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A wealth distribution model with a non-Maxwellian collision kernel

Jun Meng(孟俊)[1], Xia Zhou(周霞)[2,†], and Shaoyong Lai(赖绍永)[3]

[1] *College of Mathematics and Statistics, Kashi University, Kashi* 844006, *China*
[2] *College of Mathematics and Physics, Mianyang Teacher's College, Mianyang* 621000, *China*
[3] *School of Mathematics, Southwestern University of Finance and Economics, Chengdu* 611130, *China*

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A non-Maxwellian collision kernel is employed to study the evolution of wealth distribution in a multi-agent society. The collision kernel divides agents into two different groups under certain conditions. Applying the kinetic theory of rarefied gases, we construct a two-group kinetic model for the evolution of wealth distribution. Under the continuous trading limit, the Fokker–Planck equation is derived and its steady-state solution is obtained. For the non-Maxwellian collision kernel, we find a suitable redistribution operator to match the taxation. Our results illustrate that taxation and redistribution have the property to change the Pareto index.

Keywords: kinetic theory, non-Maxwellian collision kernel, wealth distribution, Pareto index

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1. Introduction

The study of wealth distribution has a long history. Since Pareto's discovery at the end of the 19th century that the wealth distribution in Western countries is characterized by a power law,[1] scholars have applied various techniques to construct evolutionary models of wealth distribution in an attempt to find a universal form of wealth distribution and explain what factors affect the wealth distribution in an economy. About two decades ago, scholars began to study wealth distribution in multi-agent systems with the help of the well-established kinetic theory of rarefied gases, and achieved good results. The key to apply the theory is to construct the Boltzmann equations for the evolution of wealth distribution, which is usually referred as the kinetic equations, which essentially describe the change of agents' wealth over time.[2]

The construction of the Boltzmann equation for the evolution of wealth distribution depends on the interaction rules and the frequency of interactions (i.e., collision kernels) between agents. Cordier *et al.*[3] employed a constant interaction frequency (i.e., the Maxwellian collision kernel, also called the constant collision kernel) and an interaction rule incorporating the savings propensity and the market risk to construct the Boltzmann equation for the evolution of wealth distribution, obtaining that the wealth distribution at the equilibrium state is an inverse gamma distribution. Bisi *et al.*[4] introduced taxation into the interaction rule, find a suitable redistribution operator, construct a kinetic model containing taxation and redistribution by using a constant collision kernel, and discuss the effects of taxation and redistribution on wealth distribution. Pareschi and Toscani[5] introduced knowledge into the interaction rules, and constructed a kinetic model of wealth distribution with a constant collision kernel to examine the effect of knowledge on wealth distribution. There are many works by using constant collision kernels to construct kinetic models of wealth distributions (see Refs. [6–10]).

Recently, Furioli *et al.*[11] pointed out that the constant collision kernel may ignore some behaviors of agents, and the wealth distribution model constructed with the collision kernel is somewhat different from reality. To overcome this deficiency, Furioli *et al.*[11] proposed a non-Maxwellian collision kernel (variable collision kernel), i.e.,

$$K(v,w) = \kappa(vw)^{\delta}, \tag{1}$$

where constants $\kappa > 0$ and $0 < \delta \leq 1$, v and w represent the wealth of two different agents. The collision kernel (1) is used to exclude the possibility that agents with no wealth could enter the market and enhance transactions in which the amount of money of both agents is noticeable. The results in Ref. [11] state that the constant δ changes the Pareto index of the steady-state solution and demonstrate that the solution of the resultant Fokker–Planck equation converges exponentially to equilibrium state. Subsequently, Del-Mul[12] extended the collision kernel (1) to the following form:

$$K(v,w) = \kappa v^{\alpha} w^{\beta}, \tag{2}$$

where constants $0 < \alpha \leq 1, 0 < \beta \leq 1$, and $\kappa > 0$. Indices α and β represent certain collective behavior of agents, such as the intensity of willingness to trade. If $\alpha = \beta$, equation (2) becomes the collision kernel (1). Moreover, if $\alpha = \beta = 0$, the collision kernel (2) becomes a Maxwellian collision kernel. When $\alpha = \beta$, Del-Mul[12] obtained the same result as that in Ref. [11]. However, Del-Mul[12] did not discuss the effect of the collision kernel (2) on the wealth distribution when $\alpha \neq \beta$. Ballante *et al.*[13] examined the wealth distribution of agents with uncertain trading propensities by using the collision kernel (1). Although collision kernel (1) is used in Refs. [11,13]

†Corresponding author. E-mail: xiazhou2017@163.com

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to model wealth distribution, the collision kernel (1) implies that the intensity of agents' willingness to trade is the same, and thus it does not produce a wealth distribution characterized by bimodal peaks.

The feature of the collision kernel (2) is that the intensity of each agent's willingness to trade affects the frequency of interaction. Low transaction intensity assigns a low frequency of interaction, while high transaction intensity assigns a high frequency of interaction. For this reason, different transaction intensities divide agents into different groups. Our work is to consider the wealth distribution from different agent groups and analyze the influence of different transaction intensities on wealth distribution. Our results illustrate that when two groups of agents differ largely in the intensity of their willingness to trade, it leads to a bimodal shape in the steady-state distribution of wealth. This means that if there are two completely different types of agents in a group, some of them prefer a high interaction frequency and others prefer a low one. Then the wealth distribution of the group possesses bimodal. The bimodal distribution of the actual data has been observed in the income distribution of Argentina during the economic crisis.[14]

From the work in Refs. [4,7], it is known that the wealth kinetic model with taxation and redistribution associated with the Maxwellian collision kernel is able to derive the statistical laws of wealth distribution and analyze the impact of taxation and redistribution on wealth distribution. Inspired by the ideas in Refs. [4,7], we employ a non-Maxwellian collision kernel (2) to construct the wealth distribution model with taxation and redistribution. A redistribution operator associated with the non-Maxwellian collision kernel is analyzed. The results confirm that under the non-Maxwellian collision kernel, taxation and redistribution are able to modify the Pareto index in the steady-state solution and regulate inequality in the distribution of wealth.

