_j}P(\boldsymbol{a}|\boldsymbol{x})$. Note that the following results all utilize this computational approach. It is also shown in Ref. [15] that the maximum quantum violation that can be achieved in the network is $\sqrt{2}$.

Non-linear Bell-like inequalities to detect l-QNN and l-NN for chain or star networks have been obtained in Ref. [22]. For instance, consider a star network consisting of n branch parties, $\boldsymbol{A}_1,\ldots,\boldsymbol{A}_n$, each share a bipartite system with a central node $\boldsymbol{B}$. Assume that $\boldsymbol{B}$ can perform two binary measurements B_y. Then l-QNL correlations abide by the following inequalities in this $n+1$-partite star network:

$$I^{\frac{1}{n}}+J^{\frac{1}{n}}\leq 2^{\frac{n-l}{2n}}, \quad (3)$$

where $I=\langle\Pi_{i=1}^{n}A_{x_i}^{+}B_0\rangle$ and $J=\langle\Pi_{i=1}^{n}A_{x_i}^{-}B_1\rangle$ with $A_{x_i}^{\pm}=\frac{1}{2}(A_{x_i=0}\pm A_{x_i=1})$, $A_{x_i=0(1)}$, $B_{0(1)}$ are the measurement operators for $\boldsymbol{A}_i$ and $\boldsymbol{B}$, and the correlators $\langle\Pi_{i=1}^{n}A_{x_i}B_y\rangle$ are given by $\sum_{\boldsymbol{a},b}(-1)^{\sum_{i=1}^{n}a_i+b}p(\boldsymbol{a},b|\boldsymbol{x},y)$, where $\boldsymbol{a}=a_1,\ldots,a_n$ and $\boldsymbol{x}=x_1,\ldots,x_n$ represent the inputs and outputs of party performing the measurements, respectively. The bound $2^{\frac{n-l}{2n}}$ lies in $[1,\sqrt{2}]$. It was also proved that in a chain network with n parties, for instance when n is odd, l-QNL correlations satisfy

$$|I|^{\frac{2}{n+1}}+|J|^{\frac{2}{n+1}}\leq\begin{cases}2^{\frac{n-1}{2n+2}}, & l\geq\dfrac{n-1}{2},\\ \sqrt{2}, & l<\dfrac{n-1}{2},\end{cases} \quad (4)$$

where we define $I=\langle\Pi_{\text{odd }i}A_{x_i}^{+}\Pi_{\text{even }j}A_{x_j=0}\rangle$ and $J=\langle\Pi_{\text{odd }i}A_{x_i}^{-}\Pi_{\text{even }j}A_{x_j=1}\rangle$, with $A_{x_i}^{\pm}=\frac{1}{2}(A_{x_i=0}\pm A_{x_i=1})$. For $l<(n-1)/2$ the bound in Eq. (4) arrives at the maximal quantum value $\sqrt{2}$, and thus it cannot detect this l-QNN. Compared with the HUB of Eq. (3) in a star network, these bounds in chain networks cannot explain a clear relation how l-QNN varies under l. One of the possible reasons is that chain networks generate more non-independent parities than star networks.

As we see, it is more challenging to establish the l-QNL inequalities in more complicated networks. In the paper, we try to explore HUBs of non-linear Bell-like inequalities for acyclic networks, cyclic networks and general networks, respectively. Recall that an acyclic network is an indeterminate forked tree-shaped network. In the case of acyclic networks, we consider two cases: determinate forked and indeterminate forked tree networks. Moreover, through the paper, we make

use of the inequality (2), where there is a challenging task to determine the maximal independent-node number (MINN) in the network. This is an NP-hard problem.[15] Therefore, we first always give approaches to calculating the MINNs of several kinds of networks in each of the following sections.

3. The l-QNN in k-forked tree-shaped networks

In this section, we give complete solutions for calculating the MINN of an arbitrary k-forked tree-shaped network and establish inequality criteria for its l-QNN.

The determinate forked tree network is an important and applied-extensively structure; for instance, a two-forked tree network is often used in computer sciences. So we introduce the scenario of the determinate k-forked tree-shaped network. Let $\Upsilon_{n,k}$ be a k-forked ($k>1$) n-layer quantum network as in Fig. 1, which consists of $g_1(n,k)=\frac{k-k^n}{1-k}$ sources and $g_2(n,k)=\frac{1-k^n}{1-k}$ parties.

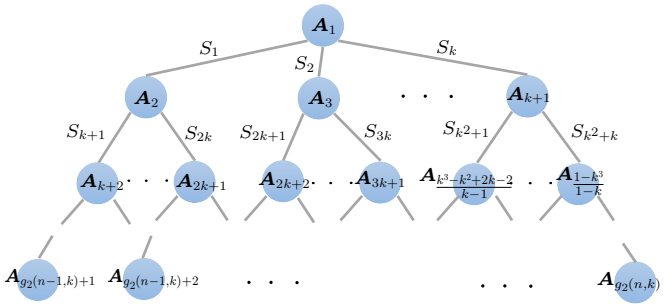


Fig. 1. An n-layer k-forked tree-shaped network consists of $g_1(n,k)=\frac{k-k^n}{1-k}$ sources, $S_1,S_2,\ldots,S_{g_1(n,k)}$, and $g_2(n,k)=\frac{1-k^n}{1-k}$ parties, $A_1,A_2,\ldots,A_{g_2(n,k)}$.

MINN for k-forked tree networks Here, the first task is to determine the maximal independence number. According to the definition of independent parties, a set of parties is independent if elements in this set cannot be linked by arbitrary identical sources. In a k-forked tree network, this means that independent parties can occur only if they lie in the same layer or in different and non-adjacent layers. Thus, we propose the rule to determine the MINN in an arbitrary k-fork tree network $\Upsilon_{n,k}$ as follows: if its layer number n is odd (even), then its MINN is the sum of node numbers in each odd (even) number layer. Let us illustrate with two examples. In Fig. 2(a), according to the aforementioned analysis, if we choose the vertex party as the independent party, then the two parties in the second layer cannot be independent. Then the four parties in the third layer are independent. Finally, the total independence number of network (a) is 6. However, if we do not choose the vertex party as independent first, then the two parties in the second layer are independent. Finally, we can determine the bigger independence number. We use orange circles to record the independent parties. So the MINN is 10 in network (a), and is 21 in (b). More generally, we can conclude that the MINN in the general network of Fig. 1 is $f_1(n,k)=\frac{k^{n+1}-1}{k^2-1}$ when n is odd, and $f_2(n,k)=\frac{k-k^{n+1}}{1-k^2}$ when n is even.

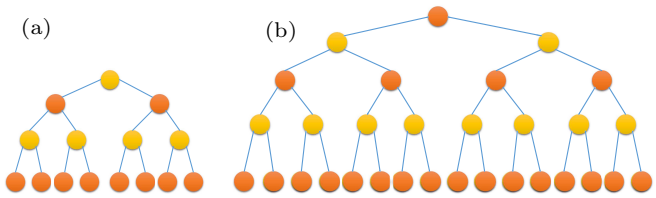


Fig. 2. The tree-shaped network scenario. Each orange circle and yellow circle respectively represent an independent party and a non-independent party. Each straight line represents one shared resource between two adjacent parties. (a) The 2-forked 4-layer tree-shaped scenario. (b) The 2-forked 5-layer tree-shaped scenario.

