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Quantum discord and its dynamics for multipartite systems

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Quantum discord, one of the famous quantum correlations, has been recently generalized to multipartite systems by Radhakrishnan *et al.* Here we give analytical solutions of the quantum discord for a family of N-qubit quantum states. For the bipartite system, we derive a zero quantum discord which will remain unchanged under the phase damping channel. For multiparitite systems, it is found that the quantum discord can be classified into three categories and the quantum discord for odd-partite systems can exhibit freezing under the phase damping channel, while the freezing does not exist in the even-partite systems.

Keywords: quantum discord, multipartite systems, decoherence channels, frozen discord

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1. Introduction

Quantum correlation is one of the core concepts in quantum information theory. A measure of quantum correlation proposed by Ollivier and Zurek[1] and Henderson and Vedral,[2] known as quantum discord, quantifies the quantum correlations in bipartite systems beyond entanglement.[3] It is defined as the minimum difference between the quantum mutual information before and after performing projective measurements in a bipartite quantum system. Since then, the quantum discord has been applied in many fields, such as quantum cryptography, remote quantum state preparation, and distributed entanglement.[4–7] To determine whether or not a quantum state has zero quantum discord is always a preliminary step to distinguish the quantum features of a bipartite state from the classical.[1,8–13]

Subsequently, some definitions of the quantum discord for tripartite or multipartite systems are proposed.[14–19] In particular, Radhakrishnan *et al.* recently defined the quantum discord for multipartite states[19] using conditional measurements, where each successive measurement is conditionally related to the previous measurement. This multipartite generalization of quantum discord is consistent with the original definition of quantum discord for bipartite systems.

It is well known that finding a general analytical solution of the quantum discord is hard work. Even for the bipartite system case, it was performed only for some special types of quantum states. For example, Luo gave the exact solution for the quantum discord of the Bell diagonal states,[20] and then more general X-states have been studied in various works.[21–29] Zhou *et al.* gave the analytical solution for the quantum discord of two non-X-states.[30] In Refs. [31–39], the lower and upper bounds of the quantum discord were given. For multipartite systems, even fewer results are derived.[40–46] For example, the quantum discord for multipartite X-states was evaluated analytically and classified into three categories by Li *et al.*, and they investigated the dynamic behavior of the quantum discord under decoherence.[45] Zhou *et al.* focused on the quantum discord for tripartite with fourteen parameters and got analytic solutions in some special cases.[46] In this paper, we obtain analytical solutions of the quantum discord for a family of N-qubit quantum states.

Furthermore, characterizing the dynamics of quantum discord under the influence of noise channels is of great importance in quantum information processing.[45,47–50] The quantum discord may tend to decrease at asymptotic time and may be frozen for a time interval, which is due to unavoidable interaction of quantum states with its environment. Freezing plays an important role in some quantum information processing, which can make task execution efficiency unaffected by the environment.[51–54] Thus we also analyzed the dynamical behavior of the quantum discord for a family of N-qubit states under the phase damping channel.

The rest of this article is organized as follows: In Section 2, we calculate the quantum discord for a family of N-qubit states. It is interesting to note that for the two-qubit states in this family, the quantum discord is zero. In fact, the two-qubit states in this family satisfy the necessary and sufficient condition for zero quantum discord.[1] However, the quantum discord in the multiparitite systems within this family is not necessarily zero. Moreover, it is shown that the quantum discord can be classified into three categories: $N = 4v-1$; $N = 2v$ $(v > 1)$; and $N = 4v+1$ with $v \in \mathbb{N}^+$. In particular, three-qubit states can be viewed as extended X-states with the anti-diagonal elements being zero.[31,32] Moreover, we illustrate the level surfaces of the quantum discord for a family of N-qubit states through images. In Section 3, we investigate the dynamic behavior of quantum discord for a family of N-qubit

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states under phase damping channel. We discuss and summarize the results of this paper in Section 4.

2. Quantum discord of *N*-qubit states

In this section, we shall first summarize the definition of the quantum discord of the N-qubit states given by Radhakrishnan *et al.*,[19] and then calculate the quantum discord for a family of the N-qubit states.

For an N-qubit state ρ, the k-partite measurement for $1 \le k \le N-1$ is

$$\Pi_j^{(k)} := \Pi_{j_1\cdots j_k}^{A_1\cdots A_k} = \Pi_{j_1}^{A_1} \otimes \Pi_{j_2|j_1}^{A_2} \otimes \cdots \otimes \Pi_{j_k|j_1\cdots j_{k-1}}^{A_k},$$

where the N subsystems are labeled as A_i and $\Pi_{j_t|j_1\cdots j_{t-1}}^{A_t}$ is a projector on subsystem A_t that is conditional on the measurement outcome of the subsystem A_1, A_2,..., and A_{t-1} with the measurements take place in the order $A_1 \to A_2 \to \cdots \to A_{t-1}$ for $2 \le t \le k$.

The quantum discord of an N-qubit state ρ is defined by

$$\begin{aligned} D_{A_1;\ldots;A_N}(\rho) = \min_{\Pi^{A_1\cdots A_{N-1}}} & [-S_{A_2\ldots A_N|A_1}(\rho) + S_{A_2|\Pi^{A_1}}(\rho) \\ & + \cdots + S_{A_N|\Pi^{A_1\cdots A_{N-1}}}(\rho)], \end{aligned} \tag{1}$$

for the measurement ordering $A_1 \to A_2 \to \ldots \to A_{N-1}$, where the conditional entropy without measurement $S_{A_2\ldots A_N|A_1}(\rho) = S_{A_1\ldots A_N}(\rho) - S_{A_1}(\rho)$ with $S_X(\rho) = -\mathrm{Tr}\rho^X \log_2 \rho^X$, the von Neumann entropy with respect to the (reduced) density matrix ρ^X on the system labeled by X, and the conditional entropy with measurement $S_{A_{k+1}|\Pi^{A_1\cdots A_k}}(\rho) = \sum_{j_1\cdots j_k} p_j^{(k)} S_{A_1\ldots A_{k+1}}(\Pi_j^{(k)}\rho\Pi_j^{(k)}/p_j^{(k)})$ with the probability $p_j^{(k)} = \mathrm{Tr}(\Pi_j^{(k)}\rho\Pi_j^{(k)})$.

