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Bipartite consensus problems of Lurie multi-agent systems over signed graphs: A contraction approach

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This paper examines the bipartite consensus problems for the nonlinear multi-agent systems in Lurie dynamics form with cooperative and competitive communication between different agents. Based on the contraction theory, some new conditions for the nonlinear Lurie multi-agent systems reaching bipartite leaderless consensus and bipartite tracking consensus are presented. Compared with the traditional methods, this approach degrades the dimensions of the conditions, eliminates some restrictions of the system matrix, and extends the range of the nonlinear function. Finally, two numerical examples are provided to illustrate the efficiency of our results.

Keywords: contraction theory, virtual system, bipartite consensus, Lurie multi-agent systems

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1. Introduction

In recent years, the coordination control of multi-agent systems has witnessed tremendous attention of scientists from different disciplines due to its wide applications in the real world. Especially the consensus problem as the basic problem of the cooperative control has formed many research results.[1–6] However, the aforementioned results are dedicated to studying the consensus problems in a cooperative network. This means that the relations between different agents in the network are cooperative. Actually, in the real world, the relation between different agents in a network is not only cooperative, but also antagonistic, such as social networks,[7] biological systems,[8] robot competitions,[9] and opinion variances.[10–12] Thus, signed graphs were proposed to depict the collaborative and antagonistic interactions between different agents in a network, where a positive edge means collaboration and a negative edge represents an antagonistic interaction.[13] Then the consensus problem is called the bipartite consensus problem, which was firstly proposed by Altafini.[14] Different from traditional consensus problems, the bipartite consensus problem makes all agents in two subgraphs converge to a common value but in opposite signs. This can meet the need of most systems in the real life. Since then, research on the consensus problems over the signed graph has been widely examined.

Following the research trace of the traditional consensus problems, bipartite consensus problems have also been addressed with the same produces. It is worthwhile noticing that bipartite consensus problems concerning general linear multi-agent systems have been widely investigated. This is due to the fact that the first-order, second-order and high-order multi-agent systems can be transformed into the linear multi-agent systems. Wen[15] employed the Lyapunov stability and graph theory to study the bipartite tracking consensus for linear multi-agent systems under the undirected topology. Under some assumptions, Zhang[16] discussed the problem in directed case. Moreover, the state feedback and output feedback controller were proposed to study bipartite consensus problems of the linear multi-agents in Ref. [13]. Two output feedback control laws were designed through reduced-order observer to consider the bipartite leaderless consensus and bipartite tracking consensus problems of the linear multi-agent systems, respectively.[17] However, compared with the linear multi-agent systems, the nonlinear multi-agent systems have much wider application range. Linear systems can be regarded as the special case of the nonlinear systems. Therefore, based on the conclusions about the bipartite consensus of the linear multi-agent systems, many scholars have paid their attention to the bipartite consensus of nonlinear systems. The existing results can be divided into two main categories. On the one hand, researchers considered the nonlinear multi-agent systems in the traditional Lipschitz case.[18–23] For example, Wu[18] investigated the bipartite tracking consensus of traditional nonlinear multi-agent systems in directed topology. In the meantime, the same nonlinear system with quantized communication was considered in Ref. [19] by using the Lyapunov technique. Ren[20] also analyzed the bipartite leader-following consensus of singular multi-agent systems with the same nonlinear function. Event-triggered controller[21,22] and adaptive controller[23] were also constructed to investigate the bipartite consensus of the nonlinear multi-agent systems. On the other hand, the nonlinear function was extended to a more

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general case. Zhang[24] firstly addressed the bipartite tracking consensus of the nonlinear multi-agent systems in generalized Lipschitz-type form. Here the odd property about the nonlinear function was removed. For this reason, the nonlinear multi-agent systems with generalized Lipschitz-type function have been explored. The system with switching topology was investigated in Refs. [25,26]. The H_∞ bipartite consensus and the exponential bipartite tracking consensus of this system with some different parameters were presented in Refs. [27,28]. Based on the aforementioned results, we can find that the existing literature about the bipartite consensus of the nonlinear multi-agent systems has the following properties.

(i) On the condition of the nonlinear function, the first one is the traditional Lipschitz dynamics, with the assumption that the nonlinear function must be odd. The other is the generalized Lipschitz-type function, then the assumption oddness can be relaxed.

(ii) On the research method, Lyapunov technique has always been in dominant position.

(iii) On the assumptions, the system matrix (A,B) must be stabilizable. This can make the Riccati equation (or inequality) have a solution.

(iv) On the communication topology, the signed graph always be required to have a spanning tree and is structurally balanced.

Actually, for the similarity between the consensus and the synchronization problem, the bipartite synchronization was also concerned in Refs. [29–33]. Lurie dynamics as the classics nonlinear system has been studied for many years, because it contains a linear feed-forward part and a nonlinear part which satisfies some sector conditions. Just as mentioned in Ref. [30], the theoretical results about the Lurie dynamics can be applied to the nonlinear systems which satisfy QUAD condition or Lipschitz condition after making some modifications. However, to our knowledge, only a few works[30–32] discussed the bipartite synchronization or consensus of the Lurie-type multi-agent systems. Most results are concerned on the traditional consensus problem. Therefore, we will aim to study the bipartite consensus of the nonlinear multi-agent systems with Lurie dynamics in this paper. The contributions of this paper are stated as follows. First, the Lyapunov stability theory is replaced by contraction theory to study the bipartite consensus of the Lurie-type multi-agent systems. This method avoids finding a proper Lyapunov function and formulating the system in compact form. This can make the condition of the bipartite consensus be only related to the dimension of the single agent. Second, it is unnecessary to guarantee the Riccati equation to have a solution, so the assumption about the system matrix can be removed. Third, different from the existing results about the consensus problems of the Lurie multi-agent systems, the range of the nonlinear function is generalized.