The l-QNN in k-forked tree networks We still use Luo's k-independence theorem,[15] and the corresponding standard Bell-type inequalities are as follows:

$$|I_1|^{\frac{1}{f_1(n,k)}}+|J_1|^{\frac{1}{f_1(n,k)}}\leq 1,\ \text{if } n \text{ is odd}, \tag{5}$$

$$|I_2|^{\frac{1}{f_2(n,k)}}+|J_2|^{\frac{1}{f_2(n,k)}}\leq 1,\ \text{if } n \text{ is even}, \tag{6}$$

where

$$I_t=\langle \Pi_{i\in\mathcal{M}_{t1}}A_{x_i}^{+}\Pi_{j\in\mathcal{M}_{t2}}A_{x_j=0}\rangle,$$
$$J_t=\langle \Pi_{i\in\mathcal{M}_{t1}}A_{x_i}^{-}\Pi_{j\in\mathcal{M}_{t2}}A_{x_j=1}\rangle,$$

with $A_{x_i}^{\pm}=\frac{1}{2}(A_{x_i=0}\pm A_{x_i=1})$ for $t=1,2$. Here, the correlations $\langle A_{x_1}\cdots A_{x_n}\rangle=\sum_{\boldsymbol{a}}(-1)^{\sum_{i=1}^{n}a_i}p(\boldsymbol{a}|\boldsymbol{x})$ with inputs $\boldsymbol{x}$ and outcomes $\boldsymbol{a}$.

In Appendix A, we give a detailed proof to establish the l-QNL and l-NL bounds for inequalities (5) and (6) as follows. Suppose that all sources $\{S_r\}_{r=1}^{g_1(n,k)}$ are composed of classical physical systems and bipartite quantum systems. If n is odd, then correlations that are l-QNL in the n-layer k-forked tree-shaped network satisfy the following inequality:

$$\begin{aligned}U_{g_1(n,k)} &= |I_1|^{\frac{1}{f_1(n,k)}}+|J_1|^{\frac{1}{f_1(n,k)}}\\ &\leq \begin{cases}2^{\frac{f_1(n,k)-1}{2f_1(n,k)}}, & l\geq\dfrac{k-k^n}{1-k^2},\\ \sqrt{2}, & l<\dfrac{k-k^n}{1-k^2};\end{cases}\end{aligned} \tag{7}$$

if n is even and corresponding correlations are l-QNL, then the following inequality holds:

$$\begin{aligned}U_{g_1(n,k)} &= |I_2|^{\frac{1}{f_2(n,k)}}+|J_2|^{\frac{1}{f_2(n,k)}}\\ &\leq \begin{cases}2^{\frac{f_2(n,k)-1}{2f_2(n,k)}}, & l\geq\dfrac{1-k^n}{1-k^2},\\ \sqrt{2}, & l<\dfrac{1-k^n}{1-k^2},\end{cases}\end{aligned} \tag{8}$$

where

$$I_t=\langle \Pi_{i\in\mathcal{M}_{t1}}A_{x_i}^{+}\Pi_{j\in\mathcal{M}_{t2}}A_{x_j=0}\rangle,$$
$$J_t=\langle \Pi_{i\in\mathcal{M}_{t1}}A_{x_i}^{-}\Pi_{j\in\mathcal{M}_{t2}}A_{x_j=1}\rangle,$$

with $A_{x_i}^{\pm}=\frac{1}{2}(A_{x_i=0}\pm A_{x_i=1})$ for $t=1,2$. Here, $\mathcal{M}_{ij}$s are index sets of independent or dependent parties. When n is odd, the index set of independent parties is $\mathcal{M}_{11}=\{1,\frac{1-k^{s-1}}{1-k}+$

$1, \frac{1-k^{s-1}}{1-k}+2, \ldots, \frac{1-k^{s}}{1-k}\}_{s=3,5,\ldots,n}$, and the index set of the rest is $\mathcal{M}_{12}=\{\frac{1-k^{s-1}}{1-k}+1, \frac{1-k^{s-1}}{1-k}+2, \ldots, \frac{1-k^{s}}{1-k}\}_{s=2,4,\ldots,n-1}$. Similarly, when n is even, the index set of the independent and remaining parties are $\mathcal{M}_{21}=\{\frac{1-k^{s-1}}{1-k}+1, \frac{1-k^{s-1}}{1-k}+2, \ldots, \frac{1-k^{s}}{1-k}\}_{s=2,4,\ldots,n}$ and $\mathcal{M}_{22}=\{1, \frac{1-k^{s-1}}{1-k}+1, \frac{1-k^{s-1}}{1-k}+2, \ldots, \frac{1-k^{s}}{1-k}\}_{s=3,5,\ldots,n-1}$. The violation of the two inequalities demonstrates l-QNN in the k-forked tree network. We show the concrete form of Eq. (7) in the 2-forked and 3-layer tree network for readers. In such a network, $k=2$ and $n=3$ in Eq. (7), one can calculate that $\frac{k-k^n}{1-k^2}=2$ and $f_1(n,k)=5$. So the Bell-like inequality for l-QNL in such a network has the following form:

$$U_{g_1(3,2)}=|I_1|^{\frac{1}{5}}+|J_1|^{\frac{1}{5}}\leq\begin{cases}2^{\frac{2}{5}}, & l\geq 2,\\ \sqrt{2}, & l<2.\end{cases} \tag{9}$$

In Eqs. (7) and (8), the two HUBs $2^{\frac{f_i(n,k)-1}{2f_i(n,k)}}<2^{\frac{1}{2}}$, $i=1,2$. For instance, in the odd case, they are available in the case $l\geq\frac{k-k^n}{1-k^2}$. Indeed, the number $\frac{k-k^n}{1-k^2}$ is less than half of the total party number. Therefore, the two inequalities can be used in most situations. In Appendix D, we realize the quantum violations with sources of pure quantum states when $l\geq\frac{k-k^n}{1-k^2}$ with odd n and $l\geq\frac{1-k^n}{1-k^2}$ with even n.

Next, we show a concrete example for quantum demonstration of Eq. (7) in a 3-layer 2-forked tree-shaped network described in Fig. 3. It is a 5-independent network. We initially assume that each of the sources distributes a bipartite Werner state $S_i=w_i|\phi^+\rangle\langle\phi^+|+\frac{1-w_i}{4}I_4$ with $|\phi^+\rangle=(|00\rangle+|11\rangle)/\sqrt{2}$ for any $i\in\{1,2,\ldots,6\}$, and each party performs measurements in the form of $A_{x_1=0}=\cos\theta_1\sigma_z^{\otimes 2}+\sin\theta_1\sigma_x^{\otimes 2}$, $A_{x_1=1}=\cos\theta_1\sigma_z^{\otimes 2}-\sin\theta_1\sigma_x^{\otimes 2}$, $A_{x_2=0}=A_{x_3=0}=\sigma_z^{\otimes 3}$, $A_{x_2=1}=A_{x_3=1}=\sigma_x^{\otimes 3}$, $A_{x_s=0}=\cos\theta_s\sigma_z+\sin\theta_s\sigma_x$, $A_{x_s=1}=\cos\theta_s\sigma_z-\sin\theta_s\sigma_x$ $(s=4,5,6,7)$, respectively. Here, $\mathcal{M}_{11}=\{1,4,5,6,7\}$ and $\mathcal{M}_{12}=\{2,3\}$, we get

$$\begin{aligned}U_{g_1(3,2)}&=|I_1|^{\frac{1}{5}}+|J_1|^{\frac{1}{5}}\\&=|\langle\Pi_{i\in\mathcal{M}_{11}}A_{x_i}^{+}\Pi_{j\in\mathcal{M}_{12}}A_{x_j=0}\rangle|^{\frac{1}{5}}\\&\quad+|\langle\Pi_{i\in\mathcal{M}_{11}}A_{x_i}^{-}\Pi_{j\in\mathcal{M}_{12}}A_{x_j=1}\rangle|^{\frac{1}{5}}=\Pi_{i=1}^{6}w_i^{\frac{1}{5}}\sqrt{2}.\end{aligned}$$

It follows from inequality (7) that there are at least two classical physical systems in this network when $U_{g_1(3,2)}\leq 2^{\frac{2}{5}}$. Thus, if $\Pi_{i=1}^{6}w_i^{\frac{1}{5}}\sqrt{2}>2^{\frac{2}{5}}$, that is, $\Pi_{i=1}^{6}w_i>2^{-\frac{1}{2}}$, this implies that there is at most one classical physical system.

