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Opinion consensus incorporating higher-order interactions in individual-collective networks

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In the current information society, the dissemination mechanisms and evolution laws of individual or collective opinions and their behaviors are the research hot topics in the field of opinion dynamics. First, in this paper, a two-layer network consisting of an individual-opinion layer and a collective-opinion layer is constructed, and a dissemination model of opinions incorporating higher-order interactions (i.e. OIHOI dissemination model) is proposed. Furthermore, the dynamic equations of opinion dissemination for both individuals and groups are presented. Using Lyapunov's first method, two equilibrium points, including the negative consensus point and positive consensus point, and the dynamic equations obtained for opinion dissemination, are analyzed theoretically. In addition, for individual opinions and collective opinions, some conditions for reaching negative consensus and positive consensus as well as the theoretical expression for the dissemination threshold are put forward. Numerical simulations are carried to verify the feasibility and effectiveness of the proposed theoretical results, as well as the influence of the intra-structure, inter-connections, and higher-order interactions on the dissemination and evolution of individual opinions. The main results are as follows. (i) When the intra-structure of the collective-opinion layer meets certain characteristics, then a negative or positive consensus is easier to reach for individuals. (ii) Both negative consensus and positive consensus perform best in mixed type of inter-connections in the two-layer network. (iii) Higher-order interactions can quickly eliminate differences in individual opinions, thereby enabling individuals to reach consensus faster.

Keywords: two-layer social networks, individual and collective opinions, higher-order interactions, consensus, Lyapunov's first method

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1. Introduction

In the information age, individual and collective opinions have an important influence on individual behaviors.[1] Due to the dissemination and evolution of opinions, information society has experienced different group phenomena, such as filter bubbles,[2] the mass consensus,[3] and the echo chamber effect.[4] On one hand, these collective phenomena can advance human scientific knowledge. On the other hand, they may distort individuals' opinions of objective phenomena. Hence, in order to investigate the mechanisms and laws of individual or collective opinions and their behaviors, the scientific community proposed opinion dynamics, which can provide how opinions are disseminated among individuals, and consequently influence the communication patterns within a group.[5] Currently, the researches related to opinion dynamics have been applied to different social issues, such as political elections,[6] consensus building,[3] disease spreading,[7] market competition,[8] group decision making,[9] and public opinion formation.[10]

The first model of opinion dynamics was proposed by French in 1956.[11] Over the next few decades, the scientific community developed different models of opinion dynamics with different evolution rules. These models can be classified as discrete models and continuous models. Discrete opinion models, such as the Ising model,[12] voter model,[13] and Sznajd model,[14] and continuous opinion models, including the DeGroot model,[15] Friedkin–Johnsen (FJ) model,[16] bounded confidence model,[17,18] and Altafini model,[19] have been developed. Generally, opinion dynamics models are composed of opinion expression formats, fusion rules, and opinion dynamics environments,[18] which focus on three stable states: consensus, polarization, and fragmentation.

As is well known, a group is composed of individuals. Individual opinions can affect whether collective opinions can reach consensus, polarization, or fragmentation. Conversely, collective opinions can also influence the evolution trend of individual opinions. Based on the continuous opinions and discrete actions model (CODA model), the difference between individual and collective opinions was explored in Ref. [20] and it was found that this difference becomes smaller with the evolution of individual opinions into stability. The authors in Ref. [21] suggested that the larger the group size, the less influence opinion leaders have on individual opinions. In Ref. [22],

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the group pressure in a bounded confidence model was considered and how group pressure affects the evolution of individual opinions was investigated. Although the above results have studied the evolutionary characteristics between individual opinions and collective opinions, they have not delved into how to use relationships between individuals, groups, or individuals and groups to handle the dissemination and evolution of opinions.

Since the introduction of small-world complex networks[23] in 1998 and scale-free complex networks[24] in 1999, the scientific community has sparked a wave of using the theory and methods of complex networks to discuss various complex systems, including the dynamics of individual and collective opinions. With complex networks, individuals or groups display different relationships, which explain the dissemination and evolution of opinions. According to different opinion diffusion models, in Refs. [25–27] the influence of the structure of complex networks on the dissemination and evolution of opinions was explored. Using opinion dynamics on complex networks, in Refs. [28–30] the consensus, polarization, and fragmentation of opinions were investigated respectively. In Ref. [31] the DeGroot model was used to describe the opinion formation processes and the evolution of self-appraisal, social power, and interpersonal relationships in a group. In Refs. [32,33], a multilayer network consisting of the DeGroot model and the SIS epidemic model was developed to analyze the asymptotic stability for the opinion equilibrium point, which has also been extended to the bounded confidence model. The above researches were all based on complex networks, which either only consider individual opinions or focus on collective opinions. In fact, there are different kinds of interactions between any two individual opinions, any two collective opinions, or individual opinions and collective opinions. How to use a complex network to exhibit these interactions and further explore the dissemination and evolution of individual opinions and collective opinions will be a highly challenging research topic.

Moreover, in social networks, the dissemination and evolution of opinions depend not only on the direct relationships between individuals and their neighbors (or first-order information), but also on the influence of the higher-order interactions (or higher-order information) of the environment in which individuals are located.[34] The higher-order interactions in networks mainly include hypergraphs and simplicial complexes as well.[34,35] In Ref. [36], higher-order information on the base of the bounded confidence model was considered and it was found that different hypergraph structures can make individual opinion consensus change drastically. In Ref. [37], the bounded confidence model was extended and it was revealed that higher-order information can promote individual consensus. Thus, introducing higher-order information into the dynamics issues of opinions will better help us reveal how the mutual influence between individual opinions and collective opinions affects the consensus of individual opinions.

According to the above analyses, a two-layer social network is constructed in this paper to analyze and explore the dissemination and evolution of individual and collective opinions by combining the DeGroot model[15] and the SIS epidemic model,[38] according to the above analyses. The main contributions of this paper are listed below.

(i) A two-layer social network with an individual-opinion layer and a collective-opinion layer is constructed, in which, when an individual belongs to a group, an inter-connection appears between the individual and the group.

(ii) A dissemination model based on higher-order information on the two-layer network, named OIHOI dissemination model, is proposed.

(iii) Some sufficient conditions for individual opinions to reach consensus and a theoretical expression of dissemination threshold are derived.

(iv) Plentiful numerical simulations verify the feasibility and effectiveness of the theoretical results presented, and show the influence of various factors in the OIHOI dissemination model on opinion consensus.

The rest of this paper is organized as follows. In Section 2, a two-layer network composed of individual and collective opinion layers is constructed, and the OIHOI dissemination model, as well as its dynamics equations is proposed. In Section 3, two theorems for individual opinions are presented to reach a negative or positive consensus by using the Lyapunov's first method. At the same time, a theorem regarding the opinion dissemination threshold is developed. In Section 4, numerical simulations are conducted to show the feasibility and effectiveness of the proposed theoretical results. Also, the influences of various factors on the dissemination and evolution of individual opinions are explored. Finally, some conclusions are drawn from the present research in Section 5.

2. Model

In Section 1, it is mentioned that many current results regarding the dissemination and evolution of opinions are based on complex networks. In this section, a new two-layer social network with an individual opinion layer and a collective opinion layer will be constructed. The interconnections between the individual layer and the collective layer can be one-to-one or one-to-many. Further, according to this network newly constructed, a new dissemination model, named the OIHOI model, is proposed.