The outline of this paper is summarized as follows. Some preliminaries about the contraction theory and signed graph theory are provided in Section 2. The discussions about the bipartite leaderless consensus and bipartite tracking consensus of the Lurie multi-agent system are presented in Section 3. Two numerical simulations are stated in Section 4. The main conclusions about this paper are drawn in Section 5.

Throughout this paper, we denote $R^{n\times m}$ as the set of $n\times m$ real matrices, $\mathrm{diag}\{d_1,d_2,\ldots,d_n\}$ as a diagonal matrix with entries $d_1,d_2,\ldots,d_n$ and sign $(\cdot)$ as the standard signum function. For a vector x, $\|x\|$ represents the Euclidean norm. For a matrix A, $\lambda_{\max}(A)$ denotes its the maximum eigenvalue, A^{T} denotes its the transpose, and $A_{\mathrm{s}}=\frac{1}{2}(A+A^{\mathrm{T}})$ denotes its the symmetric part. I_n denotes the n-dimensional identity matrix.

2. Preliminaries

To facilitate the understanding of the theoretical results about this paper, the introduction about the contraction theory will be concisely presented in this section, especially the partial contraction theory which can provide some simple and direct results about the synchronization problems of the complex networks.

2.1. Contraction theory

Consider a nonlinear system in the following form:

$$\dot{x}=f(x,t),$$

where $x(t)\in R^n$ and the nonlinear function $f:R^n\longrightarrow R^n$. Here, assume that the nonlinear function is continuously differentiable, and denote δx as a virtual displacement between two different neighboring trajectories, then the squared distance between these trajectories can be formulated as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\delta x^{\mathrm{T}}\delta x)&=2\delta x^{\mathrm{T}}\delta\dot{x}=2\delta x^{\mathrm{T}}\frac{\partial f}{\partial x}\delta x\\&=2\delta x^{\mathrm{T}}J\delta x\le 2\lambda_{\max}(x,t)\delta x^{\mathrm{T}}\delta x,\end{aligned}$$

where $J=\partial f/\partial x$, $\lambda_{\max}(x,t)$ denotes the largest eigenvalue of the matrix $J_{\mathrm{s}}=\frac{1}{2}(J+J^{\mathrm{T}})$.

Definition 1[34] If there exists a scalar $\alpha>0$, $\forall x,\forall t\ge 0$, s.t. $\partial f/\partial x\le -\alpha I<0$. As all matrix inequalities will refer to symmetric part of the square matrix involved, we can further write $\frac{1}{2}(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial x}^{\mathrm{T}})\le -\alpha I<0$, then the Jacobian matrix $J=\partial f/\partial x$ is uniformly negative definiteness (UND).

This means that if $\lambda_{\max}(x,t)$ is UND, then any infinitesimal length $\|\delta x\|$ will converge exponentially to zero. After computing path integration at the fixed time, the solutions of the system $\dot{x}=f(x,t)$ will converge exponentially to a single trajectory, independent of the initial conditions.[35,36]

2.2. Partial contraction theory

Just as stated in Ref. [37], the partial contraction theory extends the contraction analysis to a more general case, especially in investigating the stability of the large-scale systems or studying synchronization behaviors. More detail theoretical analysis about the partial contraction theory can be found in Refs. [35,36]. To obtain the main results of this paper, the most important theorem about the original system and the auxiliary system is presented in the following.

Lemma 1[35] Consider a nonlinear system of the form

$$\dot{x} = f(x,x,t)$$

and assume that the auxiliary system

$$\dot{y} = f(y,x,t)$$

is contracting with respect to y. If a particular solution of the auxiliary y-system verifies a smooth specific property, then all trajectories of the original x-system verify this property exponentially. The original system is said to be partially contracting.

It is obvious that $y(t) = x(t)$ is a particular solution of the auxiliary system. Thus, if we can make sure that the auxiliary system has a good property, then the original x-system will have this one. For this reason, some special properties of the original system can be replaced by discussing the similar properties of the auxiliary system, especially if the original system is more complex.

2.3. Signed graph theory

Without loss of generality, let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ denote the signed graph with the node set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ and the adjacency matrix $\mathcal{A} = (a_{ij})_{N\times N}$. Here the directed pair $(v_j, v_i) \in \mathcal{E}$ means that the node v_j can obtain information from the node v_i. The element of the adjacency matrix $\mathcal{A} = (a_{ij})_{N\times N}$ is defined as $a_{ij} \neq 0$, if $(v_j, v_i) \in \mathcal{E}$, otherwise $a_{ij} = 0$. Moreover, $a_{ij} > 0$ means that the relationship between the agents i and j is cooperative, then the edge (v_j, v_i) is positive. Here $a_{ij} < 0$ means that the relationship between the agents i and j is competitive, the edge (v_j, v_i) is negative. Meantime, self-edges (v_i, v_i) are not allowed in this paper, which indicates $a_{ii} = 0$. A graph $\mathcal{G}$ is called undirected if $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$, and directed otherwise. A graph $\mathcal{G}$ is called being unsigned if all edges are positive and signed if the edges can be either positive or negative. The neighbor set of node v_i is denoted as $N_i = \{v_j | (v_j, v_i) \in \mathcal{E}\}$. A (directed) path from the node v_i to the node v_j is a sequence of successive (ordered) edges in the form of $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_m}, v_j)$. The signed graph has a directed spanning tree if there exists a node called root which has a direct path to any other nodes. A (directed) signed graph is called being strongly connected if there is a (directed) path between any two distinct nodes. A signed graph is structurally balanced, if the node set can be partitioned into two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$, with $a_{ij} > 0$, $\forall (v_i, v_j \in \mathcal{V}_p, (p \in \{1,2\})$ and $a_{ij} < 0$, $\forall v_i \in \mathcal{V}_p$, $\forall v_j \in \mathcal{V}_{3-p}$, $(p \in \{1,2\})$.

The Laplacian matrix $\mathcal{L} = (l_{ij})_{N\times N}$ associated with the signed graph $\mathcal{G}$ is defined as $l_{ii} = \sum_{j=1, j\neq i}^{N} |a_{ij}|$ and $l_{ij} = -a_{ij}$, $i \neq j$. It is obvious that the sum for the ith row of the Laplacian matrix is not equal to zero.