Let us illustrate the feasibility of our inequality with a more complex example. We consider a 4-layer tree-shaped network composed of 15 parties with 14 sources (see Fig. 2(a)). Based on the above discussion, we know that its MINN is 10. Assume that each source distributes a bipartite system $S_i=|\phi_i\rangle\langle\phi_i|$ with $|\phi_i\rangle=\cos\theta_i|00\rangle+\sin\theta_i|11\rangle$, where $\theta_i\in[0,\frac{\pi}{2}]$. Suppose party $\boldsymbol{A}_1$ performs measurements $A_{x_1=0}=\sigma_z\otimes\sigma_z$, $A_{x_1=1}=\sigma_x\otimes\sigma_x$. σ_x and σ_z are Pauli operators. Any party $\boldsymbol{A}_j$ $(j\in\{4,5,6,7\})$ performs measurements $A_{x_j=0}=\sigma_z^{\otimes 3}$, $A_{x_j=0}=\sigma_x^{\otimes 3}$. The measurements performed by the parties on the second layer are $A_{x_s=0}=\cos\alpha_s\sigma_z^{\otimes 3}+\sin\alpha_s\sigma_x^{\otimes 3}$ and $A_{x_s=1}=\cos\alpha_s\sigma_z^{\otimes 3}-\sin\alpha_s\sigma_x^{\otimes 3}$ for any $s=2,3$. The parties on the last layer perform measurements $A_{x_t=0}=\cos\alpha_t\sigma_z+\sin\alpha_t\sigma_x$ and $A_{x_t=1}=\cos\alpha_t\sigma_z-\sin\alpha_t\sigma_x$ $(t=8,9,\ldots,15)$. Thus, we get $|I_1|^{\frac{1}{10}}+|J_1|^{\frac{1}{10}}=\sqrt{1+\Pi_{i=1}^{14}(2\cos\theta_i\sin\theta_i)^{\frac{1}{5}}}$ when $\cos\alpha_s=\cos\alpha_t=1/\sqrt{1+\Pi_{i=1}^{14}(2\cos\theta_i\sin\theta_i)^{\frac{1}{5}}}$ for any $s\in\{2,3\}$ and $t\in\{8,9,\ldots,15\}$. Applying Eq. (8), we can infer that there are at most 4 classical physical systems and the rest are all quantum systems when $\Pi_{i=1}^{14}(2\cos\theta_i\sin\theta_i)^{\frac{1}{5}}>2^{\frac{9}{10}}-1\approx 0.8661$.

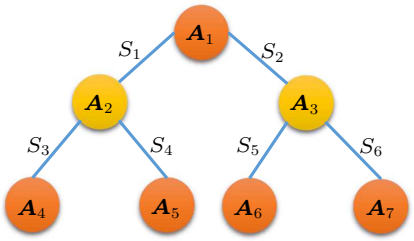


Fig. 3. A 3-layer 2-forked tree-shaped network consists of 6 sources, $S_1, S_2, \ldots, S_6$, and 7 parties, $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_7$.

To detect l-NL, the corresponding bounds of inequalities (7) and (8) are $2^{1-\frac{|S_{it}^1|}{f_1(n,k)}}$ and $2^{1-\frac{|S_{it}^2|}{f_2(n,k)}}$, where $|S_{it}^1|$ $(|S_{it}^2|)$ is the number of parties in the odd (even) layer that only receive classical systems. Details of the proof can also be found in Appendix A2.

4. The l-QNN in acyclic networks

Each acyclic network can be regarded as an indeterminate forked tree network. Here, we still first determine the MINN for an acyclic network, and then establish its l-QNN inequality criteria.

MINN for indeterminate forked tree networks We offer an idea of an approach to calculate the MINN. If the layer number is odd (even), we first designate parties in each odd (even) layer to be independent. If, among these independent parties, there is a party (record it as Λ) which connects more than 1 leaf node, then these leaf nodes will be selected to be independent ones and Λ becomes dependent. For instance, in network (a) in Fig. 4, we first regard the parties in even number layers as independent ones. However, due to the existence of the red circle in Fig. 4(b), which connects two leaf nodes. So we have Fig. 4(c), where the orange circles denote all independent parties. Finally, we conclude that the MINN of network (a) is 9.

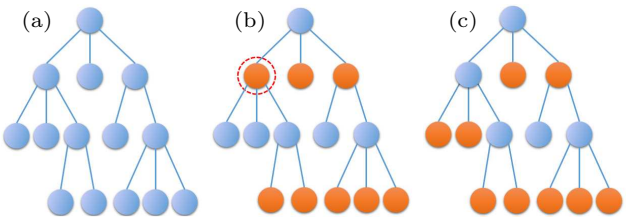


Fig. 4. The indeterminate forked tree-shaped network. Each orange circle represents an independent party.

The l-QNN for acyclic networks For an arbitrary indeterminate forked tree-shaped network consisting of n parties, denote by $f_3(n)$ as its MINN, l-QNL correlations abide by

$$|I_3'|^{\frac{1}{f_3(n)}}+|J_3'|^{\frac{1}{f_3(n)}}\leq\begin{cases}2^{\frac{f_3(n)-1}{2f_3(n)}}, & l\geq u(n),\\ \sqrt{2}, & l<u(n),\end{cases}\tag{10}$$

where $I_3'=\langle\Pi_{i\in\Gamma}A_{x_i}^{+}\Pi_{j\in\bar{\Gamma}}A_{x_j=0}\rangle$, $J_3'=\langle\Pi_{i\in\Gamma}A_{x_i}^{-}\Pi_{j\in\bar{\Gamma}}A_{x_j=1}\rangle$. Here, $u(n)$ denotes the minimal number of classical sources in the network. $l\geq u(n)$ means the existence of one independent party at least in the network which only shares classical variables. Here, we also show two concrete examples for Eq. (10). For instance, the values of $u(n)$ in Figs. 5(a) and 5(b) are 2 and 3, $f_3(n)$ are 6 and 5, respectively. Therefore, for instance, the violation of inequality (10) with bound $2^{\frac{5}{12}}$ demonstrates the l-QNN in network (a) when $l\geq 2$.

Similarly, we can derive that the bound for correlations that are l-NL is $2^{1-\frac{|S_{dt}|}{f_3(n)}}$, where $|S_{dt}|$ is the number of independent parties that only receive classical systems.