2.1. Two-layer social network with individual and collective opinions

According to Ref. [39], whenever the collective faces the problems of coordinating or cooperating interests, collective intention will arise. This collective intention represents the collective common interest, and the collective intention is most beneficial to the group. Moreover, coordinated and cooperative relationships will be generated by groups, resulting in mutual influence among groups. Therefore, the collective intention is referred to as the collective-opinion in this paper, and the mutual influence among groups is considered in our model.

In the DeGroot model,[15] individuals hold their own initial opinions at a certain event, and their opinion evolution is influenced by their neighbors in social networks. However, in real opinion dissemination, individual opinions are influenced not only by the opinions of their neighbors, but also by the opinions of the groups that they belong to. For example, when crossing a road, individuals tend to follow traffic rules. However, when their social group ignores the rules, individuals may be influenced to disregard them as well, causing their adherence to traffic rules to potentially change. Additionally, in the DeGroot model mentioned above, only the influence of the individual's neighbors on their opinion is taken into account, but not the effect of the higher-order interactions from simplicial complexes. Therefore, to reveal the relationships among individual opinions, collective opinions, first-order neighbors, and higher-order interactions on the dissemination and evolution of individual opinions in a better and more convenient way, a two-layer social network with an individual-opinion layer and a collective-opinion layer is presented (see Fig. 1).

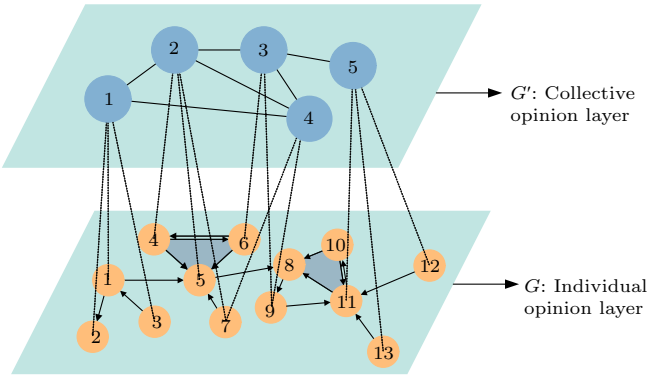


Fig. 1. Diagram of two-layer network composed of an individual-opinion layer and a collective-opinion layer.

Figure 1 is a two-layer network diagram consisting of a subnet G and a subnet G'. The subnet G (or the lower layer network) represents the individual-opinion layer $G=(V,E)$. In this layer, V is a set of all the individual opinions in the subnet G, and $|V|=N_1$, and E is a set of social relationships (or directed edges) between any two individuals in the subnet G. The dissemination and evolution of individual opinions occur on the individual-opinion layer G. For example, as shown in Fig. 1, the opinion of the individual 5 in the subnet G is influenced by its neighbors 1, 4, 6, and 7. Meanwhile, it also acts upon the opinion evolution of the individual 8. Furthermore, it is generally believed that individuals are more likely to accept the opinions of any two neighbors who know each other, among their neighbors. In order to simulate the reinforcing influence of neighboring groups on individual opinions, 2-simplex complex structures are put into use in the individual-opinion layer G. For example, a 2-simplex complex structure is formed among the individual 5, the individual 4 and the individual 6. In this case, the opinion of the individual 5 is affected not only by the individual 4 and the individual 6, but also by the reinforcing influence originating from the individual 4 and the individual 6. In this paper, we take 2-simplicial complexes as higher-order interactions. The subnet G' (or the upper layer network) indicates the collective-opinion layer $G'=(V',E')$, in which V' denotes a set of all the collective opinions in the subnet G, and $|V'|=N_2$, and E' refers to a set of all the group influences (or undirected edges) between any two groups in the subnet G'. When the individual i in the subnet G belongs to the group j in the subnet G', an edge between the individual i and the group j appears. For example, in Fig.1, individual 7 belongs to group 2 and group 4, then, there are two edges arising from the individual 7 between the individual-opinion layer G and the collective-opinion layer G'.

In addition, let $A=(a_{ij})_{N_1\times N_1}$ be the adjacency matrix, representing the intra-structure of the individual-opinion layer G, where $a_{ij}\geq 0$ refers to the intra-connection weight of the opinion influence from the individual j to the individual i, and $a_{ii}=0$ $(i=1,2,\ldots,N_1)$. Let $B=(b_{ij})_{N_2\times N_2}$ be the adjacency matrix, which is the intra-structure of the collective-opinion layer G', where $b_{ij}\geq 0$ represents the intra-connection weight of the opinion influence between the group i and the group j $(j\neq i)$, and $b_{ii}\geq 0$ denotes the intra-connection weight when maintaining the opinion of the group i $(i=1,2,\ldots,N_2)$. Further, we assume that the matrix $H=(h_{ij})_{N_1\times N_2}$ shows the inter-structure between the subnet G' and the subnet G, where $h_{ij}\geq 0$ represents the weight of the opinion influence between the individual i and the group j.

2.2. OIHOI dissemination model

According to the two-layer social network newly constructed in Subsection 2.1, in this subsection, we derive a new model, which is dissemination model of opinions including higher-order interactions, called OIHOI dissemination model. In this paper, opinions that are denoted by 1 and 0 are referred to as positive opinions and negative opinions.

Let $o_i(t)\in[0,1]$ be the opinion of the individual i in the individual-opinion layer G at time t, and $x_j(t)\in[0,1]$ denotes the proportion of individuals with positive opinions in

the group j of collective-opinion layer G' at time t. A bigger $x_j(t)$ indicates a greater number of individuals with positive opinions, while a smaller $x_j(t)$ denotes a smaller number of individuals with positive opinions. The individual opinion $o_i(t)$ in the individual-opinion layer G is determined by the opinions of its neighbors, the collective opinion to which it belongs, and its higher-order interactions. Similarly, in the collective-opinion layer G', the proportion of collective opinions $x_j(t)$ is influenced by its neighbors and the individuals the group j contains. Therefore, the dynamics equations of OIHOI dissemination model in the two-layer network can be described as follows:

$$\begin{cases} \dot{x}_j(t) = -\delta_j x_j(t)(1-\bar{o}_j(t)) + (1-x_j(t))\bar{o}_j(t)\left(\sum\limits_{k=1,k\neq j}^{N_2} b_{jk}x_k(t) + b_{jj}\right), \\ \dot{o}_i(t) = \sum\limits_{j=1}^{N_1} a_{ij}(o_j(t)-o_i(t)) + \sum\limits_{j=1}^{N_2} h_{ij}(x_j(t)-o_i(t)) + \beta_\Delta \sum\limits_{j=1,j\neq l}^{N_1}\sum\limits_{l=1}^{N_1} \mathrm{sign}(a_{ij}a_{il}a_{jl}a_{lj})\dfrac{a_{ij}+a_{il}}{2}\left(\dfrac{o_j(t)+o_l(t)}{2} - o_i(t)\right), \\ i = 1,2,\ldots,N_1, \quad j = 1,2,\ldots,N_2, \end{cases} \tag{1}$$

where $\beta_\Delta > 0$ is a higher-order reinforcement factor, $\delta_j \in [0,1]$ represents the dissemination rate of group j from positive opinions to negative ones in the collective-opinion layer G', $\delta = (\delta_1, \delta_2, \ldots, \delta_{N_2})$, and $\sum\limits_{k=1,k\neq j}^{N_2} b_{jk}x_k(t) + b_{jj}$ represents the recovery rate of the group j from negative opinions to positive opinions. The recovery rate of group j will be influenced by its neighbors. $\bar{o}_j(t)$ indicates the average opinion of individuals in the group j at time t, which satisfies

$$\bar{o}_j(t) = \frac{\sum\limits_{i=1}^{N_1} h_{ij}o_i(t)}{\sum\limits_{l=1}^{N_1} h_{lj}}, \quad j = 1,2,\ldots,N_2. \tag{2}$$

In addition, for Eq. (1), $\mathrm{sign}(a_{ij}a_{il}a_{jl}a_{lj}) = 1$ indicates that the individuals i, j, and l form a 2-simplicial complex structure, while $\mathrm{sign}(a_{ij}a_{il}a_{jl}a_{lj}) = 0$ refers to the case that a 2-simplicial complex is not formed.