Lemma 2[24] Assume that the signed directed graph $\mathcal{G}$ is structurally balanced, then there exists a matrix $W = \mathrm{diag}\{w_1, w_2, \ldots, w_N\}$ with $w_i = 1$, $i \in \mathcal{V}_1$ and $w_i = -1$, $i \in \mathcal{V}_2$ such that $W\mathcal{A}W = \overline{\mathcal{A}}$, where $\mathcal{A} = (a_{ij})_{N\times N}$ and $\overline{\mathcal{A}} = (|a_{ij}|)_{N\times N}$ are the adjacency matrices associated with the signed and its unsigned graphs, respectively.

3. Main results

3.1. Bipartite leaderless consensus of Lurie multi-agent systems

Consider a multi-agent system consisting of N agents, each agent is described by the Lurie nonlinear dynamics

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + B_1 f(y_i(t)) + Bu_i(t), \\ y_i(t) &= Cx_i(t), \end{aligned} \tag{1}$$

where $x_i \in R^n$, $u_i \in R^p$, and $y_i \in R^m$ are the state vector, control input, and the output vector of the agent i, respectively; $f(y_i(t)) = [f_1(y_{i1}(t)), \ldots, f_m(y_{im}(t))]^{\mathrm{T}} \in R^m$ is a nonlinear function, satisfying the following conditions:

$$\begin{aligned} &\text{(i) } f_l(0) = 0, \ \ y_{il} = c_l^{\mathrm{T}} x_i(t), \\ &\text{(ii) } k_l^- \leq \frac{f_l(\sigma_1) - f_l(\sigma_2)}{\sigma_1 - \sigma_2} \leq k_l^+, \ \ \forall \sigma_1 \neq \sigma_2, \ l = 1,2,\ldots,m, \\ &\text{(iii) } f_l(-\sigma) = -f_l(\sigma). \end{aligned} \tag{2}$$

$A \in R^{n\times n}$, $B_1 \in R^{n\times m}$, $B \in R^{n\times p}$, $C = (c_1, \ldots, c_m)^{\mathrm{T}} \in R^{m\times n}$ are known constant matrices.

Assumption 1 The signed graph associated with the Lurie nonlinear multi-agent network is structurally balanced and has a directed spanning tree.

Definition 2 Considering the multi-agent network described by Eqs. (1) and (2), the bipartite leaderless consensus can be achieved if

$$\begin{aligned} &\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \ \ i,j \in \mathcal{V}_p, \ \ (p = 1,2), \\ &\lim_{t\to\infty} \|x_i(t) + x_j(t)\| = 0, \ \ i \in \mathcal{V}_p, \ \ j \in \mathcal{V}_{3-p}, \ (p = 1,2). \end{aligned} \tag{3}$$

Based on Lemma 2 and Assumption 1, Eq. (3) can be equivalently expressed as

$$\lim_{t\to\infty} \|w_i x_i(t) - w_j x_j(t)\| = 0, \ \forall i,j = 1,2,\ldots,N, \ (i \neq j),$$

where $w_i = 1$, $\forall i \in \mathcal{V}_1$ and $w_i = -1$, $\forall i \in \mathcal{V}_2$.

To solve the bipartite leaderless consensus problem of the Lurie multi-agent system, a distributed consensus protocol is proposed:

$$u_i = cK\sum_{j=1}^{N}|a_{ij}|(x_i(t)-\mathrm{sgn}(a_{ij})x_j(t)), \tag{4}$$

where a_{ij} is the element of the adjacency matrix $\mathcal{A}$ about the graph $\mathcal{G}$, $c>0$ denotes the coupling strength, $K\in R^{p\times n}$ is the feedback matrix which will be determined.

Following the definition of the Laplacian matrix $\mathcal{L}$ and applying protocol (4) to the system (1), the Lurie multi-agent system can be presented as

$$\begin{aligned}\dot{x}_i(t) &= Ax_i(t)+B_1f(y_i(t))+BcK\sum_{j=1}^{N}l_{ij}x_j(t),\\ y_i(t) &= Cx_i(t).\end{aligned} \tag{5}$$

Combine Lemma 2 and Assumption 1, system (5) can be transformed into the standard form. That because the signed graph is structurally balanced, there must exist a diagonal $W=\mathrm{diag}\{w_1,w_2,\ldots,w_N\}$ such that $W\mathcal{A}W=\overline{\mathcal{A}}=(\overline{a}_{ij})=(|a_{ij}|)\geq 0$. Then the Laplacian matrix $\mathcal{L}$ associated with the adjacency matrix $\mathcal{A}$ also can be converted into $\overline{\mathcal{L}}=(\bar{l}_{ij})=(w_il_{ij}w_j)=(-|a_{ij}|)$ for all $i\neq j$ and $\bar{l}_{ii}=\sum_{j=1,j\neq i}^{N}|a_{ij}|$. It is apparent that $\overline{\mathcal{L}}$ is the traditional Laplacian matrix which has the property $\sum_{j=1}^{N}\bar{l}_{ij}=0,\forall i=1,2,\ldots,N$.