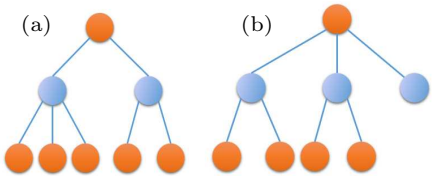


Fig. 5. Two different indeterminate forked tree-shaped networks. Each orange circle represents an independent party.

5. The l-QNN in cyclic networks

In a cyclic network (see Fig. 6), there are n parties $\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_n$ with $\boldsymbol{A}_i$ and $\boldsymbol{A}_{i+1}$ shares a source S_i for $i=1,2,\ldots,n-1$, $\boldsymbol{A}_n$ and $\boldsymbol{A}_1$ share a source S_n. In Ref. [15], it has been shown that the MINN of this network is $(n-1)/2$ when n is odd, and is $n/2$ when n is even. We can establish the l-QNL inequalities as follows (the proof is shown in Appendix B1). If n is odd,

$$Y_n=|I_3|^{\frac{2}{n-1}}+|J_3|^{\frac{2}{n-1}}\leq\begin{cases}2^{\frac{n-3}{2(n-1)}}, & l>\dfrac{n-1}{2},\\ \sqrt{2}, & l\leq\dfrac{n-1}{2}.\end{cases}\tag{11}$$

And if n ($n\geq 4$) is even,

$$Y_n=|I_3|^{\frac{2}{n}}+|J_3|^{\frac{2}{n}}\leq\begin{cases}2^{\frac{n-2}{2n}}, & l>\dfrac{n}{2},\\ \sqrt{2}, & l\leq\dfrac{n}{2},\end{cases}\tag{12}$$

where $I_3=\langle\Pi_{\text{even }i}A_{x_i}^{+}\Pi_{\text{odd }j}A_{x_j=0}\rangle$, $J_3=\langle\Pi_{\text{even }i}A_{x_i}^{-}\Pi_{\text{odd }j}A_{x_j=1}\rangle$. Notably, for the two previous cases, we can regard $\boldsymbol{A}_i$ (i is even) as an independent party. For instance, in a 5-party cyclic network, Eq. (11) has the following form:

$$Y_n=|I_3|^{\frac{1}{2}}+|J_3|^{\frac{1}{2}}\leq\begin{cases}2^{\frac{1}{4}}, & l>2,\\ \sqrt{2}, & l\leq 2.\end{cases}\tag{13}$$

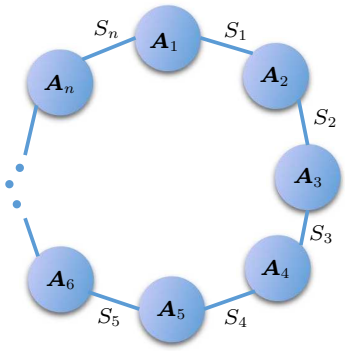


Fig. 6. The cyclic network consists of n parties and n sources.

In Appendix D, we demonstrate the quantum violation of inequalities (11) and (12) when each source is the form of $S_r=|\phi_r\rangle\langle\phi_r|$ with $|\phi_r\rangle=\cos\theta_r|00\rangle+\sin\theta_r|11\rangle$, $r=1,2,\ldots,n$ and $\theta_r\in[0,\frac{\pi}{2}]$. Moreover, l-NL in these cyclic networks can also be demonstrated by violating inequalities (11) and (12) with bounds $2^{1-\frac{2|S_c|}{n-1}}$ and $2^{1-\frac{2|S_c|}{n}}$ when n is odd and even respectively, where S_c consists of all even integers i such that party A_i only receives classical systems and $|S_c|$ denotes the cardinality of a set S_c. The details of the proof are in Appendix B2.

6. General networks

For an arbitrary finite-size quantum network with n parties, it follows from Ref. [15] that there exists an equivalent maximal-h-independent quantum network, where $\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_h$ denote independent parties and $\boldsymbol{B}$ denotes all parties in the network except for $\boldsymbol{A}_i$s. Denote by $|S_g|$ the number of all independent parties that only receive classical systems. Under this assumption, we can establish the two following inequalities:

$$W_n=|I_4|^{\frac{1}{h}}+|J_4|^{\frac{1}{h}}\leq 2^{\frac{h-|S_g|}{2h}},\tag{14}$$

$$W_n=|I_4|^{\frac{1}{h}}+|J_4|^{\frac{1}{h}}\leq 2^{\frac{h-|S_g|}{h}},\tag{15}$$

where $I_4=\langle\Pi_{x_i,i\in\{1,2,\ldots,h\}}A_{x_i}^{+}B_0\rangle$, $J_4=\langle\Pi_{x_i,i\in\{1,2,\ldots,h\}}A_{x_i}^{-}B_1\rangle$. The proof is shown in Appendix C. The violation of inequality (14) can indicate that the nonlocal correlation can be generated by these networks in which there are at most $|S_g|-1$ parties that only accept classical variables and the rest are quantum sources. The violation of inequality (15) can demonstrate the nonlocal correlation generated by networks in which there

are at most $|S_g|-1$ parties that only accept classical variables and the rest are no-signaling sources.

Example 1 (butterfly network) Let us consider the butterfly network in Fig. 7. This is a network composed of seven parties and nine bipartite sources.[24] It can be seen as a network containing three cyclic subnetworks and MINN 4, i.e., $h=4$.

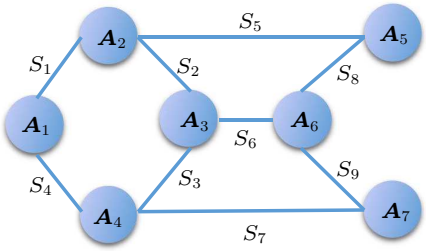


Fig. 7. The butterfly network consists of 7 parties and 9 bipartite states.

In such a case, in Eq. (14), $|S_g|$ can be 0, 1, 2, 3, or 4. The corresponding bounds are $2^{\frac{1}{2}}, 2^{\frac{3}{8}}, 2^{\frac{1}{4}}, 2^{\frac{1}{8}}$, and 1, respectively. Suppose that each physical system is a Werner state $S_i = v_i|\phi^+\rangle\langle\phi^+| + \frac{1-v_i}{4} I_4$ with $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. It is proven in Ref. [15] that there is a set of measurements performed by each party such that $|I_4|^{\frac{1}{4}} + |J_4|^{\frac{1}{4}} = \Pi_{i=1}^{9} v_i^{\frac{1}{4}} \sqrt{2}$.

Specifically, according to the inequality (14) if $\Pi_{i=1}^{9} v_i^{\frac{1}{4}} \sqrt{2} > 2^{\frac{1}{4}}$, that is $\Pi_{i=1}^{9} v_i > 1/4$, then there is at most one independent party receiving only classical systems in the network. For the butterfly network shown in Fig. 7, Fig. 8 illustrates some possible locations where classical systems may appear. Figures 8(a) and 8(b) show that there is an independent party that only receives classical physical systems. However, $|S_g| = 0$ in Figs. 8(c) and 8(d). It is worth noting that the distribution of classical physical systems here is not unique.