We consider the N-qubit states of the following type:

$$\rho = \frac{1}{2^N}\left(I^{\otimes N} + \sum_{i=1}^{3} c_i \sigma_i \otimes I^{\otimes N-1} + \sum_{i=1}^{3} d_i I \otimes \sigma_i^{\otimes N-1}\right), \tag{2}$$

where $\{\sigma_i\}_{i=1}^3$ are the Pauli matrices, I stands for the 2×2-identity operator, and $c_i, d_i \in [-1,1]$ for $i = 1,2,3$.

To calculate the conditional entropy $S_{A_2\ldots A_N|A_1}(\rho)$, we need to determine the eigenvalues of ρ and $\rho^{A_1} = \mathrm{Tr}_{A_2\ldots A_N}(\rho) = \frac{1}{2}(I + \sum_{i=1}^3 c_i\sigma_i)$. By direct calculation, the nonnegative eigenvalues of ρ^{A_1} are $\frac{1}{2}(1\pm c)$, where $c = \sqrt{c_1^2+c_2^2+c_3^2}$, and the nonnegative eigenvalues of ρ can be classified into three cases. More precisely, let $v \in \mathbb{N}^+$, $c = \sqrt{c_1^2+c_2^2+c_3^2}$, and $d = \sqrt{d_1^2+d_2^2+d_3^2}$, then we have:

(i) If $N = 2v$, the 2^{2v} nonnegative eigenvalues of ρ are given by $\frac{1}{2^{2v}}(1+d+c)$, $\frac{1}{2^{2v}}(1+d-c)$, $\frac{1}{2^{2v}}(1-d+c)$, and $\frac{1}{2^{2v}}(1-d-c)$, each with multiplicity 2^{2v-2}.

(ii) If $N = 4v-1$, the 2^{4v-1} nonnegative eigenvalues of ρ are given by $\frac{1}{2^{4v-1}}(1-d_1+d_2+d_3\pm c)$, $\frac{1}{2^{4v-1}}(1+d_1-d_2+d_3\pm c)$, $\frac{1}{2^{4v-1}}(1-d_1-d_2-d_3\pm c)$, and $\frac{1}{2^{4v-1}}(1+d_1+d_2-d_3\pm c)$, each with multiplicity 2^{4v-4};

(iii) If $N = 4v+1$, the 2^{4v+1} nonnegative eigenvalues of ρ are given by $\frac{1}{2^{4v+1}}(1+d_1+d_2+d_3\pm c)$, $\frac{1}{2^{4v+1}}(1+d_1-d_2-d_3\pm c)$, $\frac{1}{2^{4v+1}}(1-d_1-d_2+d_3\pm c)$, and $\frac{1}{2^{4v+1}}(1-d_1+d_2-d_3\pm c)$, each with multiplicity 2^{4v-2}.

Thus the conditional entropy $S_{A_2\ldots A_N|A_1}(\rho)$ is

$$\begin{aligned} S_{A_2\ldots A_N|A_1}(\rho) = & -\sum_{i=1}^{2^N} \lambda_i \log_2 \lambda_i + \frac{1+c}{2}\log_2(1+c) \\ & + \frac{1-c}{2}\log_2(1-c) - 1, \end{aligned} \tag{3}$$

where λ_i $(i = 1,\ldots,2^N)$ represents the eigenvalues of ρ given as above.

For convenience, we define the following entropy function:

$$\begin{aligned} H_\varepsilon(x) = & \frac{1}{4}[(1+\varepsilon+x)\log_2(1+\varepsilon+x) \\ & + (1+\varepsilon-x)\log_2(1+\varepsilon-x)]. \end{aligned} \tag{4}$$

Lemma 1 Let ρ be the N-qubit quantum states in Eq. (2), and let $v \in \mathbb{N}^+$, $c = \sqrt{c_1^2+c_2^2+c_3^2}$ and $d = \sqrt{d_1^2+d_2^2+d_3^2}$.

(i) If $N = 2v$, then

$$S_{A_2\ldots A_{2v}|A_1}(\rho) = -[H_d(c) + H_{-d}(c)] + 2H_0(c) + 2v - 1. \tag{5}$$

(ii) If $N = 4v-1$, then

$$\begin{aligned} S_{A_2\ldots A_{4v-1}|A_1}(\rho) = & -\frac{1}{2}[H_{-d_1+d_2+d_3}(c) + H_{d_1-d_2+d_3}(c) \\ & + H_{-d_1-d_2-d_3}(c) + H_{d_1+d_2-d_3}(c)] \\ & + 2H_0(c) + 4v - 2. \end{aligned} \tag{6}$$

(iii) If $N = 4v+1$, then

$$\begin{aligned} S_{A_2\ldots A_{4v+1}|A_1}(\rho) = & -\frac{1}{2}[H_{d_1+d_2+d_3}(c) + H_{-d_1-d_2+d_3}(c) \\ & + H_{d_1-d_2-d_3}(c) + H_{-d_1+d_2-d_3}(c)] \\ & + 2H_0(c) + 4v. \end{aligned} \tag{7}$$