Motivated by the above discussion, now we analyze the bipartite leaderless consensus problem of the network (5) via some coordinate transform. Let $\bar{x}_i(t)=w_ix_i(t)$, then one has $x_i(t)=w_i\bar{x}_i(t)$ and

$$\begin{aligned}\dot{\bar{x}}_i(t) &= w_i\dot{x}_i(t)\\ &= A\bar{x}_i(t)+B_1w_if(y_i(t))+BcK\sum_{j=1}^{N}w_il_{ij}w_j\bar{x}_j(t)\\ &= A\bar{x}_i(t)+B_1w_if(Cx_i(t))+BcK\sum_{j=1}^{N}\bar{l}_{ij}\bar{x}_j(t).\end{aligned} \tag{6}$$

However, for the nonlinear function $f(\cdot)$ is an odd function and $w_i^2=1$, then we can easily get $w_if(Cx_i(t))=w_if(Cw_i\bar{x}_i(t))=f(C\bar{x}_i(t))$. Combining it with Eq. (6), one has

$$\dot{\bar{x}}_i(t)=A\bar{x}_i(t)+B_1f(C\bar{x}_i(t))+BcK\sum_{j=1}^{N}\bar{l}_{ij}\bar{x}_j(t). \tag{7}$$

Remark 1 To solve the bipartite consensus problem of the multi-agent system with the signed graph, we will employ the contraction theory to study this problem. In the literature, the consensus problem is always transformed into the stability problem for discussing. Therefore, the Lyapunov stability theory and the linear matrix inequality are always utilized to study this problem. In most cases, the muti-agent system is expressed in compact matrix form, then the system will be in a higher dimensional state space. To simplify the calculation, contraction theory is proposed to replace the Lyapunov stability theory. In fact, a nonlinear dynamic system is called contracting if initial conditions or temporary disturbances are forgotten exponentially fast. That means all trajectories of the system converge to a unique trajectory. We only need to determine whether the displacement between any two neighboring trajectories converges to zero. The relation between the contraction theory and Krasovskii's theorem can be found in Refs. [34,37]. For the superiority of the contraction theory, we adopt it for the first time in this paper to solve the bipartite consensus problem for the nonlinear multi-agent system in Lurie dynamics. Although the method has been used to analyze the bipartite synchronization for the nonlinear complex networks, but there exists some differences between them. (i) The bipartite synchronization problem devotes to the selection of the coupling strength, but the bipartite consensus focuses on the determination of the feedback matrix. So, compared with the synchronization problem, consensus problem is more complicated. (ii) Through contraction theory, matrix measure is used to get the conditions for the bipartite synchronization. However, through constructing an appropriate virtual system, the bipartite consensus problem can be easily solved by contraction theory.

Then, in accordance with the partial contraction theory, a suitable auxiliary system of the system (7) is given as

$$\begin{aligned}\dot{\widetilde{x}}_i(t) &= A\widetilde{x}_i(t)+B_1f(C\widetilde{x}_i(t))+BcK\bar{l}_{ii}\widetilde{x}_i(t)\\ &\quad +BcK\sum_{j=1,i\neq j}^{N}\bar{l}_{ij}\bar{x}_j(t).\end{aligned} \tag{8}$$

It is apparent that $\widetilde{x}_i(t)=\bar{x}_i(t)$ is a particular solution of the system (8). Take the virtual displacement of the auxiliary system (8) as $\delta\widetilde{x}_i(t)$, then we can get

$$\begin{aligned}\delta\dot{\widetilde{x}}_i(t) &= A\delta\widetilde{x}_i(t)+B_1\frac{\partial f(\widetilde{y}_i(t))}{\partial\widetilde{x}_i(t)}\delta\widetilde{x}_i(t)+BcK\bar{l}_{ii}\delta\widetilde{x}_i(t)\\ &= \left(A+B_1\frac{\partial f(\widetilde{y}_i(t))}{\partial\widetilde{x}_i(t)}+BcK\bar{l}_{ii}\right)\delta\widetilde{x}_i(t)=J\delta\widetilde{x}_i(t).\end{aligned} \tag{9}$$

Theorem 1 Under Assumption 1, the bipartite consensus for the system (1) and (2) can be achieved if there exists a suitable feedback matrix K such that the symmetric part of the matrix J is UND.

Proof From Eq. (9), the square distance between the neighboring trajectories can be written in differential form as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\widetilde{x}_i^{\mathrm{T}}(t)\delta\widetilde{x}_i(t))=2\delta\widetilde{x}_i^{\mathrm{T}}(t)J\delta\widetilde{x}_i(t).$$

Let $\lambda_{\max}$ denote the largest eigenvalue of the matrix $J_{\mathrm{s}}=\frac{1}{2}(J^{\mathrm{T}}+J)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\widetilde{x}_i^{\mathrm{T}}(t)\delta\widetilde{x}_i(t))=2\delta\widetilde{x}_i^{\mathrm{T}}(t)J\delta\widetilde{x}_i(t)\leq 2\lambda_{\max}\delta\widetilde{x}_i^{\mathrm{T}}(t)\delta\widetilde{x}_i(t).$$

This means $\|\delta\widetilde{x}_i\| \leq \|\delta\widetilde{x}_i(0)\| \mathrm{e}^{\int_0^t \lambda_{\max}\,\mathrm{d}t}$. Because the matrix J is UND, there exists a constant number $\beta > 0$ such that $\lambda_{\max} \leq -\beta I < 0$. Thus,

$$\|\delta\widetilde{x}_i\| \leq \|\delta\widetilde{x}_i(0)\| \mathrm{e}^{\int_0^t \lambda_{\max}\,\mathrm{d}t} \leq \|\delta\widetilde{x}_i(0)\| \mathrm{e}^{-\int_0^t \beta\,\mathrm{d}t} = \mathrm{e}^{-\beta t}\|\delta\widetilde{x}_i(0)\|.$$

This implies that any infinitesimal length $\|\delta\widetilde{x}\|$ converges exponentially to zero.[28] After carrying out path integration, then all the solution trajectories of system (8) will converge exponentially to a single trajectory, independent of the initial conditions. From Lemma 1, it is clear that as the special solution of the system (8), system (7) also has the contracting behaviors. The bipartite consensus of the system (1) can be solved.

Remark 2 By utilizing the contraction analysis theory, Theorem 1 provides a new condition for the Lurie dynamics reaching bipartite consensus. It is noticeable that the condition is only related to the single agent's dimension with $n \times n$. However, the existing bipartite criteria inspired by the Lyapunov stability theory and LMIs are always in higher dimensions. In Refs. [20,24,26,33], the LMI dimensions are $2n \times 2n$, some results are even in more higher cases with $3n \times 3n$[25] or $5n \times 5n$[27] dimensions.