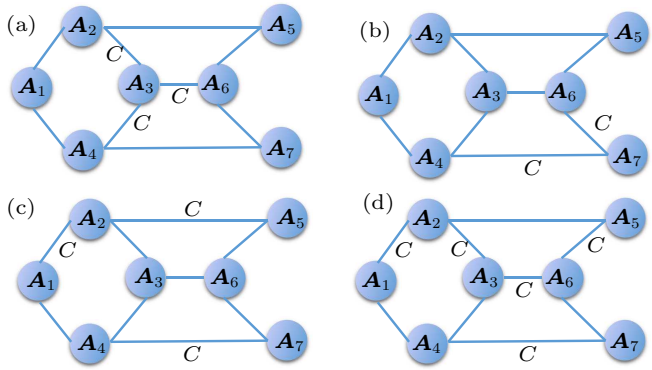


Fig. 8. Possible distributions of classical physical systems in the butterfly network, where C represents the classical physical system.

7. Discussion and conclusion

In this work, we study Bell-like inequality criteria to verify FQNN and l-QNN in more complex quantum networks, including k-fork tree networks, general acyclic networks and cyclic networks. Our key innovation is the construction of l-QNL and l-NL inequalities for more general network scenarios. Note that we also spend much time calculating the maximal independent node numbers of more complex networks since we adopt independent network local inequalities as the origin of the discussion. Indeed, it is a challenging task to determine the maximal independent node number of an arbitrary general network.

In what follows, we show a comparative analysis of the differences between acyclic networks, cyclic networks, and general networks within the proposed scheme. For acyclic networks, we first develop the approaches to determining their MINNs. These approaches are skilled and different from the cyclic network. Additionally, in an acyclic network, it is essential to determine the minimum number of classical physical systems required to ensure the existence of independent parties solely receiving classical systems. Our proposed inequality, however, can be applied to most quantum networks, especially those crucial to quantum entanglement transitions and quantum simulations.[25,26] For cyclic networks, their MINNs can be calculated like the chain network, which is a simple kind of tree network. The Bell-like inequality criteria for cyclic networks is also analogous to these chain networks. Furthermore, a general network is composed of cyclic and acyclic sub-networks. The inequality criteria that we obtain depends on the MINN. It can be determined in special networks. It seems very challenging to establish the more unified and specific Bell-like inequality criteria for general networks based on our results on cyclic and acyclic networks.

FNN and l-NN show more classes of network nonlocalities beyond standard NN, which exists in practical quantum networks. They can be applied more extensively in the development of network-based quantum information protocols. For instance, quantum information transmission needs to guarantee the absence of classical resources, which is the same requirement as FNN. Concrete protocols have been established, such as those for the quantum conference key agreement[27] and quantum secret sharing.[28,29] Our work, which contributes to verifying FNN and l-NN in more complex networks, can make such notions apply in more complex practical networks.

Indeed, we observe that the l-QNN can be more hierarchically demonstrated in high-dependence networks, such as star networks.[22] When the network structure is complicated, the l-QNN cannot be verified only if the number of l satisfies certain conditions. This will be discussed in future work. More importantly, the case $l=1$ means that it needs all sources to be non-classical in order to generate this full network nonlocality. The nonlocality seems to be the most challenging to verify among hierarchical NN in the same network configuration.

Appendix A: l-QNL inequalities for n-layer k-forked tree-shaped networks

Appendix A1: l-QNL bounds

For odd n, assume S_{it}^1 consists of integers e_1, such that party $\boldsymbol{A}_{e_1}$ only receives classical systems, and $\boldsymbol{A}_{e_1}$ is located at an odd layer. Suppose that the network contains l classical variables with $l > \frac{k-k^n}{1-k^2}$ such that $|S_{it}^1| \geq 1$. We have

$$\begin{aligned}
|I_1|^{\frac{1}{f_1(n,k)}} + |J_1|^{\frac{1}{f_1(n,k)}} &= |\langle \Pi_{i\in\mathcal{M}_{11}} A_{x_i}^{+} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=0}\rangle|^{\frac{1}{f_1(n,k)}} |\langle \Pi_{i\in\mathcal{M}_{11}} A_{x_i}^{-} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=1}\rangle|^{\frac{1}{f_1(n,k)}} \\
&\leq |\langle \Pi_{i\in S_{it}^1} A_{x_i}^{+}\rangle|^{\frac{1}{f_1(n,k)}} |\langle \Pi_{i\in\mathcal{M}_{11}; i\notin S_{it}^1} A_{x_i}^{+} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=0}\rangle|^{\frac{1}{f_1(n,k)}} \\
&\quad + |\langle \Pi_{i\in S_{it}^1} A_{x_i}^{-}\rangle|^{\frac{1}{f_1(n,k)}} |\langle \Pi_{i\in\mathcal{M}_{11}; i\notin S_{it}^1} A_{x_i}^{-} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=1}\rangle|^{\frac{1}{f_1(n,k)}},
\end{aligned} \tag{A1}$$

where $A_{x_i}^{\pm} = \frac{1}{2}(A_{x_i=0} \pm A_{x_i=1})$. It follows from $|\langle A_{x_i=0}\rangle| \leq 1$, $|\langle A_{x_i=1}\rangle| \leq 1$ $(i \in S_{it}^1)$ that we get the following inequalities:

$$|\langle \Pi_{i\in S_{it}^1} A_{x_i}^{+}\rangle \pm \langle \Pi_{i\in S_{it}^1} A_{x_i}^{-}\rangle| \leq 1, \tag{A2}$$

$$\begin{aligned}
&|\langle \Pi_{i\in\mathcal{M}_{11}; i\notin S_{it}^1} A_{x_i}^{+} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=0}\rangle|^{\frac{1}{f_1(n,k)-1}} \\
&+ |\langle \Pi_{i\in\mathcal{M}_{11}; i\notin S_{it}^1} A_{x_i}^{-} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=1}\rangle|^{\frac{1}{f_1(n,k)-1}} \leq \sqrt{2}.
\end{aligned}$$

Taking into account that

$$0 \leq |\langle \Pi_{i\in\mathcal{M}_{11}; i\notin S_{it}^1} A_{x_i}^{+} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=0}\rangle|^{\frac{1}{f_1(n,k)-1}} \leq 1,$$

$$0 \leq |\langle \Pi_{i\in\mathcal{M}_{11}; i\notin S_{it}^1} A_{x_i}^{-} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=1}\rangle|^{\frac{1}{f_1(n,k)-1}} \leq 1,$$

we have

$$|\langle \Pi_{i\in S_{it}^1} A_{x_i}^{+}\rangle| \leq \cos^2 v,$$

$$|\langle \Pi_{i\in S_{it}^1} A_{x_i}^{-}\rangle| \leq \sin^2 v,$$

$$|\langle \Pi_{i\in\mathcal{M}_{11}; i\notin S_{it}^1} A_{x_i}^{+} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=0}\rangle|^{\frac{1}{f_1(n,k)-1}} \leq \sqrt{2}\cos^2\theta,$$

$$|\langle \Pi_{i\in\mathcal{M}_{11}; i\notin S_{it}^1} A_{x_i}^{-} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=1}\rangle|^{\frac{1}{f_1(n,k)-1}} \leq \sqrt{2}\sin^2\theta$$

for some $v, \theta \in [0, \frac{\pi}{2}]$. Using the above inequalities, we can obtain

$$\begin{aligned}
|I_1|^{\frac{1}{f_1(n,k)}} + |J_1|^{\frac{1}{f_1(n,k)}} &\leq 2^{\frac{f_1(n,k)-1}{2f_1(n,k)}} \Big[(\cos v)^{\frac{2}{f_1(n,k)}} (\cos\theta)^{\frac{2(f_1(n,k)-1)}{f_1(n,k)}} \\
&\quad + (\sin v)^{\frac{2}{f_1(n,k)}} (\sin\theta)^{\frac{2(f_1(n,k)-1)}{f_1(n,k)}} \Big] \\
&\leq 2^{\frac{f_1(n,k)-1}{2f_1(n,k)}} (\cos^2\theta + \sin^2\theta) \\
&= 2^{\frac{f_1(n,k)-1}{2f_1(n,k)}}.
\end{aligned} \tag{A3}$$

Here, since the continuous function

$$\begin{aligned}
f(v) = &(\cos v)^{\frac{2}{f_1(n,k)}} (\cos\theta)^{\frac{2(f_1(n,k)-1)}{f_1(n,k)}} \\
&+ (\sin v)^{\frac{2}{f_1(n,k)}} (\sin\theta)^{\frac{2(f_1(n,k)-1)}{f_1(n,k)}}
\end{aligned}$$

has a unique extreme point $v = \theta$, inequality (A3) holds.