To calculate the conditional entropy $S_{A_{k+1}|\Pi^{A_1\cdots A_k}}(\rho)$ for $1 \le k \le N-1$, we shall first inspect the von Neumann measurement. For $k = 1$, the von Neumann measurement on the subsystem A_1 can be written as

$$\{\Pi_{j_1}^{A_1} = V^{A_1}|j_1\rangle\langle j_1|(V^{A_1})^\dagger : j_1 = 0,1\}, \tag{8}$$

where V^{A_1} is a unitary operator given by $V^{A_1} = t^{A_1}I + \sqrt{-1}\sum_{i=1}^3 y_i^{A_1}\sigma_i$ with $t^{A_1}, y_i^{A_1} \in \mathbb{R}$ $(i = 1,2,3)$ and $(t^{A_1})^2 + \sum_{i=1}^3 (y_i^{A_1})^2 = 1$. Similarly, we measure subsystem A_k based on the measurement outcomes on $A_1,\ldots,A_{k-1}$ for $2 \le k \le N-1$. Let

$$\begin{aligned} & \left\{\Pi_{j_k|j_1\cdots j_{k-1}}^{A_k} = V_{j_1\cdots j_{k-1}}^{A_k}|j_k\rangle\langle j_k|\left(V_{j_1\cdots j_{k-1}}^{A_k}\right)^\dagger : j_k = 0,1\right\}, \\ & j_1,\ldots,j_{k-1} = 0,1, \end{aligned} \tag{9}$$

be the von Neumann measurement on the subsystem A_k when the outcome of the measurement on $A_1,\ldots,A_{k-1}$ is

$j_1,\ldots,j_{k-1}$ ($j_1,\ldots,j_{k-1}=0,1$), where $V^{A_k}_{j_1\ldots j_{k-1}}=t^{A_k}_{j_1\ldots j_{k-1}}I+\sqrt{-1}\sum_{i=1}^3 y^{A_k}_{j_1\ldots j_{k-1}i}\sigma_i$ with $t^{A_k}_{j_1\ldots j_{k-1}}$, $y^{A_k}_{j_1\ldots j_{k-1}i}\in\mathbb{R}$ ($i=1,2,3$) and $(t^{A_k}_{j_1\ldots j_{k-1}})^2+\sum_{i=1}^3(y^{A_k}_{j_1\ldots j_{k-1}i})^2=1$. To simplify the further calculation, we set for $1\le k\le N-1$:

$$\begin{aligned}
\Gamma^{A_k}_{j_1\ldots j_{k-1}1}&=2\Big(-t^{A_k}_{j_1\ldots j_{k-1}}y^{A_k}_{j_1\ldots j_{k-1}2}+y^{A_k}_{j_1\ldots j_{k-1}1}y^{A_k}_{j_1\ldots j_{k-1}3}\Big),\\
\Gamma^{A_k}_{j_1\ldots j_{k-1}2}&=2\Big(-t^{A_k}_{j_1\ldots j_{k-1}}y^{A_k}_{j_1\ldots j_{k-1}1}+y^{A_k}_{j_1\ldots j_{k-1}2}y^{A_k}_{j_1\ldots j_{k-1}3}\Big),\\
\Gamma^{A_k}_{j_1\ldots j_{k-1}3}&=\Big(t^{A_k}_{j_1\ldots j_{k-1}}\Big)^2+\Big(y^{A_k}_{j_1\ldots j_{k-1}3}\Big)^2-\Big(y^{A_k}_{j_1\ldots j_{k-1}1}\Big)^2\\
&\quad-\Big(y^{A_k}_{j_1\ldots j_{k-1}2}\Big)^2.
\end{aligned}\tag{10}$$

Here, for $k=1$, we omit the term $j_1\ldots j_{k-1}$. That is,

$$\begin{aligned}
\Gamma^{A_1}_1&=2\Big(-t^{A_1}y^{A_1}_2+y^{A_1}_1y^{A_1}_3\Big),\\
\Gamma^{A_1}_2&=2\Big(-t^{A_1}y^{A_1}_1+y^{A_1}_2y^{A_1}_3\Big),\\
\Gamma^{A_1}_3&=(t^{A_1})^2+\Big(y^{A_1}_3\Big)^2-\Big(y^{A_1}_1\Big)^2-\Big(y^{A_1}_2\Big)^2.
\end{aligned}\tag{11}$$

It is easy to check $\sum_{i=1}^3(\Gamma^{A_k}_{j_1\ldots j_{k-1}i})^2=1$ and we have

$$\Pi^{A_1}_{j_1}\sigma_i\Pi^{A_1}_{j_1}=(-1)^{j_1}\Gamma^{A_1}_i\Pi^{A_1}_{j_1},\tag{12}$$

$$\Pi^{A_k}_{j_k|j_1\ldots j_{k-1}}\sigma_i\Pi^{A_k}_{j_k|j_1\ldots j_{k-1}}=(-1)^{j_k}\Gamma^{A_k}_{j_1\ldots j_{k-1}i}\Pi^{A_k}_{j_k|j_1\ldots j_{k-1}},\quad k\ge 2.\tag{13}$$

For $1\le k\le N-1$, we claim that after performing the measurement $\{\Pi^{A_1\ldots A_k}_{j_1\ldots j_k}:j_1,\ldots,j_k\in\{0,1\}\}$, the state ρ in Eq. (2) reduces to

$$\rho_{j_1\ldots j_k}=\frac{1}{2^{N-k}}\Pi^{A_1\ldots A_k}_{j_1\ldots j_k}\otimes\Big(I^{\otimes N-k}+\frac{\sum_{i=1}^3 d_i\Lambda_{j_1\ldots j_{k-1}i}\sigma_i^{\otimes N-k}}{1+(-1)^{j_1}\sum_{i=1}^3 c_i\Gamma^{A_1}_i}\Big),\tag{14}$$

with $p_{j_1\ldots j_k}=\frac{1}{2^k}(1+(-1)^{j_1}\sum_{i=1}^3 c_i\Gamma^{A_1}_i)$, where

$$\Lambda_{j_1\ldots j_{k-1}i}=\begin{cases}(-1)^{j_2+\cdots+j_k}\Gamma^{A_2}_{j_1i}\Gamma^{A_3}_{j_1j_2i}\ldots\Gamma^{A_k}_{j_1\ldots j_{k-1}i}, & k\ge 2,\\ \Lambda_i=1, & k=1,\end{cases}\qquad(i=1,2,3).\tag{15}$$