3.2. Bipartite tracking consensus of Lurie multi-agent systems

In this subsection, the bipartite tracking consensus problem of the Lurie multi-agent system will be examined. Then a leader is introduced into the system (1) with the following dynamics:

$$\begin{aligned} \dot{x}_0(t) &= Ax_0(t) + B_1 f(y_0(t)), \\ y_0(t) &= Cx_0(t). \end{aligned} \tag{10}$$

Definition 3 The bipartite tracking consensus for the system (1) and (10) can be achieved if

$$\begin{aligned} &\lim_{t\to\infty} \|x_i(t) - x_0(t)\| = 0, \quad i \in \mathcal{V}_1, \\ &\lim_{t\to\infty} \|x_i(t) + x_0(t)\| = 0, \quad i \in \mathcal{V}_2. \end{aligned} \tag{11}$$

Then, a new distributed protocol for achieving the bipartite tracking consensus is designed as

$$u_i = cK\Big(\sum_{j=1}^{N} |a_{ij}|(x_i(t) - \mathrm{sgn}(a_{ij})x_j(t)) + a_{i0}(x_i(t) - w_i x_0(t))\Big), \tag{12}$$

where $a_{i0} > 0$ means that the leader is the neighbor of agent i ($i = 1,2,\ldots,N$), and $a_{i0} = 0$ otherwise. Here w_i has the same definitions as in the foregoing analysis.

Assumption 2 For the given directed multi-agent network consisting of a leader and followers, it has a spanning tree with the leader as root node, and the communication topology described by the followers is connected and structurally balanced.

In the light of Lemma 2 and Assumption 2, Eq. (11) can also be equivalently expressed as

$$\lim_{t\to 0} \|x_i(t) - w_i x_0(t)\| = 0, \quad i = 1,2,\ldots,N.$$

Denoting the state error between the agent i and the leader as $e_i(t) = x_i(t) - w_i x_0(t)$. Let $\bar{e}_i(t) = w_i e_i(t)$, then we can get $\bar{e}_i(t) = w_i x_i(t) - x_0(t)$. Applying the distributed protocol (12) to the system (1), one can obtain

$$\begin{aligned} \dot{\bar{e}}_i(t) &= w_i\dot{x}_i(t) - \dot{x}_0(t) \\ &= Aw_i x_i(t) + B_1 w_i f(Cx_i(t)) + w_i BcK\Big[\sum_{j=1}^{N} |a_{ij}|(x_i(t) \\ &\quad - \mathrm{sgn}(a_{ij})x_j(t))\Big] + w_i BcKa_{i0}e_i(t) \\ &\quad - Ax_0(t) - B_1 f(Cx_0(t)). \end{aligned}$$

Owing to $\mathrm{sgn}(a_{ij})w_i = w_j$, one has

$$\begin{aligned} \dot{\bar{e}}_i(t) &= Aw_i x_i(t) + B_1 w_i f(Cx_i(t)) + w_i BcK\Big[\sum_{j=1}^{N} |a_{ij}|[(x_i(t) \\ &\quad - w_i x_0(t)) - \mathrm{sgn}(a_{ij})(x_j(t) - w_j x_0(t))]\Big] \\ &\quad + w_i BcKa_{i0}e_i(t) - Ax_0(t) - B_1 f(Cx_0(t)) \\ &= Aw_i x_i(t) + B_1 w_i f(Cx_i(t)) + w_i BcK\Big[\sum_{j=1}^{N} |a_{ij}|[e_i(t) \\ &\quad - \mathrm{sgn}(a_{ij})e_j(t)]\Big] + w_i BcKa_{i0}e_i(t) - Ax_0(t) \\ &\quad - B_1 f(Cx_0(t)) \\ &= Aw_i x_i(t) + B_1 w_i f(Cx_i(t)) + w_i BcK\sum_{j=1}^{N} l_{ij}e_j(t) \\ &\quad + w_i BcKa_{i0}e_i(t) - Ax_0(t) - B_1 f(Cx_0(t)) \\ &= A\bar{x}_i(t) + B_1 f(C\bar{x}_i(t)) + BcK\sum_{j=1}^{N} \bar{l}_{ij}\bar{e}_j(t) \\ &\quad + BcKa_{i0}\bar{e}_i(t) - Ax_0(t) - B_1 f(Cx_0(t)) \\ &= A\bar{e}_i(t) + B_1\eta(\bar{e}_{y_i}(t)) + BcK\sum_{j=1}^{N} \bar{l}_{ij}\bar{e}_j(t) + BcKa_{i0}\bar{e}_i(t). \end{aligned} \tag{13}$$

Here $\eta(\bar{e}_{y_i}(t)) = f(C\bar{x}_i(t)) - f(Cx_0(t))$, $\bar{e}_{y_i}(t) = C\bar{x}_i(t) - Cx_0(t) = C\bar{e}_i(t)$.

Theorem 2 Under Assumption 2, the bipartite tracking consensus for the multi-agent systems (1) and (10) by the protocol (12) can be solved if there exists an appropriate feedback matrix K such that the symmetric part of the matrix $\hat{J}$ is UND, where

$$\hat{J} = A + BcK\bar{l}_{ii} + BcKa_{i0} + B_1\frac{\partial\eta(\hat{e}_{y_i}(t))}{\partial\hat{e}_i(t)}.$$