For convenience, we denote

$$o(t) = (o_1(t), o_2(t), \ldots, o_{N_1}(t))',$$
$$x(t) = (x_1(t), x_2(t), \ldots, x_{N_2}(t))'.$$

Therefore, OIHOI dissemination model (1) can be rewritten as

$$\begin{cases} \dot{x}_j(t) = f_j(x(t), o(t)), \\ \dot{o}_i(t) = g_i(x(t), o(t)), \end{cases} i = 1,2,\ldots,N_1, \ j = 1,2,\ldots,N_2. \tag{3}$$

Let $z(t) = (x(t), o(t))'$. It is obvious that OIHOI dissemination model (1) or (3) has two equilibrium points

$$z_1^* = (0,0,\ldots,0)' \in R^{N_1+N_2}, \tag{4}$$
$$z_2^* = (1,1,\ldots,1)' \in R^{N_1+N_2}, \tag{5}$$

which correspond to all the individuals in the two-layer network holding negative and positive opinions, respectively. In other words, in these two cases, all the individuals in the two-layer network have reached a negative consensus or a positive consensus. In this paper, we are to analyze and explore what conditions all the individuals and groups in the two-layer network should meet when a negative consensus or a positive consensus is reached.

3. Theoretical analysis

In Section 2, we assumed that the individual opinion $o_i(t)$ and the proportion of collective opinions $x_j(t)$ in the OIHOI dissemination model (1) or (3) are in the interval $[0,1]$ at any time t. But, do they really change in the interval $[0,1]$? In Subsection 3.1, we first illustrate this assumption. Then, some sufficient conditions in reaching negative and positive consensus (4) and (5) for the OIHOI dissemination model (1) are established, and also the mathematical expression of dissemination threshold is given in Subsection 3.2.

3.1. Consensus in the OIHOI dissemination model

For the OIHOI dissemination model (1) or (3), assume that the initial value of the individual opinion is $o_k(0) = 0$ and $o_i(0) \in [0,1]$, $i \neq k$, $i,k = 1,2,\ldots,N_1$, and the initial value of the proportion of collective opinions is $x_j(0) \in [0,1]$, $j = 1,2,\ldots,N_2$, then, with the opinion dissemination model (1), we will have

$$\dot{o}_k(0) \geq 0, \quad k = 1,2,\ldots,N_1,$$

thus, $o_k(t) \geq 0$ when $t > 0$.

Assume that the initial value of the individual opinion is $o_k(0) = 1$ and $o_i(0) \in [0,1]$, $i \neq k$, $i,k = 1,2,\ldots,N_1$, and the initial value of the proportion of collective opinions is still $x_j(0) \in [0,1]$, $j = 1,2,\ldots,N_2$, then at this time we will have

$$\dot{o}_k(0) \leq 0, \quad k = 1,2,\ldots,N_1,$$

and $o_k(t) \leq 1$ for $t > 0$.

Since $o_k(t)$ is a continuous function with respect to time t, and $o_k(0) \in [0,1]$, $k = 1,2,\ldots,N_1$, we can conclude that if $o_k(t) = 0$, then $\dot{o}_k(t) \geq 0$; if $o_k(t) = 1$, then $\dot{o}_k(t) \leq 0$. Therefore, $o_k(t) \in [0,1]$ always holds true for $t \geq 0$.

Similarly, it can be proved that if $o_i(0) \in [0,1]$, $x_j(0) \in [0,1]$, $i = 1,2,\ldots,N_1$ and $j = 1,2,\ldots,N_2$, then $x_j(t) \in [0,1]$ holds true.

The Jacobian matrix of the OIHOI dissemination model (1) at equilibrium point $z_1^* = (0,0,\ldots,0)' \in R^{N_1+N_2}$ is

$$J(z) = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial o} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial o} \end{pmatrix}, \tag{6}$$

where functions f and g are defined, respectively, as

$$f(t) = \begin{pmatrix} f_1(x,o) \\ f_2(x,o) \\ \vdots \\ f_{N_2}(x,o) \end{pmatrix}, \quad g(t) = \begin{pmatrix} g_1(x,o) \\ g_2(x,o) \\ \vdots \\ g_{N_1}(x,o) \end{pmatrix}.$$

The partial derivatives of the functions f_j and g_i with respect to their independent variables are

$$\begin{cases} \dfrac{\partial f_j}{\partial x_j} = -\delta_j(1-\bar{o}_j) - \bar{o}_j\left(\sum\limits_{k=1,k\neq j}^{N_2} b_{jk}x_k + b_{jj}\right), \\ \dfrac{\partial f_j}{\partial x_k} = b_{jk}(1-x_j)\bar{o}_j, \\ \dfrac{\partial f_j}{\partial o_i} = \delta_j x_j \dfrac{h_{ij}}{\sum\limits_{l=1}^{N_1} h_{lj}} + (1-x_j)\dfrac{h_{ij}}{\sum\limits_{l=1}^{N_1} h_{lj}}\left(\sum\limits_{k=1,k\neq j}^{N_2} b_{jk}x_k + b_{jj}\right), \end{cases}$$
$$i = 1,2,\ldots,N_1, \quad j \neq k, \quad j,k = 1,2,\ldots,N_2, \tag{7}$$

$$\begin{cases} \dfrac{\partial g_i}{\partial x_j} = h_{ij}, \\ \dfrac{\partial g_i}{\partial o_i} = -\sum\limits_{j=1}^{N_1} a_{ij} - \sum\limits_{j=1}^{N_2} h_{ij} \\ \qquad\quad -\beta_\Delta \sum\limits_{j=1,j\neq l}^{N_1}\sum\limits_{l=1}^{N_1} \operatorname{sign}(a_{ij}a_{il}a_{jl}a_{lj})\left(\dfrac{a_{ij}+a_{il}}{2}\right), \\ \dfrac{\partial g_i}{\partial o_l} = a_{il} + \beta_\Delta \sum\limits_{j=1,j\neq l}^{N_1} \operatorname{sign}(a_{ij}a_{il}a_{jl}a_{lj})\dfrac{a_{ij}+a_{il}}{2}, \end{cases}$$
$$i \neq l, \quad i,l = 1,2,\ldots,N_1, \quad j = 1,2,\ldots,N_2. \tag{8}$$