In fact, we can prove this argument by doing induction on k. For $k=1$, the statement follows directly by Eq. (12). After performing the measurement $\{\Pi^{A_1\ldots A_{k-1}}_{j_1\ldots j_{k-1}}:j_1,\ldots,j_{k-1}\in\{0,1\}\}$, we have

$$\begin{aligned}
\rho_{j_1\ldots j_{k-1}}&=\frac{1}{2^{N-(k-1)}}\Pi^{A_1\ldots A_{k-1}}_{j_1\ldots j_{k-1}}\\
&\quad\otimes\Big(I^{\otimes N-(k-1)}+\frac{\sum_{i=1}^3 d_i\Lambda_{j_1\ldots j_{k-2}i}\sigma_i^{\otimes N-(k-1)}}{1+(-1)^{j_1}\sum_{i=1}^3 c_i\Gamma^{A_1}_i}\Big),
\end{aligned}\tag{16}$$

and $p_{j_1\ldots j_{k-1}}=\frac{1}{2^{k-1}}(1+(-1)^{j_1}\sum_{i=1}^3 c_i\Gamma^{A_1}_i)$. Performing the measurement $\{\Pi^{A_k}_{j_k|j_1\ldots j_{k-1}}:j_k=0,1\}$ on subsystem A_k, we have

$$\begin{aligned}
&(p_{j_k|j_1\ldots j_{k-1}})(\rho_{j_1\ldots j_k})\\
&=(I^{\otimes k-1}\otimes\Pi^{A_k}_{j_k|j_1\ldots j_{k-1}}\otimes I^{\otimes N-k})\rho_{j_1\ldots j_{k-1}}\\
&\quad\times(I^{\otimes k-1}\otimes\Pi^{A_k}_{j_k|j_1\ldots j_{k-1}}\otimes I^{\otimes N-k})\\
&=\frac{1}{2^{N-(k-1)}}\Pi^{A_1\ldots A_k}_{j_1\ldots j_k}\\
&\quad\otimes\Big(I^{\otimes N-k}+\frac{(-1)^{j_k}\sum_{i=1}^3 d_i\Lambda_{j_1\ldots j_{k-2}i}\Gamma^{A_k}_{j_1\ldots j_{k-1}i}\sigma_i^{\otimes N-k}}{1+(-1)^{j_1}\sum_{i=1}^3 c_i\Gamma^{A_1}_i}\Big)\\
&=\frac{1}{2^{N-(k-1)}}\Pi^{A_1\ldots A_k}_{j_1\ldots j_k}\otimes\Big(I^{\otimes N-k}+\frac{\sum_{i=1}^3 d_i\Lambda_{j_1\ldots j_{k-1}i}\sigma_i^{\otimes N-k}}{1+(-1)^{j_1}\sum_{i=1}^3 c_i\Gamma^{A_1}_i}\Big),
\end{aligned}\tag{17}$$

where $p_{j_k|j_1\ldots j_{k-1}}=\mathrm{Tr}(p_{j_k|j_1\ldots j_{k-1}}\rho_{j_1\ldots j_k})=\frac{1}{2}$. Thus Eq. (14) holds and the probability $p_{j_1\ldots j_k}=p_{j_1\ldots j_{k-1}}p_{j_k|j_1\ldots j_{k-1}}=\frac{1}{2^k}(1+(-1)^{j_1}\sum_{ix=1}^3 c_i\Gamma^{A_1}_i)$, which finishes the proof of the claim.

For the N-qubit quantum states with $N\ge 3$, moreover, by Eq. (14), we have $\mathrm{Tr}_{A_{k+2}\ldots A_N}(\rho_{j_1\ldots j_k})=\frac{1}{2}\Pi^{A_1\ldots A_k}_{j_1\ldots j_k}\otimes I$, hence

$$S_{A_1\ldots A_{k+1}}(\rho_{j_1\ldots j_k})=1,\tag{18}$$

for $1\le k\le N-2$. Then the following statement holds directly.

Lemma 2 Let $N\ge 3$ and ρ be an N-qubit state in Eq. (2). Then for $1\le k\le N-2$, we have

$$S_{A_{k+1}|\Pi^{A_1\ldots A_k}}(\rho)=\sum_{j_1,\ldots,j_k=0}^{1}p_{j_1\ldots j_k}S_{A_1\ldots A_{k+1}}(\rho_{j_1\ldots j_k})=1.\tag{19}$$

By Lemma 1 and Lemma 2, the first $N-1$ terms of the quantum discord in Eq. (1) are constants. Hence to determine the quantum discord of ρ, it suffices to find the minimum value of the conditional entropy

$$S_{A_N|\Pi^{A_1\ldots A_{N-1}}}(\rho)=\sum_{j_1,\ldots,j_{N-1}=0}^{1}p_{j_1\ldots j_{N-1}}S_{A_1\ldots A_N}(\rho_{j_1\ldots j_{N-1}}).\tag{20}$$

After performing the measurement $\{\Pi^{A_1\ldots A_{N-1}}_{j_1\ldots j_{N-1}}:j_1,\ldots,j_{N-1}\in\{0,1\}\}$, by Eq. (14), we arrive at

$$\begin{aligned}
\rho_{j_1\ldots j_{N-1}}&=\frac{1}{2}\Pi^{A_1\ldots A_{N-1}}_{j_1\ldots j_{N-1}}\otimes\Big(I+\frac{\sum_{i=1}^3 d_i\Lambda_{j_1\ldots j_{N-2}i}\sigma_i}{1+(-1)^{j_1}\sum_{i=1}^3 c_i\Gamma^{A_1}_i}\Big),\\
p_{j_1\ldots j_{N-1}}&=\frac{1}{2^{N-1}}\Big(1+(-1)^{j_1}\sum_{i=1}^{3}c_i\Gamma^{A_1}_i\Big).
\end{aligned}\tag{21}$$