The main contributions of our work are twofold. Firstly, we construct a kinetic model about the evolution of wealth distribution of two groups by using the non-Maxwell collision kernel (2) and obtain a mechanism for the bimodal formation of wealth distribution, i.e., when there is a large difference between the intensities of willingness to trade in the two groups (which leads to a large difference in the frequency of interactions), the steady-state wealth distribution takes on the shape of a bimodal peak. This mechanism is different from those used in Refs. [6,8] for the formation of bimodal distributions. Using the Maxwellian collision kernel, During and Toscani[6] considered the effect of differences in saving propensities between different countries (or groups) on wealth distribution, and Dimarco *et al.*[8] examined the impact of differences in saving propensities of different groups on wealth distribution when the COVID-19 is prevalent. The results in Refs. [6,8] showed that the wealth distribution is bimodal when differences between saving propensities of different groups are sufficiently large. Secondly, we construct a kinetic model of wealth evolution with taxation and redistribution by using the non-Maxwellian collision kernel (2). For the aim of taxation, we find a redistribution operator corresponding to Eq. (2), which is different from the redistribution operators associated with Maxwellian collision kernels in Refs. [4,7].

The organizational structure of this paper is as follows. In Section 2, employing the collision kernel (2), we construct a two-group kinetic model for the evolution of wealth. Utilizing the continuous trading limit, we obtain steady-state solutions under certain assumptions. In Section 3, we discuss the influence of taxation and redistribution on wealth distribution under the collision kernel (2). The conclusion is stated in Section 4.

2. A two-group kinetic model for the evolution of wealth

According to the kinetic theory,[2] the evolution of wealth distribution depends on the microscopic interactions between agents involved in the exchange of wealth. Agents participate in transactions with part of their wealth and each transaction is viewed as an interaction. The wealth after the interaction is usually assumed to be nonnegative.

2.1. Model setup

As stated in the introduction, agents are divided into two groups according to their different transaction intensities. That is, in the collision kernel (2), $\alpha = \beta$ means that there is only one type of agent in the market, and $\alpha \neq \beta$ indicates that there are two groups of agents in the market. We denote by $f_1(w,t)$ and $f_2(w,t)$ ($w \in R_+$), the wealth distribution at time $t \geq 0$ of agents with transaction willingness intensity of α and β, respectively. The total wealth distribution is then given by

$$f_{\rm tot}(w,t) = f_1(w,t) + f_2(w,t). \tag{3}$$

In general, the density function $f_{\rm tot}(w,t)$ satisfies $\int_{R_+} f_{\rm tot}(w,t)\,{\rm d}w = 1$.

Following the binary trading rule in Ref. [3], when two agents with wealth v and w participate a trade, the post trade wealth becomes

$$\begin{aligned} v_* &= (1-\lambda)v + \lambda w + \eta_1 v, \\ w_* &= (1-\lambda)w + \lambda v + \eta_2 w, \end{aligned} \tag{4}$$

where $\lambda \in (0,1)$ is the transaction rate parameter. The quantities η_1 and η_2 are independent and identically distributed random variables with mean zero and finite variance $\sigma > 0$. To make the post-interaction wealth non-negative, it is usually assumed that $\eta_i \geq \lambda - 1$ ($i = 1,2$). Using binary trading rules (4) and assumption of random variables η_1 and η_2, we acquire

$$\langle v_* + w_* \rangle = v + w,$$

implying that the total wealth is conserved in a statistical sense, where symbol $\langle\cdot\rangle$ indicates the mathematical expectation.

Utilizing the standard methods of kinetic theory,[2] we obtain that the evolution of wealth distribution $f_i(w,t)$ $(i=1,2)$ satisfies the Boltzmann equation

$$\frac{\partial f_i(w,t)}{\partial t} = Q(f_i,f_j)(w)+Q(f_i,f_i)(w), \quad i,j=1,2 \quad \text{and} \quad i\neq j, \tag{5}$$

where the collision operator

$$Q(f_i,f_j)(w) = \left\langle \int_{R_+} K_{ij}(v,w)(f_i(v_*)f_j(w_*)-f_i(v)f_j(w))\,\mathrm{d}v \right\rangle \tag{6}$$

represents transactions from different groups, and

$$Q(f_i,f_i)(w) = \left\langle \int_{R_+} K_{ii}(v,w)(f_i(v_*)f_i(w_*)-f_i(v)f_i(w))\,\mathrm{d}v \right\rangle \tag{7}$$

represents transactions among agents in the same group. In Eqs. (6) and (7), (v,w) denote the pre-transaction pair that generates the post-transaction pair (v_*,w_*) according to the interaction rule (4), while the collision kernels $K_{ij}(v,w)$ and $K_{ii}(v,w)$ are given by

$$\begin{aligned} &K_{12}(v,w)=\kappa w^{\alpha}v^{\beta},\\ &K_{11}(v,w)=\kappa(wv)^{\alpha}, \ \text{ for } \ i=1,\ j=2,\\ &K_{21}(v,w)=\kappa w^{\beta}v^{\alpha},\\ &K_{22}(v,w)=\kappa(wv)^{\beta}, \ \text{ for } \ i=2,\ j=1. \end{aligned}$$

In general, the weak form of the Boltzmann equation is adopted, i.e., for any smooth function $\phi(w)$ having supported set in R_+, the evolution of $f_i(w,t)(i=1,2)$ follows the equation

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+} f_i(w,t)\phi(w)\,\mathrm{d}w\\ &=\frac{1}{2}\left\langle \int_{R_+^2} K_{ij}(v,w)(\phi(w_*)+\phi(v_*)\right.\\ &\quad\left.-\phi(w)-\phi(v))f_i(w,t)f_j(v,t)\,\mathrm{d}v\mathrm{d}w\right\rangle\\ &\quad+\frac{1}{2}\left\langle \int_{R_+^2} K_{ii}(v,w)(\phi(w_*)+\phi(v_*)\right.\\ &\quad\left.-\phi(w)-\phi(v))f_i(w,t)f_i(v,t)\,\mathrm{d}v\mathrm{d}w\right\rangle\\ &=\left\langle \int_{R_+^2} K_{ij}(v,w)(\phi(w_*)-\phi(w))f_i(w,t)f_j(v,t)\,\mathrm{d}v\mathrm{d}w\right\rangle\\ &\quad+\left\langle \int_{R_+^2} K_{ii}(v,w)(\phi(w_*)-\phi(w))f_i(w,t)f_i(v,t)\,\mathrm{d}v\mathrm{d}w\right\rangle. \end{aligned} \tag{8}$$

For more details on the application of the Boltzmann equation, the reader is referred to Refs. [15–17].