After that, the values of formulae (7) and (8) at the equilibrium point $z_1^* = (0,0,\ldots,0)'$ are

$$\begin{cases} \left.\dfrac{\partial f_j}{\partial x_j}\right|_{z_1^*} = -\delta_j, \\ \left.\dfrac{\partial f_j}{\partial x_k}\right|_{z_1^*} = 0, \\ \left.\dfrac{\partial f_j}{\partial o_i}\right|_{z_1^*} = \dfrac{h_{ij}b_{jj}}{\sum\limits_{l=1}^{N_1} h_{lj}}, \end{cases}$$
$$i = 1,2,\ldots,N_1, \quad j \neq k, \quad j,k = 1,2,\ldots,N_2, \tag{9}$$

$$\begin{cases} \left.\dfrac{\partial g_i}{\partial x_j}\right|_{z_1^*} = h_{ij}, \\ \left.\dfrac{\partial g_i}{\partial o_i}\right|_{z_1^*} = -\sum\limits_{j=1}^{N_1} a_{ij} - \sum\limits_{j=1}^{N_2} h_{ij} \\ \qquad\quad -\beta_\Delta \sum\limits_{j=1,j\neq l}^{N_1}\sum\limits_{l=1}^{N_1} \operatorname{sign}(a_{ij}a_{il}a_{jl}a_{lj})\left(\dfrac{a_{ij}+a_{il}}{2}\right), \\ \left.\dfrac{\partial g_i}{\partial o_l}\right|_{z_1^*} = a_{il} + \beta_\Delta \sum\limits_{j=1,j\neq l}^{N_1} \operatorname{sign}(a_{ij}a_{il}a_{jl}a_{lj})\dfrac{a_{ij}+a_{il}}{2}, \end{cases}$$
$$i \neq l, \quad i,l = 1,2,\ldots,N_1, \quad j = 1,2,\ldots,N_2. \tag{10}$$

Therefore, using formulae (6), (9), and (10), the Jacobian matrix of the OIHOI dissemination model (1) at equilibrium point $z_1^* = (0,0,\ldots,0)'$ is given by

$$J(z_1^*) = \begin{pmatrix} \operatorname{diag}(-\delta_1,-\delta_2,\ldots,-\delta_{N_2}) & H_1 \\ H & \Omega \end{pmatrix}, \tag{11}$$

where $\Omega = (\omega_{ij})_{N_1\times N_1}$, with

$$\omega_{ij} = \begin{cases} -\sum\limits_{j=1}^{N_1} a_{ij} - \sum\limits_{j=1}^{N_2} h_{ij} - \beta_\Delta \sum\limits_{j=1,j\neq l}^{N_1}\sum\limits_{l=1}^{N_1} \operatorname{sign}(a_{ij}a_{il}a_{jl}a_{lj})\dfrac{a_{ij}+a_{il}}{2}, & \text{if} \quad i = j, \\ a_{ij} + \beta_\Delta \sum\limits_{l=1,l\neq j}^{N_1} \operatorname{sign}(a_{ij}a_{il}a_{jl}a_{lj})\dfrac{a_{ij}+a_{il}}{2}, & \text{if} \quad i \neq j, \end{cases} \tag{12}$$

$$H_1 = \operatorname{diag}\left(\frac{b_{11}}{\sum\limits_{l=1}^{N_1} h_{l1}}, \frac{b_{22}}{\sum\limits_{l=1}^{N_1} h_{l2}}, \ldots, \frac{b_{N_2N_2}}{\sum\limits_{l=1}^{N_1} h_{lN_2}}\right) H'. \tag{13}$$

Evidently, in Jacobian matrix (11), all the diagonal elements are less than zero, while all the off-diagonal elements are greater than or equal to zero, which shows that $J(z_1^*)$ is a Metzler matrix. According to Ref. [40], $J(z_1^*)$ is a Hurwitz matrix, if it is a Metzler matrix and there exists a vector $\xi \in R^n$ such that each element of ξ is positive and each element of $J(z_1^*)\xi$ is negative. Thus, we denote

$$\xi_1 = [1_{N_2}, \alpha_1 1_{N_1}]^{\mathrm{T}},$$

where 1_{N_2} and 1_{N_1} are two vectors, whose elements are 1, and $\alpha_1 > 0$ is a constant. To make each element of the vector $J(z_1^*)\xi_1$ be negative, the following conditions must be met:

$$-\delta_j + \alpha_1 b_{jj} < 0, \quad (1-\alpha_1)\sum_{l=1}^{N_2} h_{il} < 0. \tag{14}$$

Obviously, if $\alpha_1 > 1$ and the condition (14) holds, the Jacobian matrix $J(z_1^*)$ is a Hurwitz matrix. In this case, the maximum eigenvalue of the matrix $J(z_1^*)$ is negative. According to Lyapunov's first method,[40] the individual opinions and collective opinions of the OIHOI dissemination model (1) reach a local negative consensus at the equilibrium point $z_1^* = (0,0,\ldots,0)'$.

Similarly, according to Eqs. (7) and (8), the partial derivatives of the functions f_j and g_i at the equilibrium point $z_2^* = (1,1,\ldots,1)'$ are given by

$$\begin{cases} \left.\dfrac{\partial f_j}{\partial x_j}\right|_{z_2^*} = -\sum\limits_{k=1}^{N_2} b_{jk}, \\ \left.\dfrac{\partial f_j}{\partial x_k}\right|_{z_2^*} = 0, \\ \left.\dfrac{\partial f_j}{\partial o_i}\right|_{z_2^*} = \dfrac{h_{ij}\delta_j}{\sum\limits_{l=1}^{N_1} h_{lj}}, \end{cases}$$
$$i = 1,2,\ldots,N_1, \quad j \neq k, \quad j,k = 1,2,\ldots,N_2, \tag{15}$$

$$\begin{cases} \left.\dfrac{\partial g_i}{\partial x_j}\right|_{z_2^*} = h_{ij}, \\ \left.\dfrac{\partial g_i}{\partial o_i}\right|_{z_2^*} = -\sum\limits_{j=1}^{N_1} a_{ij} - \sum\limits_{j=1}^{N_2} h_{ij} \\ \qquad\qquad -\beta_\Delta \sum\limits_{j=1, j\neq l}^{N_1}\sum\limits_{l=1}^{N_1} \operatorname{sign}(a_{ij}a_{il}a_{jl}a_{lj})\dfrac{a_{ij}+a_{il}}{2}, \\ \left.\dfrac{\partial g_i}{\partial o_l}\right|_{z_2^*} = a_{il} + \beta_\Delta \sum\limits_{j=1, j\neq l}^{N_1} \operatorname{sign}(a_{ij}a_{il}a_{jl}a_{lj})\dfrac{a_{ij}+a_{il}}{2}, \end{cases}$$
$$i \neq l, \quad i,l = 1,2,\ldots,N_1, \quad j = 1,2,\ldots,N_2. \tag{16}$$