Thus, we have

$$\begin{aligned}
&S_{A_1\ldots A_N}(\rho_{j_1\ldots j_{N-1}})\\
&=-\Big(\frac{1+(-1)^{j_1}z+\alpha_{j_1\ldots j_{N-2}}}{2(1+(-1)^{j_1}z)}\Big)\log_2\Big(\frac{1+(-1)^{j_1}z+\alpha_{j_1\ldots j_{N-2}}}{2(1+(-1)^{j_1}z)}\Big)
\end{aligned}$$

$$-\Big(\frac{1+(-1)^{j_1}z-\alpha_{j_1\cdots j_{N-2}}}{2(1+(-1)^{j_1}z)}\Big)\log_2\Big(\frac{1+(-1)^{j_1}z-\alpha_{j_1\cdots j_{N-2}}}{2(1+(-1)^{j_1}z)}\Big), \tag{22}$$

where $z=\sum_{i=1}^{3}c_i\Gamma_i^{A_1}$ and

$$\alpha_{j_1\cdots j_{N-2}} = \sqrt{(d_1\Lambda_{j_1\cdots j_{N-2}1})^2+(d_2\Lambda_{j_1\cdots j_{N-2}2})^2+(d_3\Lambda_{j_1\cdots j_{N-2}3})^2}. \tag{23}$$

Here, for $N=2$, we omit $j_1\ldots j_{N-2}$, that is, $\alpha=\sqrt{d_1^2+d_2^2+d_3^2}=d$ since $\Lambda_i=1$ $(i=1,2,3)$ by Eq. (15). Inserting Eqs. (21) and (22) into Eq. (20) yields

$$\begin{aligned}
&S_{A_N|\Pi^{A_1\cdots A_{N-1}}}(\rho)\\
=&-\frac{1}{2^N}\sum_{j_1,\ldots,j_{N-1}=0}^{1}(1+(-1)^{j_1}z+\alpha_{j_1\cdots j_{N-2}})\\
&\times\log_2(1+(-1)^{j_1}z+\alpha_{j_1\cdots j_{N-2}})\\
&-\frac{1}{2^N}\sum_{j_1,\ldots,j_{N-1}=0}^{1}(1+(-1)^{j_1}z-\alpha_{j_1\cdots j_{N-2}})\\
&\times\log_2(1+(-1)^{j_1}z-\alpha_{j_1\cdots j_{N-2}})\\
&+\frac{1}{2}(1+z)\log_2(1+z)+\frac{1}{2}(1-z)\log_2(1-z)+1.
\end{aligned}\tag{24}$$

In particular, for $N=2$, we have

$$\begin{aligned}
&S_{A_2|\Pi^{A_1}}(\rho)=\sum_{j_1=0}^{1}p_{j_1}S_{A_1A_2}(\rho_{j_1})\\
=&-\frac{1}{4}(1+z+d)\log_2(1+z+d)-\frac{1}{4}(1+z-d)\log_2(1+z-d)\\
&-\frac{1}{4}(1-z+d)\log_2(1-z+d)-\frac{1}{4}(1-z-d)\log_2(1-z-d)\\
&+\frac{1}{2}(1+z)\log_2(1+z)+\frac{1}{2}(1-z)\log_2(1-z)+1\\
=&-H_d(z)-H_{-d}(z)+2H_0(z)+1.
\end{aligned}\tag{25}$$

Setting $F(z)=S_{A_2|\Pi^{A_1}}(\rho)$, it is easy to see that $F(z)$ is an even function of z. Thus to determine the minimum value of $F(z)$, we only need to consider the case: $z\geq 0$. Since

$$\frac{\mathrm{d}F}{\mathrm{d}z}=-\frac{1}{4}\log_2\frac{1-(\frac{d}{1+z})^2}{1-(\frac{d}{1-z})^2}\leq 0,$$

$F(z)$ is a decreasing function of z for $z\geq 0$. Thus, we need to determine the maximum value of z. As $z=\sum_{i=1}^{3}c_i\Gamma_i^{A_1}$ and $\sum_{i=1}^{3}(\Gamma_i^{A_1})^2=1$, we consider

$$\begin{aligned}
&f(\Gamma_1^{A_1},\Gamma_2^{A_1},\Gamma_3^{A_1},\mu)\\
&=c_1\Gamma_1^{A_1}+c_2\Gamma_2^{A_1}+c_3\Gamma_3^{A_1}+\mu\Big(1-\sum_{i=1}^{3}(\Gamma_i^{A_1})^2\Big),
\end{aligned}$$

where μ is a parameter. Setting $\partial f/\partial\Gamma_i^{A_1}=c_i-2\mu\Gamma_i^{A_1}=0$ for $i=1,2,3$ and $\partial f/\partial\mu=0$, we have $\mu=\pm c/2$ and $\Gamma_i^{A_1}=\pm c_i/c$. Hence the maximum value of z for $z\geq 0$ is c, and the minimum of $F(z)=S_{A_2|\Pi^{A_1}}(\rho)$ is taken at $z=c$. Therefore, the minimum of $S_{A_2|\Pi^{A_1}}(\rho)$ is

$$\min S_{A_2|\Pi^{A_1}}(\rho)=-H_d(c)-H_{-d}(c)+2H_0(c)+1. \tag{26}$$

Similarly, for $N\geq 3$, it is easy to check that $S_{A_N|\Pi^{A_1\cdots A_{N-1}}}(\rho)$ decreases with respect to each variable $\alpha_{j_1\cdots j_{N-2}}$, where $j_1,\ldots,j_{N-2}\in\{0,1\}$. Setting $\tilde{d}=\max\{|d_1|,|d_2|,|d_3|\}$, we have

$$\begin{aligned}
&\alpha_{j_1\cdots j_{N-2}}\\
\leq&\sqrt{(\tilde{d}\Lambda_{j_1\cdots j_{N-2}1})^2+(\tilde{d}\Lambda_{j_1\cdots j_{N-2}2})^2+(\tilde{d}\Lambda_{j_1\cdots j_{N-2}3})^2}\leq\tilde{d}.
\end{aligned}\tag{27}$$

Without loss of generality, let $|d_1|\geq|d_2|\geq|d_3|$. Then the above equality holds for the following cases:

(i) $|d_1|=|d_2|=|d_3|$: For any fixed $i\in\{1,2,3\}$, $\Gamma_{j_1i}^{A_2}=\cdots=\Gamma_{j_1\cdots j_{N-2}i}^{A_{N-1}}=1$ and $\Gamma_{j_1l}^{A_2}=\cdots=\Gamma_{j_1\cdots j_{N-2}l}^{A_{N-1}}=0$, where $l\in\{1,2,3\}$ and $l\neq i$.