In Eq. (8), when $\phi(w)=1$, mass conservation is obtained, meaning that the following equation holds:

$$\begin{aligned} &\int_{R_+} f_1(w,t)\,\mathrm{d}w=\int_{R_+} f_1(w,0)\,\mathrm{d}w=\rho_1,\\ &\int_{R_+} f_2(w,t)\,\mathrm{d}w=\int_{R_+} f_2(w,0)\,\mathrm{d}w=\rho_2, \end{aligned}$$

where the positive constants ρ_1 and ρ_2 satisfy $\rho_1+\rho_2=1$. If $\phi(w)=w$, utilizing Eq. (4), for any $K_{ij}(v,w)$ and $K_{ii}(v,w)$, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+} wf_i(w,t)\,\mathrm{d}w\\ &=\frac{1}{2}\left\langle \int_{R_+^2} K_{ij}(v,w)(w_*+v_*-w-v)f_i(w,t)f_j(v,t)\,\mathrm{d}v\mathrm{d}w\right\rangle\\ &\quad+\frac{1}{2}\left\langle \int_{R_+^2} K_{ii}(v,w)(w_*+v_*-w-v)f_i(w,t)f_i(v,t)\,\mathrm{d}v\mathrm{d}w\right\rangle\\ &=0, \end{aligned}$$

indicating that the average wealth is conservation. For the sake of the subsequent discussion, we assume that first three moments of the solution $f_i(w,t)$ of Eq. (8) are bounded.

2.2. Continuous trading limit and steady-state solutions

Generally speaking, it is very difficult to solve the Boltzmann equation directly. For this reason, the Fokker–Planck equation is used as its approximate equation to analyze the large-time behavior of transaction. It is assumed that most transactions correspond to a very small exchange of money, that is, in binary trading rule (4), the transaction rate parameter λ is selected to be small enough, but at the same time it keeps certain traces of the phenomena that affect the market rules at the macro level such as market risk, taxation, and redistribution. This kind of asymptotic analysis is called "continuous trading limit" (see Refs. [3,7,18]). It is achieved by introducing the scaling

$$\lambda\to\xi\lambda, \quad \eta_1\to\sqrt{\xi}\eta_1, \quad \eta_2\to\sqrt{\xi}\eta_2, \tag{9}$$

where ξ is a small positive parameter.

Using expression (9) yields

$$\begin{aligned} &\langle w_*-w\rangle=\xi\lambda(v-w),\\ &\langle(w_*-w)^2\rangle=\xi^2\lambda^2(v-w)^2+\xi\sigma w^2. \end{aligned}$$

For a smooth function $\phi(w)$, utilizing the Taylor series $\phi(w_*)$ around $\phi(w)$, we obtain

$$\begin{aligned} &\langle\phi(w_*)-\phi(w)\rangle\\ &=\xi\left[\lambda\phi'(w)(v-w)+\frac{\sigma w^2}{2}\phi''(w)\right]+\langle R_\xi(v,w)\rangle, \end{aligned} \tag{10}$$

where $R_\xi(v,w)$ denotes the remainder term including ξ^n $(n\geq 3/2)$.

Inserting Eq. (10) into Eq. (8) gives rise to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+} f_1(w,t)\phi(w)\,\mathrm{d}w$$

$$
\begin{aligned}
&= \xi\kappa\int_{R_+^2} w^\alpha v^\beta(\lambda\phi'(w)(v-w)\\
&\quad+\frac{\sigma w^2}{2}\phi''(w))f_1(w,t)f_2(v,t)\,\mathrm{d}v\mathrm{d}w\\
&\quad+\xi\kappa\int_{R_+^2}(wv)^\alpha(\lambda\phi'(w)(v-w)\\
&\quad+\frac{\sigma w^2}{2}\phi''(w))f_1(w,t)f_1(v,t)\,\mathrm{d}v\mathrm{d}w\\
&\quad+\widetilde{R}_\xi(v,w),
\end{aligned}\tag{11}
$$

where

$$
\begin{aligned}
\widetilde{R}_\xi(v,w) &= \kappa\Bigg[\int_{R_+^2} w^\alpha v^\beta\langle R_\xi(v,w)\rangle f_1(w,t)f_2(v,t)\,\mathrm{d}v\mathrm{d}w\\
&\quad+\int_{R_+^2}(wv)^\alpha\langle R_\xi(v,w)\rangle f_1(w,t)f_1(v,t)\,\mathrm{d}v\mathrm{d}w\Bigg].
\end{aligned}
$$

Now, we set

$$
\tau=\xi t,\quad g_1(w,\tau)=f_1(w,t),\quad g_2(w,\tau)=f_2(w,t),
$$

which implies $g_1(w,0)=f_1(w,0)$ and $g_2(w,0)=f_2(w,0)$. Letting $\xi\to 0$, the remainder $\widetilde{R}_\xi(v,w)\to 0$ (see Ref. [19]), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+} g_1(w,\tau)\phi(w)\,\mathrm{d}w\\
&= \kappa\int_{R_+^2} w^\alpha v^\beta(\lambda\phi'(w)(v-w)\\
&\quad+\frac{\sigma w^2}{2}\phi''(w))g_1(w,\tau)g_2(v,\tau)\,\mathrm{d}v\mathrm{d}w\\
&\quad+\kappa\int_{R_+^2}(wv)^\alpha(\lambda\phi'(w)(v-w)\\
&\quad+\frac{\sigma w^2}{2}\phi''(w))g_1(w,\tau)g_1(v,\tau)\,\mathrm{d}v\mathrm{d}w.
\end{aligned}\tag{12}
$$