For even n, one can consider the set S_{it}^2, S_{it}^2 consists of integers e_2, such that party $\boldsymbol{A}_{e_2}$ only receives classical systems, and $\boldsymbol{A}_{e_2}$ is located at an even layer. If the number of classical variables $l > \frac{1-k^n}{1-k^2}$, then $|S_{it}^2| \geq 1$. Similar to the discussion of the case of odd n, we can get the inequality (8).

Appendix A2: l-NL bounds

In the following, let us first consider an odd layer tree-shaped network with l $(l \geq 1)$ classical variables and no-signaling systems. As in the previous proof, the properties of classical variables are utilized. So, we obtain that

$$\begin{aligned}
|I_1|^{\frac{1}{f_1(n,k)}} + |J_1|^{\frac{1}{f_1(n,k)}} &\leq |\langle \Pi_{i\in S_{it}^1} A_{x_i}^{+}\rangle|^{\frac{1}{f_1(n,k)}} |\langle \Pi_{i\in\mathcal{M}_{11}; i\notin S_{it}^1} A_{x_i}^{+} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=0}\rangle|^{\frac{1}{f_1(n,k)}} \\
&\quad + |\langle \Pi_{i\in S_{it}^1} A_{x_i}^{-}\rangle|^{\frac{1}{f_1(n,k)}} |\langle \Pi_{i\in\mathcal{M}_{11}; i\notin S_{it}^1} A_{x_i}^{-} \Pi_{j\in\mathcal{M}_{12}} A_{x_j=1}\rangle|^{\frac{1}{f_1(n,k)}}, \\
&\leq |\langle \Pi_{i\in S_{it}^1} A_{x_i}^{+}\rangle|^{\frac{1}{f_1(n,k)}} + |\langle \Pi_{i\in S_{it}^1} A_{x_i}^{-}\rangle|^{\frac{1}{f_1(n,k)}} \\
&\leq \Pi_{i\in S_{it}^1} (\cos v_i)^{\frac{2}{f_1(n,k)}} + \Pi_{i\in S_{it}^1} (\sin v_i)^{\frac{2}{f_1(n,k)}} \\
&\leq \sin^{\frac{2|S_{it}^1|}{f_1(n,k)}} \left(\frac{\pi}{2} - \frac{1}{|S_{it}^1|} \sum_{i\in S_{it}^1} v_i \right) + \sin^{\frac{2|S_{it}^1|}{f_1(n,k)}} \left(\frac{1}{|S_{it}^1|} \sum_{i\in S_{it}^1} v_i \right) := (\cos v)^{\frac{2|S_{it}^1|}{f_1(n,k)}} + (\sin v)^{\frac{2|S_{it}^1|}{f_1(n,k)}} \\
&\leq 2^{1-\frac{|S_{it}^1|}{f_1(n,k)}},
\end{aligned} \tag{A4}$$

where the third line of inequality (A4) holds because the bound of no-signaling systems is 1, i.e.,

$$|\langle\Pi_{i\in\mathcal{M}_{11};i\notin S_{it}^1}A_{x_i}^+\Pi_{j\in\mathcal{M}_{12}}A_{x_j=0}\rangle|\leq 1, \tag{A5}$$

$$|\langle\Pi_{i\in\mathcal{M}_{11};i\notin S_{it}^1}A_{x_i}^-\Pi_{j\in\mathcal{M}_{12}}A_{x_j=1}\rangle|\leq 1; \tag{A6}$$

the fourth line of inequality (A4) comes from $|\langle A_{x_i}^+\rangle|\leq\cos^2 v_i$, $|\langle A_{x_i}^-\rangle|\leq\sin^2 v_i$ for some $v_i\in[0,\frac{\pi}{2}]$; the fifth line of inequality (A4) holds because $\sum_{i=1}^k\sin^{\frac{1}{k}}v_i\leq\sin(k^{-1}\sum_{i=1}^k v_i)$ where $k=|S_{it}^1|$, write $v=\frac{1}{|S_{it}^1|}\sum_{i\in S_{it}^1}v_i$ and the last line of inequality (A4) comes from the fact that the continuous function $f(v)=(\cos v)^{\frac{2|S_{it}^1|}{f_1(n,k)}}+(\sin v)^{\frac{2|S_{it}^1|}{f_1(n,k)}}$ reaches its maximum value at the unique extreme point $v=\pi/4$.

Utilizing a similar approach, we can get a corresponding bound for l-NL inequality in the case of even n.

Appendix B: l-QNL inequalities for cyclic networks

Appendix B1: l-QNL bounds

For odd n, assume that there are l classical variables in this cyclic network. When $l\geq\frac{n-1}{2}$, $|S_c|\geq 1$, where S_c consists of all even integers i such that party A_i only receives classical systems and $|S_c|$ denotes the cardinality of a set S_c. It follows

$$\begin{aligned}&|I_3|^{\frac{2}{n-1}}+|J_3|^{\frac{2}{n-1}}\\ \leq&|\langle\Pi_{i\in S_c}A_{x_i}^+\rangle|^{\frac{2}{n-1}}|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^+\Pi_{\text{odd }j}A_{x_j=0}\rangle|^{\frac{2}{n-1}}\\ &+|\langle\Pi_{i\in S_c}A_{x_i}^-\rangle|^{\frac{2}{n-1}}|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^-\Pi_{\text{odd }j}A_{x_j=1}\rangle|^{\frac{2}{n-1}},\end{aligned} \tag{B1}$$

where

$$I_3=\langle\Pi_{\text{even }i}A_{x_i}^+\Pi_{\text{odd }j}A_{x_j=0}\rangle,$$

$$J_3=\langle\Pi_{\text{even }i}A_{x_i}^-\Pi_{\text{odd }j}A_{x_j=1}\rangle,$$

and $A_{x_i}^{\pm}=\frac{1}{2}(A_{x_i=0}\pm A_{x_i=1})$. Thus we have

$$\begin{aligned}&|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^+\Pi_{\text{odd }j}A_{x_j=0}\rangle|^{\frac{2}{n-1}}\\ &+|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^-\Pi_{\text{odd }j}A_{x_j=1}\rangle|^{\frac{2}{n-1}}\leq\sqrt{2},\end{aligned} \tag{B2}$$

and one can write that

$$|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^+\Pi_{\text{odd }j}A_{x_j=0}\rangle|^{\frac{2}{n-1}}\leq\sqrt{2}\cos^2\theta, \tag{B3}$$

$$|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^-\Pi_{\text{odd }j}A_{x_j=1}\rangle|^{\frac{2}{n-1}}\leq\sqrt{2}\sin^2\theta \tag{B4}$$

for some $\theta\in[0,\frac{\pi}{2}]$.