Proof Formulating the virtual system for the system (13) as

$$\dot{\hat{e}}_i(t) = A\hat{e}_i(t) + BcK\bar{l}_{ii}\hat{e}_i(t) + BcK\sum_{j=1,i\neq j}^{N} \bar{l}_{ij}\bar{e}_j(t)$$

$$+BcKa_{i0}\hat{e}_i(t)+B_1\eta(\hat{e}_{y_i}(t)). \tag{14}$$

When $\hat{e}_i(t)=\bar{e}_i(t)$, system (14) converts into the system (13). Therefore, system (13) is a special solution of the system (14). Defining $\delta\hat{e}_i(t)$ as the virtual displacement of the system (14), it can be expressed in differential form as

$$\begin{aligned}\delta\dot{\hat{e}}_i(t)&=A\delta\hat{e}_i(t)+BcK\bar{l}_{ii}\delta\hat{e}_i(t)+BcKa_{i0}\delta\hat{e}_i(t)\\&\quad+B_1\frac{\partial\eta(\hat{e}_{y_i}(t))}{\partial\hat{e}_i(t)}\delta\hat{e}_i(t)\\&=\Big(A+BcK\bar{l}_{ii}+BcKa_{i0}+B_1\frac{\partial\eta(\hat{e}_{y_i}(t))}{\partial\hat{e}_i(t)}\Big)\delta\hat{e}_i(t)\\&=\hat{J}\delta\hat{e}_i(t).\end{aligned}\tag{15}$$

Then, the square distance between the neighboring trajectories of the system (15) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\hat{e}_i^{\mathrm{T}}(t)\delta\hat{e}_i(t))=2\delta\hat{e}_i^{\mathrm{T}}(t)\hat{J}\delta\hat{e}_i(t).$$

Denoting $\hat{\lambda}_{\max}$ as the largest eigenvalue of the matrix $\hat{J}_{\mathrm{s}}=\frac{1}{2}(\hat{J}^{\mathrm{T}}+\hat{J})$, we can derive

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\hat{e}_i^{\mathrm{T}}(t)\delta\hat{e}_i(t))=2\delta\hat{e}_i^{\mathrm{T}}(t)\hat{J}\delta\hat{e}_i(t)\le 2\hat{\lambda}_{\max}\delta\hat{e}_i^{\mathrm{T}}(t)\delta\hat{e}_i(t).$$

Owing to the matrix $\hat{J}$ being UND, there must exist a constant number $\hat{\beta}>0$ such that $\hat{J}_{\mathrm{s}}\le-\hat{\beta}I<0$. Then we can get

$$\|\delta\hat{e}_i\|\le\|\delta\hat{e}_i(0)\|\,\mathrm{e}^{\int_0^t\hat{\lambda}_{\max}\mathrm{d}t}\le\|\delta\hat{e}_i(0)\|\,\mathrm{e}^{-\int_0^t\hat{\beta}\,\mathrm{d}t}=\mathrm{e}^{-\hat{\beta}t}\|\delta\hat{e}_i(0)\|.$$

Thus, the system (15) will be global exponential stable, independent of the initial conditions. Then all the solution trajectories of system (14) converge exponentially to each other. As a special solution of the system (14), the system (13) will also have this property. This means that the bipartite tracking consensus problem of the systems (1) and (10) can be solved.

4. Numerical examples

In this section, two examples are provided to illustrate the effectiveness of the proposed results.

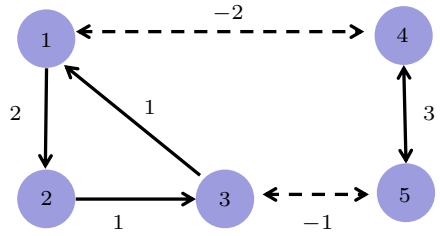


Fig. 1. Signed graph $\mathcal{G}_1$ as example 1.

Example 1 Consider a multi-agent system composed by five agents, the interactions between these agents are depicted by a signed directed digraph $\mathcal{G}_1$ shown in Fig. 1. We can find that the graph is structurally balanced and has a spanning tree. Obviously, Assumption 1 is satisfied. All nodes are divided into two parts with $\mathcal{V}_1=\{1,2,3\}$, $\mathcal{V}_2=\{4,5\}$. Furthermore, $\mathcal{W}=\mathrm{diag}\{1,1,1,-1,-1\}$, the adjacency matrix $\mathcal{A}$ and the Laplacian matrix $\mathcal{L}$ are

$$\mathcal{A}=\begin{bmatrix}0&0&1&-2&0\\2&0&0&0&0\\0&1&0&0&-1\\-2&0&0&0&3\\0&0&-1&3&0\end{bmatrix},\quad\mathcal{L}=\begin{bmatrix}3&0&-1&2&0\\-2&2&0&0&0\\0&-1&2&0&1\\2&0&0&5&-3\\0&0&1&-3&4\end{bmatrix}.$$

Each agent is modeled by the nonlinear Lurie dynamics with the following parameters:

$$A=\begin{bmatrix}-1.5&0.1\\0.1&-1.5\end{bmatrix},\ B_1=\begin{bmatrix}2&-0.1\\-5&3\end{bmatrix},\ C=\begin{bmatrix}1&0\\0&1\end{bmatrix},$$
$$B=\begin{bmatrix}1\\0\end{bmatrix},\ f(y_i)=\tanh(y_i).$$

Then from Theorem 1, we can obtain

$$\begin{aligned}J&=A+B_1\frac{\partial f(C\widetilde{x}_i(t))}{\partial\widetilde{x}_i(t)}+BcK\bar{l}_{ii}\\&=\begin{bmatrix}-1.5+c\bar{l}_{ii}k_1+2M_1&0.1-0.1M_1+c\bar{l}_{ii}k_2\\0.1-5M_1&-1.5+3M_1\end{bmatrix},\end{aligned}$$

where $M_1=\frac{\partial f(\widetilde{y}_{il})}{\partial\widetilde{y}_{il}}\in[0,1]$, $l=1,2$.