In Eq. (12), integration by parts yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+} g_1(w,\tau)\phi(w)\,\mathrm{d}w\\
&= \kappa\frac{\sigma}{2}(m_{\beta,2}(\tau)+m_{\alpha,1}(\tau))\\
&\quad\times\int_{R_+}\phi(w)\frac{\partial^2}{\partial w^2}(w^{2+\alpha}g_1(w,\tau))\,\mathrm{d}w\\
&\quad+\kappa\lambda(m_{\beta,2}(\tau)+m_{1+\alpha,1}(\tau))\\
&\quad\times\int_{R_+}\phi(w)\frac{\partial}{\partial w}(w^{1+\alpha}g_1(w,\tau))\,\mathrm{d}w\\
&\quad-\kappa\lambda(m_{1+\beta,2}(\tau)+m_{1+\alpha,1}(\tau))\\
&\quad\times\int_{R_+}\phi(w)\frac{\partial}{\partial w}(w^{\alpha}g_1(w,\tau))\,\mathrm{d}w,
\end{aligned}
$$

where

$$
\begin{aligned}
m_{\beta,2}(\tau)&=\int_{R_+} v^\beta g_2(v,\tau)\,\mathrm{d}v,\\
m_{\alpha,1}(\tau)&=\int_{R_+} v^\alpha g_1(v,\tau)\,\mathrm{d}v,\\
m_{1+\beta,2}(\tau)&=\int_{R_+} v^{1+\beta} g_2(v,\tau)\,\mathrm{d}v,\\
m_{1+\alpha,1}(\tau)&=\int_{R_+} v^{1+\alpha} g_1(v,\tau)\,\mathrm{d}v.
\end{aligned}
$$

Note that the first subscript of $m_{\epsilon,j}$ represents the ϵ-th moment of $g_i(w,\tau)$, and the second subscript denotes the group to which the agent belongs.

Consequently, the evolution of $g_1(w,\tau)$ satisfies the Fokker–Planck equation

$$
\begin{aligned}
\frac{\partial g_1(w,\tau)}{\partial\tau} &= \frac{\kappa\sigma}{2}\frac{\partial^2}{\partial w^2}[(m_{\beta,2}(\tau)+m_{\alpha,1}(\tau))w^{2+\alpha}g_1(w,\tau)]\\
&\quad+\kappa\lambda\frac{\partial}{\partial w}[(m_{\beta,2}(\tau)+m_{\alpha,1}(\tau))w^{1+\alpha}g_1(w,\tau)\\
&\quad-(m_{1+\beta,2}(\tau)+m_{1+\alpha,1}(\tau))w^{\alpha}g_1(w,\tau)].
\end{aligned}\tag{13}
$$

In the same way, we have

$$
\begin{aligned}
\frac{\partial g_2(w,\tau)}{\partial\tau} &= \frac{\kappa\sigma}{2}\frac{\partial^2}{\partial w^2}[(m_{\alpha,1}(\tau)+m_{\beta,2}(\tau))w^{2+\beta}g_2(w,\tau)]\\
&\quad+\kappa\lambda\frac{\partial}{\partial w}[(m_{\alpha,1}(\tau)+m_{\beta,2}(\tau))w^{1+\beta}g_2(w,\tau)\\
&\quad-(m_{1+\alpha,1}(\tau)+m_{1+\beta,2}(\tau))w^{\beta}g_2(w,\tau)].
\end{aligned}\tag{14}
$$

In Eq. (13), because the diffusion term and drift term contain $m_{\alpha,1}(\tau)$ and $m_{1+\alpha,1}(\tau)$, it is difficult to obtain the analytical solution of Eq. (13). In Eq. (14), the same is true. If we assume $Q(f_i,f_i)(w)=0$, that is, agents from the same group do not trade, equations (13) and (14) become

$$
\begin{aligned}
\frac{\partial g_1(w,\tau)}{\partial\tau} &= \frac{\kappa\sigma}{2}\frac{\partial^2}{\partial w^2}(m_{\beta,2}(\tau)w^{2+\alpha}g_1(w,\tau))\\
&\quad+\kappa\lambda\frac{\partial}{\partial w}[m_{\beta,2}(\tau)w^{1+\alpha}g_1(w,\tau)\\
&\quad-m_{1+\beta,2}(\tau)w^{\alpha}g_1(w,\tau)],
\end{aligned}\tag{15}
$$

$$
\begin{aligned}
\frac{\partial g_2(w,\tau)}{\partial\tau} &= \frac{\kappa\sigma}{2}\frac{\partial^2}{\partial w^2}(m_{\alpha,1}(\tau)w^{2+\beta}g_2(w,\tau))\\
&\quad+\kappa\lambda\frac{\partial}{\partial w}[m_{\alpha,1}(\tau)w^{1+\beta}g_2(w,\tau)\\
&\quad-m_{1+\alpha,1}(\tau)w^{\beta}g_2(w,\tau)],
\end{aligned}\tag{16}
$$

respectively.

For $0<\beta\le 1$, according to the Jensen's inequality

$$
\int_{R_+} v^\beta g_2(v,\tau)\,\mathrm{d}v\le\left(\int_{R_+} v g_2(v,\tau)\,\mathrm{d}v\right)^\beta,
$$

we have

$$
m_{\beta,2}(\tau)\le(m_{1,2}(\tau))^\beta.
$$

If the initial average wealth is not zero, it is known from the conservation of average wealth that the average wealth will not be zero at each subsequent moment. Therefore, we obtain $m_{\beta,2}(\tau)\neq 0$. $m_{\alpha,1}(\tau)\neq 0$ is derived by the same method.

The steady-state solution of Eq. (15) is determined by the following equation:

$$
m_{\beta,2}(\infty)\frac{\partial}{\partial w}\left(\frac{\sigma}{2}w^{2+\alpha}g_{1,\infty}(w)\right)
$$

$$+\lambda m_{\beta,2}(\infty)\left(w^{\alpha}\left(w-\frac{m_{1+\beta,2}(\infty)}{m_{\beta,2}(\infty)}\right)g_{1,\infty}(w)\right)=0. \quad (17)$$

Using $\int_{R_+} g_1(w,t)\,\mathrm{d}w=\rho_1$ and solving Eq. (17), we obtain

$$g_{1,\infty}(w) = \frac{\rho_1(2\lambda\widetilde{m}_\beta/\sigma)^{1+\alpha+2\lambda/\sigma}}{\Gamma(1+\alpha+2\lambda/\sigma)}w^{-2-\alpha-2\lambda/\sigma} \times \exp\left(-\frac{2\lambda\widetilde{m}_\beta}{\sigma w}\right), \quad (18)$$

where $\widetilde{m}_\beta=\frac{m_{1+\beta,2}(\infty)}{m_{\beta,2}(\infty)}$. From Eq. (16), employing the same calculation, the steady-state solution $g_{2,\infty}(w)$ is given by

$$g_{2,\infty}(w) = \frac{\rho_2(2\lambda\widetilde{m}_\alpha/\sigma)^{1+\beta+2\lambda/\sigma}}{\Gamma(1+\beta+2\lambda/\sigma)}w^{-2-\beta-2\lambda/\sigma} \times \exp\left(-\frac{2\lambda\widetilde{m}_\alpha}{\sigma w}\right), \quad (19)$$

where $\widetilde{m}_\alpha=\frac{m_{1+\alpha,1}(\infty)}{m_{\alpha,1}(\infty)}$.