Furthermore, it follows from the classical bound $|\langle\Pi_{i\in S_c}A_{x_i}^+\pm\Pi_{i\in S_c}A_{x_i}^-\rangle|\leq 1$ and $|\langle\Pi_{i\in S_c}A_{x_i}^+\rangle|\leq 1$ that we can write $|\langle\Pi_{i\in S_c}A_{x_i}^+\rangle|\leq\cos^2 v$ and $|\langle\Pi_{i\in S_c}A_{x_i}^-\rangle|\leq\sin^2 v$. Substituting this and (B3), (B4) into (B1), we get

$$\begin{aligned}&|I_3|^{\frac{2}{n-1}}+|J_3|^{\frac{2}{n-1}}\\ \leq&(\cos v)^{\frac{4}{n-1}}(\sqrt{2}\cos\theta)^{\frac{n-3}{n-1}}+(\sin v)^{\frac{4}{n-1}}(\sqrt{2}\sin\theta)^{\frac{n-3}{n-1}}\\ \leq&2^{\frac{n-3}{2(n-1)}}.\end{aligned} \tag{B5}$$

For even n, by using the classical bound and the inequalities (B3), (B4), one obtains

$$\begin{aligned}&|I_3|^{\frac{2}{n}}+|J_3|^{\frac{2}{n}}\\ \leq&|\langle\Pi_{i\in S_c}A_{x_i}^+\rangle|^{\frac{2}{n}}|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^+\Pi_{\text{odd }j}A_{x_j=0}\rangle|^{\frac{2}{n}}\\ &+|\langle\Pi_{i\in S_c}A_{x_i}^-\rangle|^{\frac{2}{n}}|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^-\Pi_{\text{odd }j}A_{x_j=1}\rangle|^{\frac{2}{n}}\\ \leq&2^{\frac{n-1}{2n}}[(\cos v)^{\frac{4}{n}}(\cos\theta)^{\frac{2(n-2)}{n}}+(\sin v)^{\frac{4}{n}}(\sin\theta)^{\frac{2(n-2)}{n}}]\\ \leq&2^{\frac{n-1}{2n}},\end{aligned} \tag{B6}$$

where the last line of inequality (B6) comes from a unique extreme point $v=\theta$ of the continuous function $f(v)=(\cos v)^{\frac{4}{n}}(\cos\theta)^{\frac{2(n-2)}{n}}+(\sin v)^{\frac{4}{n}}(\sin\theta)^{\frac{2(n-2)}{n}}$.

Appendix B2: l-NL bounds

We first deal with the odd n case. Similar to inequalities (A5) and (A6), it follows from the no-signaling principle that

$$|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^+\Pi_{\text{odd }j}A_{x_j=0}\rangle|\leq 1, \tag{B7}$$

$$|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^-\Pi_{\text{odd }j}A_{x_j=1}\rangle|\leq 1. \tag{B8}$$

For the case of $l\geq\frac{n-1}{2}$, one finds

$$\begin{aligned}&|I_3|^{\frac{2}{n-1}}+|J_3|^{\frac{2}{n-1}}\\ \leq&|\langle\Pi_{i\in S_c}A_{x_i}^+\rangle|^{\frac{2}{n-1}}|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^+\Pi_{\text{odd }j}A_{x_j=0}\rangle|^{\frac{2}{n-1}}\\ &+|\langle\Pi_{i\in S_c}A_{x_i}^-\rangle|^{\frac{2}{n-1}}|\langle\Pi_{\text{even }i;i\notin S_c}A_{x_i}^-\Pi_{\text{odd }j}A_{x_j=1}\rangle|^{\frac{2}{n-1}}\\ \leq&|\langle\Pi_{i\in S_c}A_{x_i}^+\rangle|^{\frac{2}{n-1}}+|\langle\Pi_{i\in S_c}A_{x_i}^-\rangle|^{\frac{2}{n-1}}\\ \leq&\Pi_{i\in S_c}(\cos v_i)^{\frac{4}{n-1}}+\Pi_{i\in S_c}(\sin v_i)^{\frac{4}{n-1}}\\ \leq&\left[\sin\left(\frac{\pi}{2}-\frac{1}{|S_c|}\sum_{i=1}^{|S_c|}v_i\right)\right]^{\frac{4|S_c|}{n-1}}+\left[\sin\left(\frac{1}{|S_c|}\sum_{i=1}^{|S_c|}v_i\right)\right]^{\frac{4|S_c|}{n-1}}\\ =&\left[\sin\left(\frac{\pi}{2}-v\right)\right]^{\frac{4|S_c|}{n-1}}+(\sin v)^{\frac{4|S_c|}{n-1}}\\ :=&(\cos v)^{\frac{4|S_c|}{n-1}}+(\sin v)^{\frac{4|S_c|}{n-1}}\\ \leq&2^{1-\frac{2|S_c|}{n-1}},\end{aligned} \tag{B9}$$

where the fifth line of inequality (B9) comes from the inequality $|\langle A_{x_i}^+\rangle|\leq\cos^2 v_i$ and $|\langle A_{x_i}^-\rangle|\leq\sin^2 v_i$. The sixth line of inequality (B9) can be proved similar to the case of (A4). Write $v=\frac{1}{|S_c|}\sum_{i=1}^{|S_c|}v_i$. By calculating the extreme value of a continuous function $f(v)=(\cos v)^{\frac{4|S_c|}{n-1}}+(\sin v)^{\frac{4|S_c|}{n-1}}$, we can obtain the inequality (B9) when we take $v=\pi/4$.

Note that, for the case of even n, substituting inequalities (B7) and (B8) into $|I_3|^{\frac{2}{n}}+|J_3|^{\frac{2}{n}}$, a similar inequality can be obtained, i.e., inequality (12) holds.

Appendix C: Inequalities for h-independent quantum network

Now we deal with the more general finite-size networks, i.e., there exists an equivalent network with h independent ob-

servers and $h \geq 2$. This kind of network is called h independent. Suppose that the set S_g refers to the set of indicators r such that spacelike separated party $\boldsymbol{A}_r$ only receives classical systems and $|S_g|$ is the number of parties meeting conditions.

Recall the quantum bound for the $h-|S_g|$-independent quantum network in Ref. [22], we have

$$|\langle \Pi_{x_i, i\notin S_g} A_{x_i}^{+} B_0\rangle|^{\frac{1}{h-|S_g|}} + |\langle \Pi_{x_i, i\notin S_g} A_{x_i}^{-} B_1\rangle|^{\frac{1}{h-|S_g|}} \leq \sqrt{2}. \quad \text{(C1)}$$

Considering that

$$|\langle \Pi_{x_i, i\notin S_g} A_{x_i}^{+} B_0\rangle|^{\frac{1}{h-|S_g|}} \leq 1,$$

$$|\langle \Pi_{x_i, i\notin S_g} A_{x_i}^{-} B_1\rangle|^{\frac{1}{h-|S_g|}} \leq 1,$$

it follows that

$$|\langle \Pi_{x_i, i\notin S_g} A_{x_i}^{+} B_0\rangle|^{\frac{1}{h-|S_g|}} \leq \sqrt{2}\cos^2\theta, \quad \text{(C2)}$$

$$|\langle \Pi_{x_i, i\notin S_g} A_{x_i}^{-} B_1\rangle|^{\frac{1}{h-|S_g|}} \leq \sqrt{2}\sin^2\theta \quad \text{(C3)}$$

for some $\theta \in [0, \frac{\pi}{2}]$.