(ii) $|d_1|>|d_2|\geq|d_3|$: $\Gamma_{j_11}^{A_2}=\cdots=\Gamma_{j_1\cdots j_{N-2}1}^{A_{N-1}}=1$ and $\Gamma_{j_1i}^{A_2}=\cdots=\Gamma_{j_1\cdots j_{N-2}i}^{A_{N-1}}=0$, where $i=2,3$.

(iii) $|d_1|=|d_2|>|d_3|$: $\Gamma_{j_13}^{A_2}=\cdots=\Gamma_{j_1\cdots j_{N-2}3}^{A_{N-1}}=0$, $\Gamma_{j_1i}^{A_2}=\cdots=\Gamma_{j_1\cdots j_{N-2}i}^{A_{N-1}}=1$ and $\Gamma_{j_1l}^{A_2}=\cdots=\Gamma_{j_1\cdots j_{N-2}l}^{A_{N-1}}=0$, where $i,l\in\{1,2\}$ and $l\neq i$.

Now we define

$$\begin{aligned}
&G(z)\\
=&-\frac{1}{4}(1+z+\tilde{d})\log_2(1+z+\tilde{d})-\frac{1}{4}(1+z-\tilde{d})\log_2(1+z-\tilde{d})\\
&-\frac{1}{4}(1-z+\tilde{d})\log_2(1-z+\tilde{d})-\frac{1}{4}(1-z-\tilde{d})\log_2(1-z-\tilde{d})\\
&+\frac{1}{2}(1+z)\log_2(1+z)+\frac{1}{2}(1-z)\log_2(1-z)+1.
\end{aligned}\tag{28}$$

Then we have $S_{A_N|\Pi^{A_1\cdots A_{N-1}}}(\rho)\geq G(z)$, where the equality holds if one of the above cases achieves. Hence, to calculate the minimum of $S_{A_N|\Pi^{A_1\cdots A_{N-1}}}(\rho)$, it suffices to find the minimum of $G(z)$. In comparison of Eqs. (25) and (28), it is easy to see that $F(z)$ and $G(z)$ differ only on the constant terms: d and $\tilde{d}$, hence $G(z)$ should also reach the minimum at $z=c$. Thus

$$\begin{aligned}
\min S_{A_N|\Pi^{A_1\cdots A_{N-1}}}(\rho)&=G(c)\\
&=-H_{\tilde{d}}(c)-H_{-\tilde{d}}(c)+2H_0(c)+1.
\end{aligned}\tag{29}$$

Combining Lemma 1, Lemma 2, Eqs. (1), (25), (26) and (29), we obtain the following theorem.

Theorem 1 Let $v\in\mathbb{N}^+$, $c=\sqrt{c_1^2+c_2^2+c_3^2}$, $d=\sqrt{d_1^2+d_2^2+d_3^2}$, and $\tilde{d}=\max\{|d_1|,|d_2|,|d_3|\}$. For the family of N-qubit quantum states (2), we classify the quantum discord into four cases:

(i) If $N=2$, then $D_{A_1;A_2}(\rho)=0$.

(ii) If $N=2v$ with $v>1$, then

$$D_{A_1;\ldots;A_{2v}}(\rho)=H_d(c)+H_{-d}(c)-H_{\tilde{d}}(c)-H_{-\tilde{d}}(c). \tag{30}$$

(iii) If $N = 4v-1$, then

$$D_{A_1;\ldots;A_{4v-1}}(\rho) = \frac{1}{2}[H_{-d_1+d_2+d_3}(c)+H_{d_1-d_2+d_3}(c) +H_{-d_1-d_2-d_3}(c)+H_{d_1+d_2-d_3}(c) -2H_{\tilde{d}}(c)-2H_{-\tilde{d}}(c)]. \quad (31)$$

(iv) If $N = 4v+1$, then

$$D_{A_1;\ldots;A_{4v+1}}(\rho) = \frac{1}{2}[H_{d_1+d_2+d_3}(c)+H_{-d_1-d_2+d_3}(c) +H_{d_1-d_2-d_3}(c)+H_{-d_1+d_2-d_3}(c) -2H_{\tilde{d}}(c)-2H_{-\tilde{d}}(c)]. \quad (32)$$

Figure 1 illustrates the level surfaces of the quantum discord for $D(\rho) = 0.03, 0.15, 0.3$ with $c = 0.1$. The level surfaces consist of three intersecting "tubes" running along the three Cartesian axes. As discord decreases, the tubes shrink toward the Cartesian axes. As discord increases, the tubes expand until only a few vertices remained, where (f) has level surfaces in eight corners, while (c) and (i) have only four corners left.