Therefore, by using formulae (6), (15), and (16), the Jacobian matrix of the OIHOI dissemination model (1) at the equilibrium point $z_2^* = (1,1,\ldots,1)'$ is obtained as follows:

$$J(z_2^*) = \begin{pmatrix} \underset{H}{\operatorname{diag}(-\sum\limits_{j=1}^{N_2} b_{1k}, -\sum\limits_{j=1}^{N_2} b_{2k}, \ldots, -\sum\limits_{j=1}^{N_2} b_{N_2k})} & \underset{\Omega}{H_2} \end{pmatrix},$$

where H_2 satisfies

$$H_2 = \operatorname{diag}\left(\frac{\delta_1}{\sum\limits_{l=1}^{N_1} h_{l1}}, \frac{\delta_2}{\sum\limits_{l=1}^{N_1} h_{l2}}, \ldots, \frac{\delta_{N_2}}{\sum\limits_{l=1}^{N_1} h_{lN_2}}\right) H'.$$

Like the matrix $J(z_1^*)$, the matrix $J(z_2^*)$ is also a Metzler matrix. $J(z_2^*)$ is a Hurwitz matrix if it is a Metzler matrix and there exists a vector $\xi \in R^n$ such that each element of ξ is positive and each element of $J(z_2^*)\xi$ is negative. Thus, let

$$\xi_2 = (1_{N_2}, \alpha_2 1_{N_1})',$$

where $\alpha_2 > 0$ is a constant. To make each element of vector $J(z_2^*)\xi_2$ be negative, we must have

$$-\sum_{k=1}^{N_2} b_{jk} + \alpha_2\delta_j < 0, \quad (1-\alpha_2)\sum_{l=1}^{N_2} h_{il} < 0. \tag{17}$$

When parameter $\alpha_2 > 1$ and condition (17) is met, Jacobian matrix $J(z_2^*)$ is a Hurwitz matrix. In this case, the maximum eigenvalue of matrix $J(z_2^*)$ is negative. When parameter $\alpha_2 > 1$ and condition (17) is met, the individual opinions and collective opinions of the OIHOI dissemination model (1) reach a local positive consensus at the equilibrium point $z_2^* = (1,1,\ldots,1)'$.

As is well known, in the two-layer social network, the greater the dissemination rate of negative opinions δ_j, the greater the probability that individuals will adopt negative opinions. According to conditions (14) and (17), it is evident that as the dissemination rate δ_j increases, conditions (14) are more likely to be satisfied, which causes all the individuals to hold negative opinions faster. Conversely, with the decrease of dissemination rate δ_j, conditions (17) are easier to meet, such that all the individuals reach the positive consensus faster.

3.2. Opinion dissemination threshold

Sufficient conditions (14) and (17) reflect that if all the individuals in the individual-opinion layer G reaching consensus depend only on the dissemination rate $\delta_j\,(j = 1,2,\ldots,N_2)$ and the intra-structure of the collective-opinion layer G', but not on the higher-order reinforcement factor β_Δ nor the intra-structure A of the individual-opinion layer nor the inter-structure H. So, are these factors unrelated to the opinion dissemination in a two-layer network? In order to answer this question, in this subsection, we assume that the dissemination rate $\delta_j = \delta\,(j = 1,2,\ldots,N_2)$, and discuss the dissemination threshold δ_c of the OIHOI dissemination model (1).

For the negative consensus $z_1^* = (0,0,\ldots,0)'$, when all the individuals arrive at negative opinions, positive opinions definitely will not be disseminated in the individual-opinion layer G. When negative consensus is satisfied, the right-hand side of the OIHOI dissemination model (1) becomes zero. At this time, we have

$$\begin{cases} -\delta x_j(1-\bar{o}_j) + (1-x_j)\bar{o}_j\left(\sum\limits_{k=1,k\neq j}^{N_2} b_{jk}x_k + b_{jj}\right) = 0, \\ \sum\limits_{j=1}^{N_1} a_{ij}(o_j - o_i) + \sum\limits_{j=1}^{N_2} h_{ij}(x_j - o_i) \\ +\beta_\Delta \sum\limits_{j=1, j\neq l}^{N_1}\sum\limits_{l=1}^{N_1} \operatorname{sign}(a_{ij}a_{il}a_{jl}a_{lj}) \\ \dfrac{a_{ij}+a_{il}}{2}\left(\dfrac{o_j+o_l}{2} - o_i\right) = 0, \end{cases}$$
$$i = 1,2,\ldots,N_1, \quad j = 1,2,\ldots,N_2. \tag{18}$$

As the dissemination rate of opinions δ approaches the dissemination threshold δ_c, individual opinions and collective opinions tend to become negative, i.e., $0 < x_j \ll 1$, $0 < o_i \ll 1$, $i = 1,2,\ldots,N_1$, $j = 1,2,\ldots,N_2$. Therefore, the first term on the left-hand side of Eq. (18) can be approximated as

$$-\delta x_j + b_{jj}\bar{o}_j \approx 0. \tag{19}$$

It can be inferred from Eqs. (2) and (19) that

$$x_j = \frac{b_{jj}}{\delta\sum\limits_{l=1}^{N_1} h_{lj}}\left(h_{1j}, h_{2j}, \ldots, h_{N_1 j}\right) o. \tag{20}$$

Then, equation (18) can be expressed in the form of matrix as

$$x = \frac{1}{\delta} H_1 o, \tag{21}$$

$$\begin{pmatrix} H & \Omega \end{pmatrix} \begin{pmatrix} x \\ o \end{pmatrix} = 0. \tag{22}$$

Substituting Eq. (21) into the second term on the left-hand side of Eq. (18), we have matrix form

$$\left(\Omega + \frac{1}{\delta} H H_1 \right) o = 0. \tag{23}$$

Assuming Ω to be invertible, equation (23) can be rewritten as

$$-\Omega^{-1} H H_1 o = \delta o. \tag{24}$$

Obviously, equation (24) indicates that δ is an eigenvalue of the matrix $-\Omega^{-1} H H_1$ and o is the corresponding eigenvector. Because $\lambda_{\max}\left(-\Omega^{-1} H H_1\right)$ is the largest eigenvalue of the matrix $-\Omega^{-1} H H_1$, according to Lyapunov's first method, the dissemination threshold of opinions is

$$\delta_{\mathrm{c}} = \lambda_{\max}\left(-\Omega^{-1} H H_1\right). \tag{25}$$

When the dissemination rate of opinions $\delta > \delta_{\mathrm{c}}$, the negative opinions will be disseminated faster.

Equation (25) characterizes the minimum value of the dissemination rate of opinions, which is related to the intra-structure B of collective-opinion layer G', intra-structure A of individual-opinion layer G and inter-structure H, as well as higher-order interactions.

4. Numerical simulations

In order to verify the effectiveness and feasibility of the theorems presented in Section 3, and to explore the influences of intra-connections, inter-connections, and higher-order interactions on the dissemination of individual opinion consensus, in this section numerical simulations are conducted. For all the numerical simulations, the individual-opinion layer G is a Barabási–Albert (BA)[24] scale-free network with 1000 nodes. Before its growth, the network G has 100 randomly connected nodes. And a new node is added each time, resulting in 10 new edges. The collective-opinion layer G' is a Watts–Strogatz (WS) small world network[23] with 50 nodes. Its initial network consists of 50 nodes, and each node links to its nearest $K = 5$ neighbors. Meanwhile, in the network G', edges are randomly connected with probability $p = 0.5$. The inter-connection weight of each edge between the individual-opinion layer G and the collective-opinion layer G' belongs to $[0,1]$. Additionally, we assume the dissemination rate of opinions $\delta_j = \delta$ $(j = 1,2,\ldots,N_2)$.