To make sure that the matrix

$$J_{\mathrm{s}}=\begin{bmatrix}-1.5+c\bar{l}_{ii}k_1+2M_1&\dfrac{0.2-5.1M_1+c\bar{l}_{ii}k_2}{2}\\\dfrac{0.2-5.1M_1+c\bar{l}_{ii}k_2}{2}&-1.5+3M_1\end{bmatrix}$$

is UND, the feedback matrix K must satisfy the following conditions:

$$\begin{aligned}&\text{(i)}\quad-1.5+c\bar{l}_{ii}k_1+2M_1\le0;\\&\text{(ii)}\quad(-1.5+c\bar{l}_{ii}k_1+2M_1)(-1.5+3M_1)\\&\qquad-\frac{(0.2-5.1M_1+c\bar{l}_{ii}k_2)^2}{4}\ge0.\end{aligned}$$

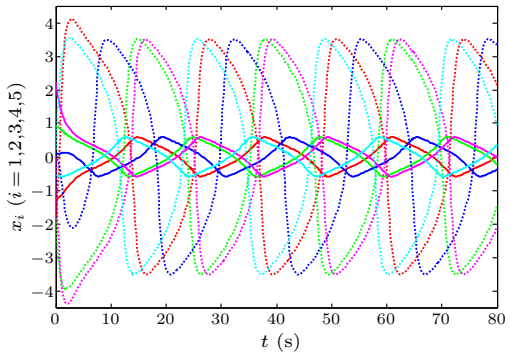


Fig. 2. Behaviors of all agents without the distributed controller.

Thus, any values of the k_i $(i=1,2)$ satisfying the above conditions would ensure the system to reach the bipartite consensus. Without loss of generality, if let $c\bar{l}_{ii}k_2=-0.2$, then

$\bar{l}_{ii}k_1 < -1.5$. Without the distributed controller, the behaviors of all agents are shown in Fig. 2. Take $c = 1$ and $\bar{l}_{ii} = \text{mindeg}(i) = 2$, we can get $K = \begin{bmatrix}-1.9 & -0.1\end{bmatrix}$. Then, the trajectories of all agents in Lurie form under the protocol (4) are shown in Fig. 3. Obviously, the agents are separated into two subgroups, the bipartite consensus of the Lurie dynamics multi-agents under directed signed topology can be solved. The initial states of the system (4) are selected randomly.

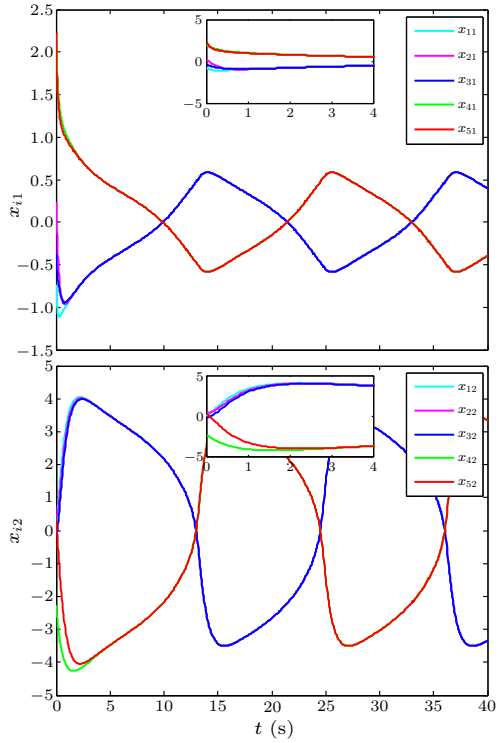


Fig. 3. Trajectories of Lurie multi-agent system under the protocol (4).

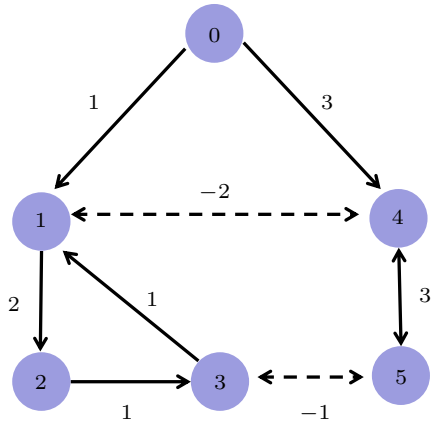


Fig. 4. Signed graph $\mathcal{G}_2$ as example 2.

Example 2 Now a leader is introduced into example 1. Then, consider a Lurie dynamical multi-agent system containing one leader labeled as 0 and five followers with some parameters in Chua's circuits form. The directed signed communication topology between six agents as Refs. [18,24] is shown in Fig. 4. The parameters of Chua's circuit are selected as

$$A = \begin{bmatrix}-2.574 & 9 & 0\\ 1 & -1 & 1\\ 0 & -14 & -0.01\end{bmatrix},\ B_1 = \begin{bmatrix}-5.284\\ 0\\ 0\end{bmatrix},$$

$$C = \begin{bmatrix}1 & 0 & 0\end{bmatrix},\ B = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix},$$

$$f(y_i(t)) = \frac{1}{2}(|x_{i1}(t)-1| - |x_{i1}(t)+1|).$$

Following Theorem 2, we can reach

$$\hat{J} = A + BcK\bar{l}_{ii} + BcKa_{i0} + B_1\frac{\partial\eta(\hat{e}_{y_i}(t))}{\partial\hat{e}_i(t)} = \begin{bmatrix}-2.574+ck_1s_0-5.284M_2 & 9+ck_2s_0 & ck_3s_0\\ 1 & -1 & 1\\ 0 & -14 & -0.01\end{bmatrix},$$

where $s_0 = \bar{l}_{ii} + a_{i0}$ and $M_2 = \frac{\partial\eta(\hat{e}_{y_{i1}}(t))}{\partial e_{y_{i1}}(t)} \in [-1,0]$.