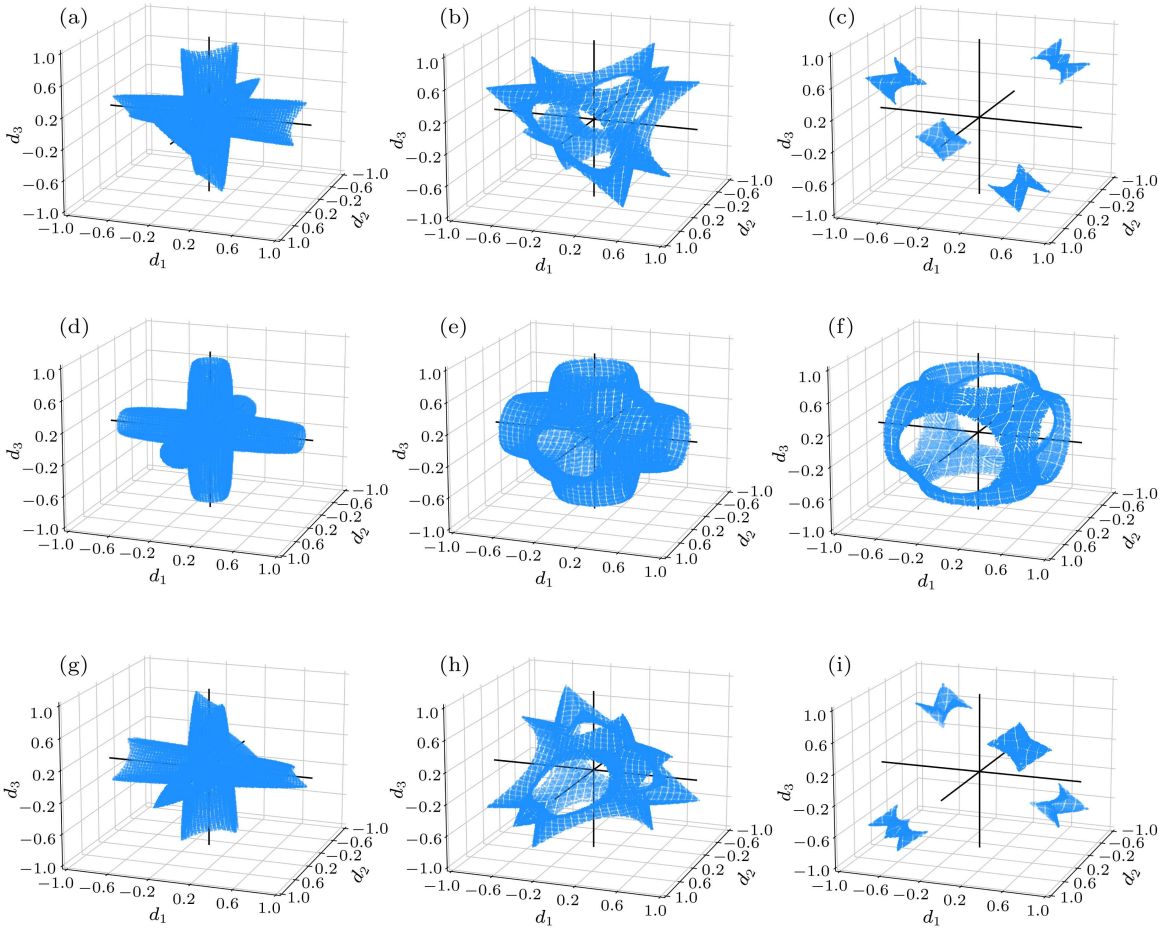


Fig. 1. Level surfaces of quantum discord for given constant $c = 0.1$. $N = 4v-1$ for (a), (b), and (c), with $D(\rho) = 0.03$, 0.15, 0.3, respectively. $N = 2v$ $(v > 1)$ for (d), (e), and (f), with $D(\rho) = 0.03$, 0.15, 0.3, respectively. $N = 4v+1$ for (g), (h), and (i), with $D(\rho) = 0.03$, 0.15, 0.3, respectively.

Remark 1 In the proof of Theorem 1(i) we obtained the zero quantum discord in the bipartite system by calculation. Actually one can also directly check that $\rho = \sum_{j_1=0}^{1}(\Pi_{j_1}^{A_1} \otimes I)\rho(\Pi_{j_1}^{A_1} \otimes I)$, which shows that this state satisfies the necessary and sufficient condition for zero quantum discord.[1] Thus this also shows in some extent that the consistency of definition of Radhakrishnan *et al.* if quantum discord with the conventional one in bipartite system. Moreover, there is one zero quantum discord state in Ref. [36], which can be viewed as a special case of our two-qubit states.

In Refs. [31,32], extended X-states were introduced, but the explicit analytic formulas of the quantum discord for this extended X-states were not completely given. Since our three-qubit states can be viewed as a special form of the extended X-states with the anti-diagonal elements being zero, we have given analytical solution of the quantum discord for some special extended X-states.

3. Dynamics of quantum discord under phase damping channel

In this section, we discuss the behavior of the quantum discord for the family of N-qubit quantum states (2) through

the phase damping channels[55] with the Kraus operators $K_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$ and $K_1 = \sqrt{\gamma}|1\rangle\langle 1|$, where $\gamma \in [0,1]$ is the decoherence rate and $\sum_i K_i^\dagger K_i = I$. Under the phase damping channel ρ evolves into

$$\begin{aligned}\varepsilon(\rho) &= \sum_{i_1,\ldots,i_N\in\{0,1\}} (K_{i_1}^{A_1}\otimes\cdots\otimes K_{i_N}^{A_N})\rho(K_{i_1}^{A_1}\otimes\cdots\otimes K_{i_N}^{A_N})^\dagger \\ &= \frac{1}{2^N}\Big(I^{\otimes N} + \sum_{i=1}^{2}\sqrt{1-\gamma}c_i\sigma_i\otimes I^{\otimes N-1} + c_3\sigma_3\otimes I^{\otimes N-1} \\ &\quad + \sum_{i=1}^{2}(1-\gamma)^{\frac{N-1}{2}}d_i I\otimes\sigma_i^{\otimes N-1} + d_3 I\otimes\sigma_3{}^{\otimes N-1}\Big). \quad (33)\end{aligned}$$

The parameter γ determines the influence of the noise in the channel to the quantum discord. Obviously, if $\gamma = 0$, the quantum discord is preserved. It is interesting to find some non-trivial cases, in which the quantum discord will also be kept unchanged. According to Theorem 1, we shall distinguish four cases from each other.

(i) $N = 2$.

In this case, it is easy to check that $D_{A_1;A_2}(\varepsilon(\rho)) = 0$.

(ii) $N = 2v$, $v \in \mathbb{N}^+$ and $(v > 1)$.