Similar to the proof of inequality (A3), we have

$$\begin{aligned}
&|I_4|^{\frac{1}{h}} + |J_4|^{\frac{1}{h}} \\
&= |\langle \Pi_{x_i, i\in\{1,2,\ldots,h\}} A_{x_i}^{+} B_0\rangle|^{\frac{1}{h}} + |\langle \Pi_{x_i, i\in\{1,2,\ldots,h\}} A_{x_i}^{-} B_1\rangle|^{\frac{1}{h}} \\
&\leq |\langle \Pi_{i\in S_g} A_{x_i}^{+}\rangle|^{\frac{1}{h}} |\langle \Pi_{x_i, i\notin S_g} A_{x_i}^{+} B_0\rangle|^{\frac{1}{h}} \\
&\quad + |\langle \Pi_{i\in S_g} A_{x_i}^{-}\rangle|^{\frac{1}{h}} |\langle \Pi_{x_i, i\notin S_g} A_{x_i}^{-} B_1\rangle|^{\frac{1}{h}} \\
&\leq (\cos v)^{\frac{2}{h}} (\sqrt{2}\cos^2\theta)^{\frac{h-|S_g|}{h}} + (\sin v)^{\frac{2}{h}} (\sqrt{2}\sin^2\theta)^{\frac{h-|S_g|}{h}} \\
&= 2^{\frac{h-|S_g|}{2h}} \left[(\cos v)^{\frac{2}{h}} (\cos\theta)^{\frac{2(h-|S_g|)}{h}} + (\sin v)^{\frac{2}{h}} (\sin\theta)^{\frac{2(h-|S_g|)}{h}} \right] \\
&\leq 2^{\frac{h-|S_g|}{2h}}.
\end{aligned} \quad \text{(C4)}$$

Here, the last inequality comes from the fact that the maximum value of the binary continuous function $f(v,\theta) = (\cos v)^{\frac{2}{h}} (\cos\theta)^{\frac{2(h-|S_g|)}{h}} + (\sin v)^{\frac{2}{h}} (\sin\theta)^{\frac{2(h-|S_g|)}{h}}$ is 1 when $(v,\theta) = (0,0)$.

Let us assume that the network consists of classical physical systems and non-signaling systems, then we get

$$|\langle \Pi_{x_i, i\notin S_g} A_{x_i}^{+} B_0\rangle| \leq 1, \quad \text{(C5)}$$

$$|\langle \Pi_{x_i, i\notin S_g} A_{x_i}^{-} B_1\rangle| \leq 1. \quad \text{(C6)}$$

Similar to the previous proofs, we have the following inequalities:

$$\begin{aligned}
&|I_4|^{\frac{1}{h}} + |J_4|^{\frac{1}{h}} \\
&\leq |\langle \Pi_{i\in S_g} A_{x_i}^{+}\rangle|^{\frac{1}{h}} + |\langle \Pi_{i\in S_g} A_{x_i}^{-}\rangle|^{\frac{1}{h}} \\
&\leq \Pi_{i\in S_g} (\cos v_i)^{\frac{2}{h}} + \Pi_{i\in S_g} (\sin v_i)^{\frac{2}{h}} \\
&\leq \left[\sin\left(\frac{\pi}{2} - v\right) \right]^{\frac{2|S_g|}{h}} + (\sin v)^{\frac{2|S_g|}{h}} \\
&:= (\cos v)^{\frac{2|S_g|}{h}} + (\sin v)^{\frac{2|S_g|}{h}} \\
&\leq 2^{\frac{h-|S_g|}{h}},
\end{aligned} \quad \text{(C7)}$$

where $v = \frac{1}{|S_g|} \sum_{i=1}^{|S_g|} v_i$, the third line of inequality (C7) follows from the inequality $|\langle A_{x_i}^{+}\rangle \pm \langle A_{x_i}^{-}\rangle| \leq 1$, which means $|\langle A_{x_i}^{+}\rangle| \leq \cos^2 v_i$ and $|\langle A_{x_i}^{-}\rangle| \leq \sin^2 v_i$; the last line of inequality (C7) arises from the computation of the extreme point of $g(v) = (\cos v)^{\frac{2|S_g|}{h}} + (\sin v)^{\frac{2|S_g|}{h}}$.

Appendix D: Quantum violations

Here we show quantum violations of l-QNL inequalities in n-layer k-forked tree-shaped quantum networks or cyclic networks. Assume that each source can be written in the form of $S_r = |\phi_r\rangle\langle\phi_r|$ with $|\phi_r\rangle = \cos\theta_r|00\rangle + \sin\theta_r|11\rangle$.

In the tree network, for odd layer number $n \geq 3$, using the quantum observables $\{A_{x_r}\}_{r=1}^{g_1(n,k)}$ shown in Ref. [15], we can get

$$|I_1|^{\frac{1}{f_1(n,k)}} + |J_1|^{\frac{1}{f_1(n,k)}} = (1 + \Pi_{r=1}^{g_1(n,k)} (\sin 2\theta_r)^{\frac{2}{f_1(n,k)}})^{\frac{1}{2}}. \quad \text{(C8)}$$

Thus, if $\Pi_{r=1}^{g_1(n,k)} \sin 2\theta_r > (2^{\frac{f_1(n,k)-1}{f_1(n,k)}} - 1)^{\frac{f_1(n,k)}{2}}$, then inequality (7) can be violated and this implies that there exists l-QNN for tree-shaped networks with $l > \frac{k-k^n}{1-k^2}$. For even n, using the same observables $\{A_{x_r}\}_{r=1}^{g_1(n,k)}$, we have

$$|I_2|^{\frac{1}{f_2(n,k)}} + |J_2|^{\frac{1}{f_2(n,k)}} = (1 + \Pi_{r=1}^{g_1(n,k)} (\sin 2\theta_r)^{\frac{2}{f_2(n,k)}})^{\frac{1}{2}}, \quad \text{(C9)}$$

and prove that there is l-QNN for tree-shaped networks with $l > \frac{k^2-k^{n+1}}{1-k^2}$ when $\Pi_{r=1}^{g_1(n,k)} \sin 2\theta_r > (2^{\frac{f_2(n,k)-1}{f_2(n,k)}} - 1)^{\frac{f_2(n,k)}{2}}$.

For cyclic networks, it follows from

$$|I_3|^{\frac{2}{n-1}} + |J_3|^{\frac{2}{n-1}} = (1 + \Pi_{i=1}^{n} (\sin 2\theta_i)^{\frac{4}{n-1}})^{\frac{1}{2}}, \quad \text{(C10)}$$

$$|I_3|^{\frac{2}{n}} + |J_3|^{\frac{2}{n}} = (1 + \Pi_{i=1}^{n} (\sin 2\theta_i)^{\frac{4}{n}})^{\frac{1}{2}} \quad \text{(C11)}$$

that we obtain a violation of inequality (11) when $\Pi_{i=1}^{n} \sin 2\theta_i > (2^{\frac{n-3}{n-1}} - 1)^{\frac{n-1}{4}}$, and a violation inequality (12) when $\Pi_{i=1}^{n} \sin 2\theta_i > (2^{\frac{n-1}{n}} - 1)^{\frac{n}{4}}$.

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