So the symmetric part of Jacobian $\hat{J}_s = \frac{1}{2}(\hat{J}^{\mathrm{T}} + \hat{J})$ can be presented as

$$\hat{J}_s = \begin{bmatrix}-2.574+ck_1s_0-5.284M_2 & \frac{10+ck_2s_0}{2} & \frac{ck_3s_0}{2}\\ \frac{10+ck_2s_0}{2} & -1 & \frac{-13}{2}\\ \frac{ck_3s_0}{2} & \frac{-13}{2} & -0.01\end{bmatrix}.$$

To ensure the system achieve consensus, the following conditions must be satisfied:

(i) $-2.574+ck_1s_0-5.284M_2 \le 0$;

(ii) $(2.574-ck_1s_0+5.284M_2) - \dfrac{(10+ck_2s_0)^2}{4} \ge 0$;

(iii) $0.01(-2.574+ck_1s_0-5.284M_2) - \dfrac{13ck_3s_0(10+ck_2s_0)}{4} + \dfrac{(ck_3s_0)^2}{4} - \dfrac{169(-2.574+ck_1s_0-5.284M_2)}{4} + \dfrac{0.01(10+ck_2s_0)^2}{4} \le 0.$

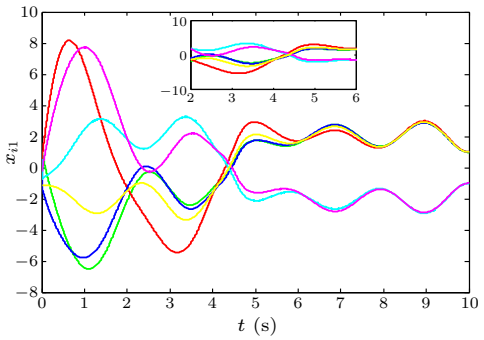


Fig. 5. Bipartite tracking consensus of $x_{i1}(t)$ $(i = 0,1,2,\ldots,5)$ in Chua's circuit.

Therefore, if the feedback matrix $K = \begin{bmatrix}k_1 & k_2 & k_3\end{bmatrix}$ satisfies the above conditions, the contracting behaviors of the virtual system (14) can be guaranteed, all followers will

trace the leader's state. Without loss of generality, select $cl_{ii}k_2 = -42$, $cl_{ii}k_3 = -208$ and $-258.71 \leq cl_{ii}k_1 \leq -253.426$. Then let $c = 100$, the feedback matrix K is chosen as $\begin{bmatrix}-1.28 & -0.21 & -1.04\end{bmatrix}$. The behaviors of all agents with the leader are shown in Figs. 5–7.

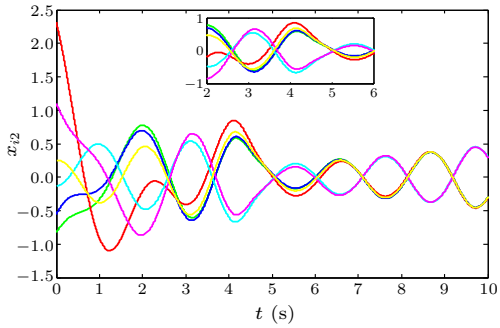


Fig. 6. Bipartite tracking consensus of $x_{i2}(t)$ $(i = 0,1,2,\ldots,5)$ in Chua's circuit.

It can be observed from Figs. 5–7 that the behaviors of all agents are different at the initial state. With the controller (12), the agents 1–3 follow the leader's trajectories, but the agents 4 and 5 follow the opposite values of the leader. Figure 8 depicts the errors between the leader and the followers, it is evident that the errors between them converge to zero. This means that the bipartite tracking consensus can be successfully achieved under the protocol (12).

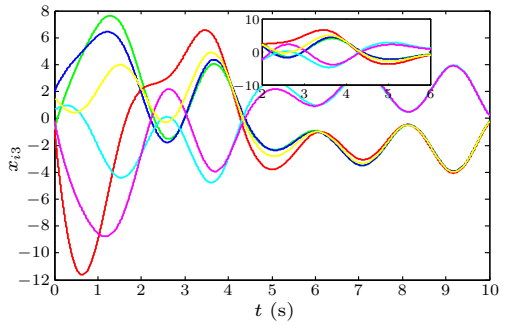


Fig. 7. Bipartite tracking consensus of $x_{i3}(t)$ $(i = 0,1,2,\ldots,5)$ in Chua's circuit.

Remark 3 Compared to the existing results, such as Refs. [30,31], the solution of the linear inequality matrix belongs to $R^{(N-1)n\times(N-1)n}$ or $R^{Nn\times Nn}$, whereas we only consider the bipartite consensus problem in $R^{n\times n}$. The calculation can be simplified.

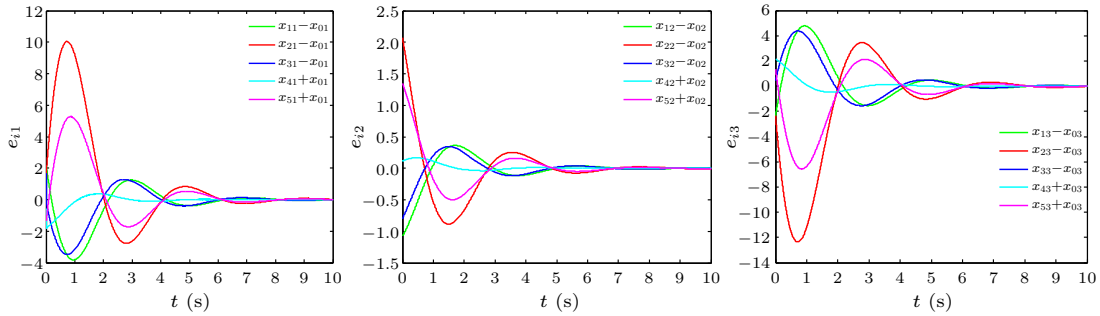


Fig. 8. Errors between the leader and the followers.

5. Conclusions

Distributed bipartite consensus problems for the nonlinear Lurie-type multi-agent systems with or without a leader have been investigated in this paper. Unlike some previous results, the contraction analysis theory has been employed to guarantee the multi-agent systems to achieve bipartite consensus. First, without constructing the Lyapunov function and solving the Riccati equation, the bipartite consensus problems have been transformed into the contracting behaviors of the virtual systems. Second, it is unnecessary to recast the multi-agent systems in compact form, we only consider the system in lower-dimensional case. Third, the range of the nonlinear function is more general. Furthermore, some simulations have been given to verify the theoretical analysis. Future work will focus on discussing the bipartite consensus problems for the Lurie-type multi-agent systems with dynamic topology and time-delay.

Acknowledgments

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