When $\alpha=\beta$, equations (15) and (16) are the Fokker–Planck equations corresponding to the bilinear model in Ref. [11]. For more research on the application of the Fokker–Planck equation and its large time behavior, please refer to Refs. [19–21]. In Eq. (18), if $w\to+\infty$, we have $g_{1,\infty}(w)\sim w^{-2-\alpha-2\lambda/\sigma}$. The Pareto index is $1+\alpha+2\lambda/\sigma$, implying that wealth inequality for the first group decreases as α increases. Similarly, from Eq. (19) we obtain that wealth inequality for the second group decreases with increasing β. In this case, the total wealth distribution is

$$\begin{aligned} g_{\mathrm{tot},\infty}(w) = {} & \frac{\rho_1(2\lambda\widetilde{m}_\beta/\sigma)^{1+\alpha+2\lambda/\sigma}}{\Gamma(1+\alpha+2\lambda/\sigma)}w^{-2-\alpha-2\lambda/\sigma} \\ & \times\exp\left(-\frac{2\lambda\widetilde{m}_\beta}{\sigma w}\right) \\ & +\frac{\rho_2(2\lambda\widetilde{m}_\alpha/\sigma)^{1+\beta+2\lambda/\sigma}}{\Gamma(1+\beta+2\lambda/\sigma)}w^{-2-\beta-2\lambda/\sigma} \\ & \times\exp\left(-\frac{2\lambda\widetilde{m}_\alpha}{\sigma w}\right), \end{aligned} \quad (20)$$

which has a bimodal distribution. Indeed, considering a simple case of $\widetilde{m}_\beta=\widetilde{m}_\alpha=\widetilde{m}$, $\rho_1=\rho_2=\rho$ and letting $\mu=2\lambda/\sigma$, formula (20) becomes

$$\begin{aligned} g_{\mathrm{tot},\infty}(w) = {} & \frac{\rho}{\mu\widetilde{m}}\left[\left(\frac{\mu\widetilde{m}}{w}\right)^{2+\alpha+\mu}\frac{1}{\Gamma(1+\alpha+\mu)}\right. \\ & \left.+\left(\frac{\mu\widetilde{m}}{w}\right)^{2+\beta+\mu}\frac{1}{\Gamma(1+\beta+\mu)}\right] \\ & \times\exp\left\{-\frac{\mu\widetilde{m}}{w}\right\}. \end{aligned} \quad (21)$$

Letting $y=w/(\mu\widetilde{m})$, we obtain

$$g_{\mathrm{tot},\infty}(y) = \frac{\rho}{\mu\widetilde{m}}\left[\frac{y^{-(2+\alpha+\mu)}}{\Gamma(1+\alpha+\mu)}+\frac{y^{-(2+\beta+\mu)}}{\Gamma(1+\beta+\mu)}\right] \times\exp\left\{-\frac{1}{y}\right\}. \quad (22)$$

The extreme cases in Eq. (22) depend on the solution of the equation

$$\begin{aligned} & \frac{\rho}{\mu\widetilde{m}}\left[-\frac{(2+\alpha+\mu)y^{-(3+\alpha+\mu)}}{\Gamma(1+\alpha+\mu)}-\frac{(2+\beta+\mu)y^{-(3+\beta+\mu)}}{\Gamma(1+\beta+\mu)}\right. \\ & \left.+\frac{y^{-(4+\alpha+\mu)}}{\Gamma(1+\alpha+\mu)}+\frac{y^{-(4+\beta+\mu)}}{\Gamma(1+\beta+\mu)}\right]\exp\left\{-\frac{1}{y}\right\}=0. \end{aligned} \quad (23)$$

Clearly, $y=0$ is an extremum (a minimum). If $y>0$ and $\beta>\alpha$, the remaining extreme cases are determined by the equation

$$\begin{aligned} \varphi(y) = {} & -\frac{(2+\alpha+\mu)y^{1+(\beta-\alpha)}}{\Gamma(1+\alpha+\mu)}-\frac{(2+\beta+\mu)y}{\Gamma(1+\beta+\mu)} \\ & +\frac{y^{(\beta-\alpha)}}{\Gamma(1+\alpha+\mu)}+\frac{1}{\Gamma(1+\beta+\mu)}=0. \end{aligned} \quad (24)$$

Using the expression of $\varphi(y)$, we obtain $\varphi(0)>0$ and $\varphi(+\infty)<0$. Therefore, the curve $z=\varphi(y)$ touches the axis $z=0$ either in a single point or in three points or others. If there is only one maximum, $g_{\mathrm{tot},\infty}(w)$ presents a single peak shape (see Fig. 1(b)). When it has three points, two maximum and one minimum are obtained except $y=0$ and $g_{\mathrm{tot},\infty}(w)$ shows a bimodal distribution (see Fig. 1(a)). As Ballante *et al.*[13] stated, the bimodal distribution of wealth occurs in times of poor economic conditions, such as the income distribution due to the Argentinean economic crisis around the year 2002,[14] and the wealth distribution under the influence of COVID-19 epidemic in recent years.[8] From our analysis and the results in Refs. [6,8], it is known that the emergence of the bimodal wealth distribution is the inconsistency in agents' behaviors in the system due to the shocks to the economy caused by some unforeseen events, resulting in the appearance of different transaction types of agents, such as some people with a high frequency of interactions (or high savings rate) and some people with a low frequency of interactions (or low savings rate). In this case, parts of the agents are extremely rich and some are extremely poor, and the total wealth distribution in the system shows a bimodal peak.