For one-to-one connections, each group randomly establishes connections with an individual in order until all groups have connections with different individuals. For one-to-many connections, there is a 0.5 probability of connection between each individual and each group based on one-to-one connections. For mixed-type connections, there is a 0.5 probability of connection between each individual and each group. And the inter-connection weight of each edge between the individual-opinion layer G and the collective-opinion layer G' belongs to $[0,1]$.

According to Subsection 3.1, for all the numerical simulations, the initial values of all the nodes in the two-layer network are randomly selected from the interval $[0,1]$. Due to the fact that when all the individual opinions in the network G reach negative consensus 0 or positive consensus 1, the proportion of individuals with positive opinions in the network G' must be 0 or 1. Therefore, the following simulations mainly focus on the evolution of individual opinions, but not on the evolution of collective opinions.

4.1. Influence of collective-opinion layer G' on opinion consensus

In this subsection, we will validate the feasibility and effectiveness of Theorem 2 of the OIHOI dissemination model. For all the numerical simulations, let the inter-connections between the subnets G and G' be one-to-many, and the higher-order reinforcement factor $\beta_\Delta = 0.5$. In addition, we suppose that $\delta = 0.6$ or $\delta = 0.15$ to examine whether the individuals in the network reach negative consensus or positive consensus. Further, if the difference between the opinions of all the individuals and the negative opinion 0 or the positive opinion 1 is less than 0.05, in this subsection it is believed that all the individuals have reached a local consensus of negative opinions or positive opinions. In Eq. (14), if $\max\{b_{ii}, i = 1,2,\ldots,N_2\} < \delta$, there must be $\alpha_1 > 1$ to make Eq. (14) hold. Similarly, if $\delta < \min\left\{\sum_{j=1}^{N_2} b_{ij}, i = 1,2,\ldots,N_2\right\}$, there must be $\alpha_2 > 1$ to make inequality (17) hold true. Therefore, denote

$$\begin{aligned} b_1 &= \max\{b_{ii}, i = 1,2,\ldots,N_2\}, \\ b_2 &= \min\left\{\sum\nolimits_{j=1}^{N_2} b_{ij}, i = 1,2,\ldots,N_2\right\}, \end{aligned} \tag{26}$$

then we will have $b_1 = 0.19$ and $b_2 = 0.4$. According to Subsection 3.1, there exists a constant $\alpha_1 = 1.2$ satisfying condition (14) such that all the individuals can reach a negative consensus. Similarly, there exists a constant $\alpha_2 = 1.2$ meeting condition (17) such that all the individuals can reach a positive consensus.

Figure 2 illustrates the evolution of individual opinions in the individual-opinion layer G with time t. Each line in Fig. 2 represents the evolution of different individual opinion with time. Obviously, the opinions of each individual in Fig. 2 evolve in the interval $[0,1]$. Figures 2(a) and 2(b) show that all the individuals in the individual-opinion layer G rapidly

reach negative opinion consensus and positive opinion consensus, respectively. The simulation results in Fig. 2 are evidently consistent with the theoretical results mentioned above.

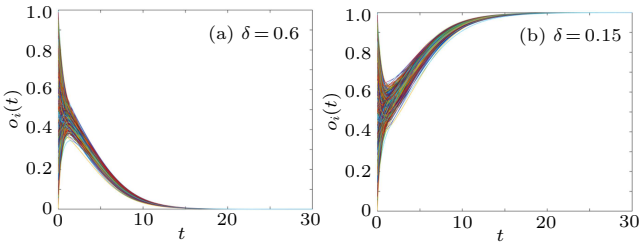


Fig. 2. Evolution of individual opinion with time t for (a) negative opinion consensus and (b) positive opinion consensus.

To further reveal the influence of the intra-connections in the collective-opinion layer G' on the evolution of individual opinions, in this subsection 50 WS networks G' are randomly generated with the probability $p = 0.5$. Using Eq. (26), the edge weights in these 50 networks are selected from the interval $[0,1]$ randomly, such that $b_1 = 0.02 : 0.02 : 1$ and $b_2 = 0.02 : 0.02 : 1$. At the same time, it is assumed that the intra-structure of the individual-opinion layer G and the inter-connections matrix H remain unchanged. Figure 3(a) displays the evolutionary characteristics of individual opinions at $t = 30$ with dissemination rates $\delta = 0.1 : 0.1 : 1$ and $b_1 = 0.02 : 0.02 : 1$. Figure 3(b) exhibits the evolutionary characteristics of individual opinions at $t = 30$ with dissemination rates $\delta = 0.1 : 0.1 : 1$ and $b_2 = 0.02 : 0.02 : 1$. It should be noted that for each simulation, the collective-opinion layer G' is generated 50 times randomly, so the 50 collective-opinion layers G' used in Figs. 3(a) and 3(b) may not be the same.

In Fig. 3(a), as b_1 decreases, it becomes easier for the individuals in the opinion layer G to reach negative consensus. Conversely, as b_1 increases, a larger dissemination rate δ is required for the individuals in the individual-opinion layer G to reach negative consensus. They are consistent with the condition (14). When $b_1 > 0.34$, no matter how large the value of δ is, the individuals in the individual-opinion layer G can only reach positive consensus. Similarly, in Fig. 3(b), as b_2 increases, it becomes easier for the individuals in the network G to reach positive consensus. With b_2 decreasing, a smaller dissemination rate δ is needed for the individuals in the network G to meet positive consensus. They are consistent with the condition (17). When $b_2 < 0.13$, no matter how small the value of δ is, the individuals in the network G always reach negative consensus.

Additionally, in Fig. 3, apart from blue area and yellow area, there are a few small red areas. Generally, with relatively small parameters b_1, b_2, and δ, the red areas emerge. It should be noted that although the red areas imply that the individuals in opinion layer G do not reach consensus within 30, we cannot conclude that they will not reach consensus forever.

In a word, figure 3 shows that changing the intra-structure of the collective-opinion layer G' can influence the consensus of the individual opinions in the individual-opinion layer G. From Fig. 3, it can be observed that in order to reach a negative consensus for all the individuals, a smaller b_1 performs better. Conversely, a larger b_2 is preferred to reach a positive consensus. Based on these findings, for the following simulations, we assume that $b_1 = 0.1$ when discussing negative opinion consensus and $b_2 = 0.9$ when investigating positive opinion consensus.

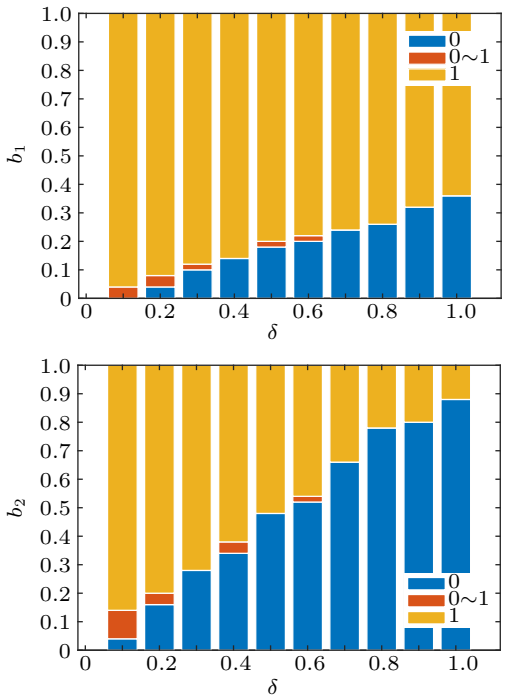


Fig. 3. Evolutionary characteristics of individual opinions at $t = 30$ with different dissemination rates and collective-opinion layer structures. Orange color represents individuals reaching positive consensus, blue color denotes individuals reaching the negative consensus, and red color refers to individuals reaching neither negative consensus nor positive consensus.