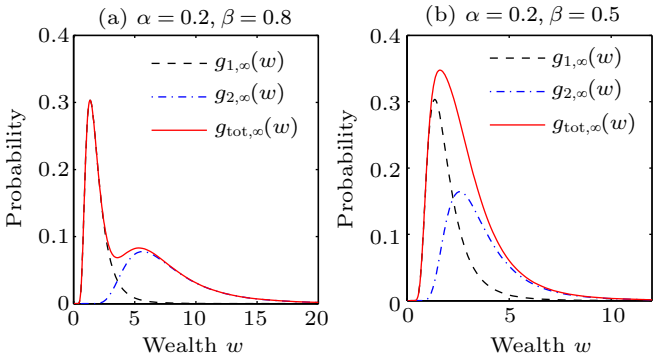


Fig. 1. Steady-state probability $g_{1,\infty}(w), g_{2,\infty}(w)$, and $g_{\mathrm{tot},\infty}(w)$. (a): $\rho_1=\rho_2=0.5$, $\alpha=0.2$, $\beta=0.8$, $\lambda=0.2$, $\sigma=0.1$, $m_{11}(0)=1$, and $m_{12}(0)=4$. (b): $\rho_1=\rho_2=0.5$, $\alpha=0.2$, $\beta=0.5$, $\lambda=0.2$, $\sigma=0.1$, $m_{11}(0)=1$, and $m_{12}(0)=3$.

3. Taxation and redistribution under non-Maxwellian collision kernel

Bisi *et al.*[4] introduced taxation into binary trading rules, selected appropriate redistribution operators under the Maxwellian collision kernel, and obtained that taxation and redistribution affect the Pareto index. Bisi[7] reviewed and studied a kinetic model with taxation and redistribution of wealth. In this section, under the non-Maxwellian collision kernel (2), we introduce taxation into the transaction rule and seek an appropriate redistribution operator to analyze how taxation and redistribution policy affect the Pareto index.

As it is known, the main function of taxation and redistribution is to adjust the wealth gap and decrease the inequality of wealth distribution. In terms of narrowing the gap between the rich and the poor, the rich should be taxed more and the poor should pay less, but economists are more concerned about which taxation rate can maximize social welfare. Mankiw *et al.*[22] pointed out that the "flat taxation" (that is, the same taxation rate relies on different income levels) may be close to optimal.

Following the idea in Ref. [22], we introduce flat taxation into the transaction rule, i.e.,

$$w_* = (1-\lambda-s)w + \lambda v + \eta w, \tag{25}$$

where w and w_* denote pre- and post-interaction wealth of the agent, respectively. $0<\lambda<1$ is a transaction rate parameter, $0<s<1$ is flat taxation rate, η is a central random variable with mean zero and finite variance $\sigma>0$. To ensure that the wealth after trade is positive and the taxation rate does not exceed the transaction rate, we assume $s \le \min\{(1-\lambda+\eta),\lambda\}$ (see Ref. [4]). In the interaction rule (25), the first term indicates the remaining wealth after the agent participates in market trading and pays taxes. The second term represents the wealth obtained by agents for trading, where $v \in R_+$ denotes the wealth owned by agents in the market with a certain distribution $H(v)$. The third term implies the wealth obtained by agents due to risks in the market.

Denote by $f(w,t)$ the wealth distribution of agents with wealth $w>0$ at the moment $t \ge 0$. According to the kinetic theory,[2] the evolution of $f(w,t)$ obeys the Boltzmann equation

$$\frac{\partial f(w,t)}{\partial t} = Q(f,f)(w,t), \tag{26}$$

where $Q(f,f)(w,t)$ is a collision operator which describes the change of $f(w,t)$ under non-Maxwellian collision kernel (2) and linear transaction rule (25). The weak form of Eq. (26) is generally adopted, that is, for any smooth function $\phi(w)$, the evolution of $f(w,t)$ follows the equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+} f(w,t)\phi(w)\,\mathrm{d}w\\ &= \left\langle \int_{R_+^2} K(v,w)(\phi(w_*)-\phi(w))H(v)f(w,t)\,\mathrm{d}v\mathrm{d}w \right\rangle, \end{aligned}\tag{27}$$

where $K(v,w)$ is the collision kernel defined in Eq. (2). Set $M_\alpha = \int_{R_+} v^\alpha H(v)\,\mathrm{d}v < +\infty$ $(0 \le \alpha \le 4)$, $m_\gamma(t) = \int_{R_+} w^\gamma f(w,t)\,\mathrm{d}w$ $(\gamma>0)$ represents the moment of order γ of $f(w,t)$.

In Eq. (27), we choose $\phi(w)=w$ and obtain that the average wealth evolution in the presence of taxation satisfies the equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+} wf(w,t)\,\mathrm{d}w\\ &= \kappa\left[\lambda M_{1+\alpha}\int_{R_+} w^\beta f(w,t)\,\mathrm{d}w\right.\\ &\quad \left.-(\lambda+s)M_\alpha\int_{R_+} w^{1+\beta}f(w,t)\,\mathrm{d}w\right]. \end{aligned}\tag{28}$$

Utilizing the Jensen's inequality, we get

$$\frac{\mathrm{d}m_1(t)}{\mathrm{d}t} \le \kappa[\lambda M_{1+\alpha}m_1(t)^\beta - (\lambda+s)M_\alpha m_1(t)^{1+\beta}],$$

which results in the inequality

$$m_1(t) \le \max\left\{m_1(0), \frac{\lambda M_{1+\alpha}}{(\lambda+s)M_\alpha}\right\},$$

meaning that the average wealth is bounded if its initial average wealth is bounded. In Eq. (25), If $s=0$, i.e., agents do not pay taxes, the evolutionary equation for average wealth is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+} wf(w,t)\,\mathrm{d}w &= \kappa\left[\lambda M_{1+\alpha}\int_{R_+} w^\beta f(w,t)\,\mathrm{d}w\right.\\ &\quad \left.-\lambda M_\alpha\int_{R_+} w^{1+\beta}f(w,t)\,\mathrm{d}w\right]. \end{aligned}\tag{29}$$

According to the idea in Ref. [7], the wealth obtained through taxation is returned to agents by appropriate redistribution policy. In such a way, the total wealth remains unchanged. The amount of wealth owned by agents after trade under collision kernel (2) is different from the amount of wealth owned by agents after trade under constant collision kernel. Furthermore, comparing Eqs. (28) and (29), it is found that the effect of taxation on the evolution of average wealth is captured by the term $-\kappa s M_\alpha m_{1+\beta}(t)$, where $m_{1+\beta}(t) = \int_{R_+} w^{1+\beta}f(w,t)\,\mathrm{d}w$. Thus, to ensure matching the collision kernel (2), we use the redistribution operator

$$R(f)(w,t) = -\kappa s M_\alpha m_{1+\beta}(t)\frac{\partial}{\partial w}(\Psi(w)f(w,t)), \tag{30}$$

where

$$\Psi(w) = \frac{1}{m_{1+\beta}(t)}(-\chi w^{1+\beta} + (\chi+1)m_{1+\beta}(t)). \tag{31}$$

The function $\Psi(w)$ denotes the fraction of the obtained taxation that is distributed to each agent with wealth w, also called redistribution function, and its value could be positive or negative. When it is negative, meaning that a certain part of the population is further taxed. If $\beta=0$, then $R(f)(w,t)$ and

$\Psi(w)$ are the redistribution operators and redistribution functions in Ref. [7], respectively. χ in the redistribution function is a free parameter, which represents the redistribution policy and is given by policy maker. The meaning of redistribution policy is consistent with that in Ref. [7] if $\beta = 0$. The meaning of redistribution policy should be determined according to specific values of β if $\beta \neq 0$. Because β takes in the interval $(0,1]$, we only analyze the meaning of redistribution policy when $\beta = 1$ and $\Psi(w) < 0$, and the analysis of other β values is similar. When $\chi < -1$, for $w \in I_1 = \left(0, \sqrt{\frac{\chi+1}{\chi} m_2(t)}\right)$, we have $\Psi(w) < 0$. Therefore, the agents with wealth $w \in I_1$ supply additional resources to the agents with $w \notin I_1$. When $\chi > 0$, for $w \in I_2 = (\sqrt{\frac{\chi+1}{\chi} m_2(t)}, +\infty)$, we have $\Psi(w) < 0$. In this case, the agents with wealth $w \notin I_2$ are favored, while the agents with $w \in I_2$ are taxed more.

Assume that $f(w,t)$ satisfies boundary condition $f(0,t) = 0$. The redistribution operator satisfies following constraints:

$$\int_{R_+} wR(f)(w,t)\,\mathrm{d}w = \kappa s M_\alpha m_{1+\beta}(t),$$
$$\int_{R_+} R(f)(w,t)\,\mathrm{d}w = 0,$$

and the redistribution function fulfills

$$\int_{R_+} \Psi(w) f(w,t)\,\mathrm{d}w = 1.$$

Now we consider the kinetic model with taxation and redistribution mechanism. That is, for the non-Maxwellian collision kernel (2), the linear trading rule (25), and the redistribution operator (30), the evolution of $f(w,t)$ satisfies the Boltzmann equation

$$\frac{\partial f(w,t)}{\partial t} = Q(f,f)(w,t) + R(f)(w,t), \tag{32}$$

where $Q(f,f)(w,t)$ is a collision operator. For any smooth function $\phi(w)$, the weak form of the Boltzmann equation with taxation and redistribution becomes

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_+} \phi(w) f(w,t)\,\mathrm{d}w \\
&= \left\langle \kappa \int_{R_+^2} v^\alpha w^\beta (\phi(w_*) - \phi(w)) H(v) f(w,t)\,\mathrm{d}v\,\mathrm{d}w \right\rangle \\
&\quad - \kappa s M_\alpha \int_{R_+} \phi'(w)[\chi w^{1+\beta} - (\chi+1) m_{1+\beta}] f(w,t)\,\mathrm{d}w,
\end{aligned} \tag{33}$$

where the second term on the right hand in Eq. (33) is obtained by integration by parts for $R(f)(w,t)$, bearing in mind that $f(0,t) = 0$. The redistribution operator has been constructed to compensate the losses of wealth caused by taxation. For this reason, $m_{1+\beta}$ is a constant (independent of t) in Eq. (33).

Similarly, following the idea of continuous trading limit, we use the Fokker–Planck equation as an approximation of the Boltzmann equation (33) to discuss the large-time behavior of wealth distribution. For a very small positive parameter $\xi \ll 1$, we set $\lambda \to \xi\lambda$, $s \to \xi s$, and $\eta \to \sqrt{\xi}\eta$. Then a second-order Taylor expansion of $\phi(w)$ around w yields

$$\begin{aligned}
\langle \phi(w_*) - \phi(w) \rangle &= \xi \left[\phi'(w)(\lambda(v-w) - sw) + \frac{\sigma w^2}{2} \phi''(w) \right] \\
&\quad + \langle R_{s\xi}(v,w) \rangle.
\end{aligned} \tag{34}$$

Inserting Eq. (34) into Eq. (33), and letting $\tau = \xi t$, $g(w,\tau) = f(w,t)$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+} \phi(w) g(w,\tau)\,\mathrm{d}w \\
&= \kappa \int_{R_+^2} v^\alpha w^\beta \left[\phi'(w)(\lambda(v-w) - sw) + \frac{\phi''(w)}{2} \sigma w^2 \right] \\
&\quad \times H(v) g(w,\tau)\,\mathrm{d}v\,\mathrm{d}w - \kappa s M_\alpha \int_{R_+} \phi'(w) \\
&\quad \times [\chi w^{1+\beta} - (\chi+1) m_{1+\beta}] g(w,\tau)\,\mathrm{d}w + \widetilde{R}_{s\xi}(v,w),
\end{aligned} \tag{35}$$

where

$$\widetilde{R}_{s\xi}(v,w) = \frac{1}{\xi} \int_{R_+^2} v^\alpha w^\beta \langle R_{s\xi}(v,w) \rangle H(v) g(w,\tau)\,\mathrm{d}v\,\mathrm{d}w.$$