4.2. Influence of inter-connections on the evolution of individual opinions.

When changing the inter-connections between the individual-opinion layer G and the collective-opinion layer G', how are the individual opinions disseminated and evolved? In this subsection, regarding the dissemination and evolution of individual opinions, three types of inter-connections between individuals and groups are considered to explore what kind of phenomenon will take place. The first type of inter-connections is one-to-one inter-connections occurring between each individual and each group. The second type represents one-to-many inter-connections, where each individual belongs to multiple group. The third type mixes the first type and the second type. Assuming that the inter-connections satisfy $\sum_{j=1}^{N_2} h_{ij} = 1$ $(i = 1,2,\ldots,N_1)$, figure 4 exhibits the dissemination process and evolution process of individual opinions with time t for the three types of inter-connections be-

tween individuals and groups mentioned above. Like the scenario in Subsection 4.1, the dissemination rate $\delta = 0.15$ and $\delta = 0.6$ are involved in the negative opinions and positive opinions of the OIHOI dissemination model in Figs. 4(a) and 4(b), respectively.

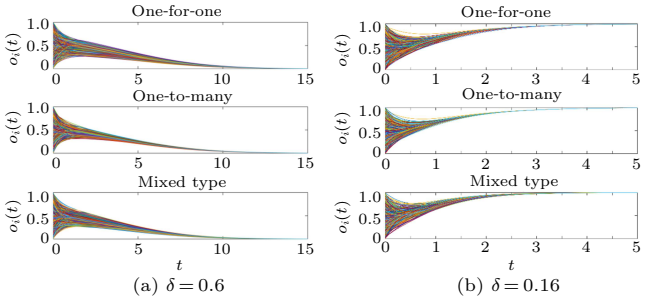


Fig. 4. Variations of dissemination and evolution of individual opinions with time t for three types of inter-connections, for (a) negative opinions with $\delta = 0.6$ and $b_1 = 0.1$, and (b) positive opinions with $\delta = 0.15$ and $b_2 = 0.9$, and with higher-order reinforcement factor being $\beta_\Delta = 0.5$.

Evidently, no matter what type of inter-connections takes place, all the individual opinions in the individual-opinion layer G can eventually reach either negative consensus or positive consensus. Further, in Fig. 4(a) or 4(b), when $\delta = 0.6$ or $\delta = 0.15$, the third type, i.e., mixed-type inter-connections in the two-layer network reaches negative consensus or positive consensus earlier than the first type and the second type of inter-connections. Therefore, the two-layer network with mixed-type inter-connections performs best, which will speed up reaching consensus and save time. In the following numerical simulations, we consistently assume that the inter-connections are of mixed type, that is to say, there are not only one-to-one inter-connections but also one-to-many inter-connections in the two-layer network.

4.3. Influence of higher-order reinforcement factor β_Δ on evolution of individual opinions.

In this subsection, we discuss the influences of higher-order interactions (or 2-simplex complex) in the individual-opinion layer G on the dissemination and evolution of individual opinions. Moreover, in this subsection, all individuals in the individual-opinion layer G are considered to have reached a consensus, if the difference in opinion between any two individuals is less than 0.05.

It should be noted that the influence of higher-order interactions in this paper is characterized by the higher-order reinforcement factor β_Δ. Thus, figure 5 shows the evolution of individual opinions in the individual-opinion layer G with time when the higher-order reinforcement factor $\beta_\Delta = 0$, $\beta_\Delta = 0.5$, and $\beta_\Delta = 1$. In Fig. 5(a), it is evident that all the individuals in the individual-opinion layer G eventually reach negative consensus. Moreover, the bigger the value of β_Δ, the faster the individuals reach negative consensus. Similarly, in Fig. 5(b), all the individuals in the network G eventually reach positive consensus. Then, the bigger the value of β_Δ, the faster the individuals reach positive consensus. In general, the presence of the higher-order reinforcement factor reduces the disagreement or heterogeneity of individual opinions and accelerates the speed of consensus.

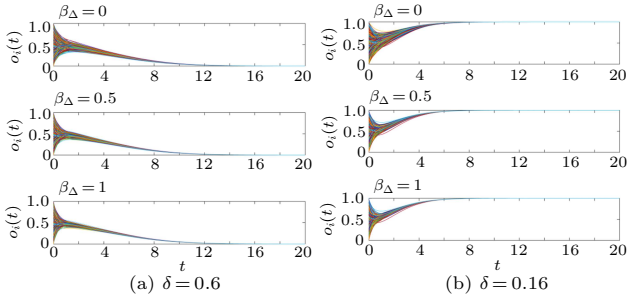


Fig. 5. Time evolution of individual opinions with different values of β_Δ for (a) negative opinions with $\delta = 0.6$ and $b_1 = 0.1$ and (b) positive opinions with $\delta = 0.15$ and $b_2 = 0.9$.

Furthermore, in order to examine the heterogeneity of the individual opinions in Fig. 5, the standard deviation of individual opinions in the individual-opinion layer G is newly defined, that is,

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (o_i - \bar{o})^2}{N_1}}, \tag{27}$$

where $\bar{o}$ represents the average opinion of all the individuals in the OIHOI dissemination model. Expression (27) characterizes the variability of the opinions between any two individuals in opinion layer G. A larger σ implies greater variability, while a smaller σ indicates less variability. With expression (27), the simulation results will be very similar for positive or negative consensus. Therefore in this subsection, we will only consider the case of positive consensus. In addition, based on Fig. 3(b), with the dissemination rate $\delta = 0.4$ and the collective-opinion layer intra-structure $b_2 = 0.36$, all the individual opinions in the individual-opinion layer G reach neither positive consensus nor negative consensus. Therefore, this case will be investigated in this subsection.

When the high-order reinforcement factors $\beta_\Delta = 0$, $\beta_\Delta = 0.3$, $\beta_\Delta = 0.6$, and $\beta_\Delta = 1$, figure 6 illustrates the trajectories of the standard deviation σ in the individual-opinion layer G. These trajectories are represented by the black, the blue, the green, and the red lines, respectively. In Fig. 6(a), it can be seen that the black line shows the slowest decrease, followed by the blue line. And the red line shows a fastest descent, followed by the green line. Therefore, it can be inferred that with the increase of high-order reinforcement factor β_Δ, the standard deviation σ tending towards zeros speeds up, which indicates that the individuals are more likely to reach positive consensus. In Fig. 6(b), when $t \leq 30$, the bigger the reinforcement factor β_Δ, the smaller the standard deviation σ of individual opinions in the OIHOI dissemination model is. Thus, in general, figure 6 displays that a higher reinforcement factor β_Δ makes it easier for individuals to reach a consensus.