When $\xi \to 0$, $\widetilde{R}_{s\xi}(v,w) \to 0$. Then, equation (35) becomes

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{R_+} \phi(w) g(w,\tau)\,\mathrm{d}w \\
&= \kappa \int_{R_+^2} v^\alpha w^\beta \left[\phi'(w)(\lambda(v-w) - sw) + \frac{\phi''(w)}{2} \sigma w^2 \right] \\
&\quad \times H(v) g(w,\tau)\,\mathrm{d}v\,\mathrm{d}w - \kappa s M_\alpha \int_{R_+} \phi'(w) \\
&\quad \times [\chi w^{1+\beta} - (\chi+1) m_{1+\beta}] g(w,\tau)\,\mathrm{d}w.
\end{aligned} \tag{36}$$

Integrating the right side of Eq. (36) by parts and combining with boundary conditions, we acquire the Fokker–Planck equation

$$\begin{aligned}
&\frac{\partial g(w,\tau)}{\partial \tau} \\
&= \frac{\sigma\kappa}{2} M_\alpha \frac{\partial^2}{\partial w^2}(w^{2+\beta} g(w,\tau)) \\
&\quad + \kappa M_\alpha \frac{\partial}{\partial w}\left[\left((\lambda+s) w^{1+\beta} - \lambda \frac{M_{1+\alpha}}{M_\alpha} w^\beta \right) g(w,\tau) \right] \\
&\quad + \kappa s M_\alpha \frac{\partial}{\partial w}[(\chi w^{1+\beta} - (\chi+1) m_{1+\beta}) g(w,\tau)].
\end{aligned} \tag{37}$$

To facilitate analysis, we set $\sigma\kappa M_\alpha/2 = 1$ and $M_{1+\alpha}/M_\alpha = m$. Obviously, $\kappa M_\alpha = 2/\sigma$. Thus, equation (37) becomes

$$\begin{aligned}
\frac{\partial g(w,\tau)}{\partial \tau} &= \frac{\partial^2}{\partial w^2}(w^{2+\beta} g(w,\tau)) \\
&\quad + \frac{2}{\sigma}\frac{\partial}{\partial w}[((\lambda+s) w^{1+\beta} - \lambda m w^\beta) g(w,\tau)] \\
&\quad + \frac{2s}{\sigma}\frac{\partial}{\partial w}[(\chi w^{1+\beta} - (\chi+1) m_{1+\beta}) g(w,\tau)].
\end{aligned} \tag{38}$$

When $\tau \to \infty$, we get the steady-state solution $g_\infty(w)$ satisfying the equation

$$\frac{\partial}{\partial w}(w^{2+\beta} g(w,\tau))$$

$$+\frac{2}{\sigma}[((\lambda+s)w^{1+\beta}-\lambda m w^{\beta})g(w,\tau)]$$
$$+\frac{2s}{\sigma}[(\chi w^{1+\beta}-(\chi+1)m_{1+\beta})g(w,\tau)]=0. \quad (39)$$

Solving Eq. (39), we obtain

$$g_\infty(w) = C\cdot w^{-(2+\beta+\frac{2\lambda}{\sigma}+\frac{2s(\chi+1)}{\sigma})} \times \exp\left(-\frac{2\lambda m}{\sigma w}-\frac{2s(\chi+1)m_{1+\beta}}{(1+\beta)\sigma}\frac{1}{w^{1+\beta}}\right), \quad (40)$$

where C satisfies $\int_{R_+} g_\infty(w)\,\mathrm{d}w=1$.

In Eq. (40), if $s=0$ or $\chi=-1$, The steady-state solution $g_\infty(w)$ is the equilibrium wealth distribution without the presence of taxation and redistribution. When $w\to+\infty$, we have $g_\infty(w)\sim w^{-(2+\beta+\frac{2\lambda}{\sigma}+\frac{2s(\chi+1)}{\sigma})}$. Thus, the Pareto index is $1+\beta+\frac{2\lambda}{\sigma}+\frac{2s(\chi+1)}{\sigma}$. This result demonstrates that the taxation and redistribution affect the Pareto index under the non-Maxwellian collision kernel (2). Figure 2(a) gives a graph of the steady-state solution $g_\infty(w)$ as the taxation rate s varies. It is observed that when the rich are taxed (i.e., $\chi>0$), the inequality of wealth distribution decreases as the taxation rate increases. Figure 2(b) shows a picture of the steady-state solution $g_\infty(w)$ as the redistribution parameter χ varies. It is seen that, if other parameters in the system do not change, the inequality of wealth distribution decreases with an increase of redistributive parameter χ. According to the above analysis, the choice of redistribution policy would affect the value of redistribution function. Which type of agents (the poor or the rich or others) will benefit should be considered by policy makers.

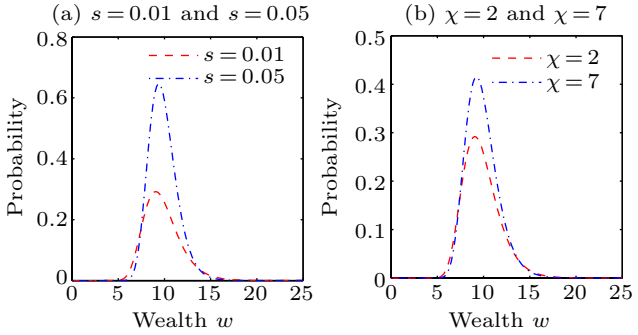


Fig. 2. (a) Steady-state probability $g_\infty(w)$ with different s for $\lambda=0.2$, $\sigma=0.02$, $\beta=0.5$, and $\chi=2$. (b) Steady-state probability $g_\infty(w)$ with different χ for $\lambda=0.2$, $\sigma=0.02$, $\beta=0.5$, and $s=0.01$.

4. Conclusion

In this work, we employ a non-Maxwellian collision kernel to study the evolution of wealth distribution in a multi-agent society. According to the characteristics of the collision kernel (2), we construct a two-group kinetic model for the evolution of wealth distribution. Using the continuous trading limit, we obtain a corresponding Fokker–Planck equation. Under certain assumptions, we derive a steady-state solution, illustrating that the total wealth distribution presents a bimodal shape when different groups of agents have significantly different intensities of willingness to trade. In addition, under the non-Maxwellian collision kernel (2), we find the redistribution operator that matches the taxation and construct a wealth distribution model with taxation and redistribution. The results show that taxation and redistribution have the property to change the Pareto index under the non-Maxwellian collision kernel (2).

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