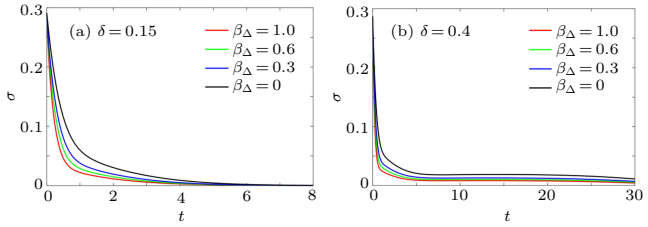


Fig. 6. Trajectories of standard deviation of individual opinions, for (a) reaching positive opinion consensus with $\delta = 0.15$ and $b_2 = 0.9$, and (b) neither positive consensus nor negative consensus is guaranteed with $\delta = 0.4$ and $b_2 = 0.36$.

Figure 7 shows the relationship between the reinforcement factor β_Δ and time t of reaching consensus. The red, the blue, and the yellow lines represent the trajectories when the dissemination rate $\delta = 0.3$, $\delta = 0.6$, and $\delta = 0.9$, respectively. From Fig. 7, it can be observed that no matter what the value δ is, a larger β_Δ leads to a shorter time taken for the individuals reaching opinion consensus. Specifically, when $\delta = 0.6$ and $\beta_\Delta = 0$, the value δ needs approximately 20, whereas when $\beta_\Delta = 0.6$, it only takes around 5 to reach consensus. Therefore, in the OIHOI dissemination model, it is significant for introducing the higher-order reinforcement factors β_Δ to eliminate divergences of opinions, thereby reaching opinion consensus quickly. In the following simulations, we assume the high-order reinforcement factor $\beta_\Delta = 1$, which will lead to a shortest time for reaching a consensus.

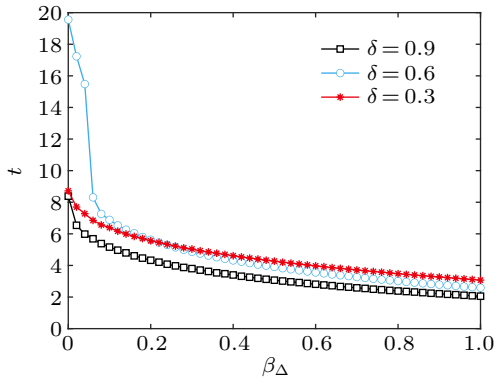


Fig. 7. Trajectories of time taken for reaching consensus *versus* reinforcement factor.

4.4. Opinion dissemination threshold

In this subsection, numerical simulations are conducted on the dissemination threshold (Eq. (25)) of the OIHOI dissemination model (1) proposed in Subsection 3.2. All the numerical simulations in this subsection have the same topology structures as in Subsection 4.1. The inter-connections of the two-layer network is still of mixed type.

In the following numerical simulations, the initial proportion of negative opinions in the individual-opinion layer G is assumed to be 10%. In other words, a randomly selected 10% of nodes satisfies $o_i(0) = 0$, while the remaining individual opinions are randomly selected in the interval $(0, 1]$. The edge weights of collective opinion networks G' are assigned values in the interval $[0, 1]$, which needs to satisfy $b_1 = 0.1$. And, if the difference in opinion among all the individuals in the individual-opinion layer G is less than 0.05, we regard the individual's opinion as negative consensus (0).

Figure 8 exhibits the evolution of the proportion of negative opinions ρ^{neg} in the individual-opinion layers G with δ changing. Overall, figure 8 shows that there is an eruption of the proportion of negative opinions around $\delta = 0.5$. Approximately at $\delta = 0.6$, 100% of the individuals in the individual-opinion layer G reach negative opinion consensus. The detailed trajectories of the proportion of negative opinions ρ^{neg} when $0.545 \leq \delta \leq 0.563$ are enlarged in the inset of Fig. 8. It can be observed from the inset that the proportion of negative opinions ρ^{neg} gradually increases from 0 when δ is around 0.5452. In the meantime, the proportion of negative opinions then starts to erupt in the OIHOI dissemination model (1). In addition, according to Eq. (25), we have

$$\lambda_{\max}\left(-\Omega^{-1}HH_1\right) = 0.515,$$

which confirms the theoretical result of the dissemination threshold $\delta_c \approx 0.515$. This result is very close to the numerical simulation result $\delta = 0.5452$, and thereby demonstrating the correctness of the theoretical result in Subsection 3.2.

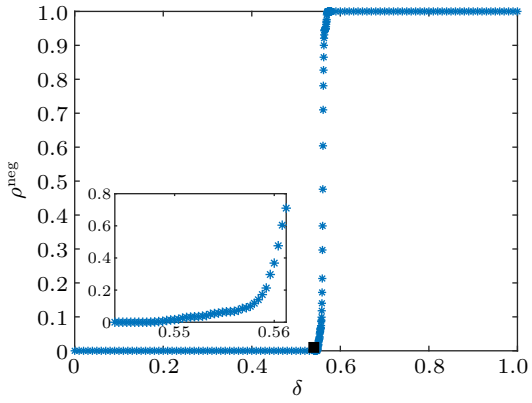


Fig. 8. Trajectories of proportion of negative opinions in individual-opinion layer, with high-order reinforcement factor $\beta_\Delta = 1$.

5. Conclusions

In the process of opinion dissemination and evolution, individual opinions are influenced not only by the opinions of their neighbors, but also by the opinions of the groups they belong to. Moreover, the inter-connections between individuals and groups are complex. Therefore, in order to investigate the mutual influence between any two individual opinions, between any two collective opinions, and between individuals and groups, a two-layer network consisting of an individual-opinion layer and a collective-opinion layer is constructed, and its corresponding OIHOI dissemination model is newly proposed. Next, the dynamic equations of opinion dissemination for the OIHOI dissemination model are established. Thus,

some theoretical sufficient conditions related to the two equilibrium points (i.e. negative consensus and positive consensus) of the OIHOI dissemination model are proposed for individuals to reach opinion consensus. It is found that reaching consensus depends on the structural characteristics of the intra-connections in the collective-opinion layer and the opinion dissemination rate. Then, in order to investigate the influences of intra-structure, inter-structure, and higher-order interactions on opinion dissemination, the mathematical expression of the opinion dissemination threshold is presented. Finally, numerical simulations are carried out to validate the feasibility and effectiveness of the proposed theoretical results, as well as the effects of intra-structures, inter-connections, and higher-order interactions on the dissemination and evolution of individual opinions. From those findings in the present research the following conclusions can be drawn. (i) The smaller the maximum value on the main diagonal of the adjacency matrix of the intra-connections in the collective-opinion layer, the easier it is to reach negative consensus; while the larger the minimum value of the sums of rows in the adjacency matrix, the easier it is to reach positive consensus. (ii) For the negative and the positive consensus, the two-layer network with a mixed type of inter-connections performs best. (iii) In the OIHOI dissemination model, higher-order interactions will help to eliminate disagreement between any two individual opinions, leading to faster consensus. Further, based on the achieved results mentioned above, the correctness of the proposed expression for the dissemination threshold is verified.

The findings of this paper will provide theoretical support and practical guidance for the regulation and management of individual and collective opinions. However, in this paper, the two-layer social network constructed is static, and the issue of opinion dissemination just focuses on consensus. In our real lives, the relationship between any two individuals or groups may be changed. Meanwhile, the dissemination of opinions may lead not only to consensus, but also to fragmentation or polarization. Therefore, in the future, we will focus on exploring opinion dynamics in two-layer dynamic networks.

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