In this case, by Eq. (30), we have

$$\begin{aligned}D_{A_1;\ldots;A_{2v}}(\varepsilon(\rho)) &= H_\omega(\theta) + H_{-\omega}(\theta) - H_\varphi(\theta) - H_{-\varphi}(\theta) \\ &= \frac{1}{4}(1+\omega+\theta)\log_2(1+\omega+\theta) \\ &\quad + \frac{1}{4}(1+\omega-\theta)\log_2(1+\omega-\theta) \\ &\quad + \frac{1}{4}(1-\omega+\theta)\log_2(1-\omega+\theta) \\ &\quad + \frac{1}{4}(1-\omega-\theta)\log_2(1-\omega-\theta) \\ &\quad - \frac{1}{4}(1+\varphi+\theta)\log_2(1+\varphi+\theta) \\ &\quad - \frac{1}{4}(1+\varphi-\theta)\log_2(1+\varphi-\theta) \\ &\quad - \frac{1}{4}(1-\varphi+\theta)\log_2(1-\varphi+\theta) \\ &\quad - \frac{1}{4}(1-\varphi-\theta)\log_2(1-\varphi-\theta), \quad (34)\end{aligned}$$

where

$$\begin{aligned}\theta &= \sqrt{(1-\gamma)(c_1^2+c_2^2)+c_3^2}, \\ \omega &= \sqrt{(1-\gamma)^{2v-1}(d_1^2+d_2^2)+d_3^2}, \\ \varphi &= \max\{(1-\gamma)^{\frac{2v-1}{2}}|d_1|, (1-\gamma)^{\frac{2v-1}{2}}|d_2|, |d_3|\}.\end{aligned}$$

We may assume $\omega > \varphi > 0$, since $D_{A_1;\ldots;A_{2v}}(\varepsilon(\rho)) = 0$ if $\omega = \varphi$, which is not our interesting case here.

Then for $\theta > 0$, the derivative of $D_{A_1;\ldots;A_{2v}}(\varepsilon(\rho))$ with respect to γ is

$$\begin{aligned}&D'_{A_1;\ldots;A_{2v}}(\varepsilon(\rho)) \\ &= \frac{1}{4}\theta'\log_2\frac{((1+\theta)^2-\omega^2)((1-\theta)^2-\varphi^2)}{((1-\theta)^2-\omega^2)((1+\theta)^2-\varphi^2)} \\ &\quad + \frac{1}{4}\omega'\log_2\frac{(1+\omega)^2-\theta^2}{(1-\omega)^2-\theta^2} - \frac{1}{4}\varphi'\log_2\frac{(1+\varphi)^2-\theta^2}{(1-\varphi)^2-\theta^2}, \quad (35)\end{aligned}$$

where

$$\begin{aligned}\theta' &= -\frac{c_1^2+c_2^2}{\theta} \le 0, \\ \omega' &= -\frac{(2v-1)(1-\gamma)^{2v-2}(d_1^2+d_2^2)}{2\omega} < 0. \quad (36)\end{aligned}$$

In the first term of $D'_{A_1;\ldots;A_{2v}}(\varepsilon(\rho))$, we have

$$\begin{aligned}&\log_2\frac{((1+\theta)^2-\omega^2)((1-\theta)^2-\varphi^2)}{((1-\theta)^2-\omega^2)((1+\theta)^2-\varphi^2)} \\ &= \log_2\frac{(1-\theta^2)^2+\omega^2\varphi^2-(1+\theta^2)(\omega^2+\varphi^2)+2\theta(\omega^2-\varphi^2)}{(1-\theta^2)^2+\omega^2\varphi^2-(1+\theta^2)(\omega^2+\varphi^2)-2\theta(\omega^2-\varphi^2)} \\ &< 0, \quad (37)\end{aligned}$$

hence the first term of $D'_{A_1;\ldots;A_{2v}}(\varepsilon(\rho)) \le 0$. Next, we shell verify that the rest part of $D'_{A_1;\ldots;A_{2v}}(\varepsilon(\rho))$ is less than 0, and hence $D'_{A_1;\ldots;A_{2v}}(\varepsilon(\rho)) < 0$.

In fact, if $\varphi = |d_3|$, then $\varphi' = 0$, and it is easy to check that $D'_{A_1;\ldots;A_{2v}}(\varepsilon(\rho)) < 0$. Otherwise, we introduce a new function $f(x) = \frac{1}{x}\log_2\frac{(1+x)^2-\theta^2}{(1-x)^2-\theta^2}$, which is a strictly increasing function on $(0,1)$, since

$$\begin{aligned}f'(x) &= -\frac{1}{x^2}\log_2\frac{1+\theta+x}{1+\theta-x} + \frac{2}{x\ln 2}\frac{1+\theta}{(1+\theta)^2-x^2} \\ &\quad - \frac{1}{x^2}\log_2\frac{1-\theta+x}{1-\theta-x} + \frac{2}{x\ln 2}\frac{1-\theta}{(1-\theta)^2-x^2} \\ &= \frac{2}{x(1+\theta)\ln 2}\Big(\sum_{n=0}^{\infty}\Big(\frac{x}{1+\theta}\Big)^{2n} - \sum_{n=0}^{\infty}\frac{\Big(\frac{x}{1+\theta}\Big)^{2n}}{2n+1}\Big) \\ &\quad + \frac{2}{x(1-\theta)\ln 2}\Big(\sum_{n=0}^{\infty}\Big(\frac{x}{1-\theta}\Big)^{2n} - \sum_{n=0}^{\infty}\frac{\Big(\frac{x}{1-\theta}\Big)^{2n}}{2n+1}\Big) > 0. \quad (38)\end{aligned}$$

If $\varphi = (1-\gamma)^{\frac{2v-1}{2}}|d_1|$, then $\varphi' = -\frac{2v-1}{2}(1-\gamma)^{\frac{2v-3}{2}}|d_1|$. Hence, combining with Eq. (38) yields

$$\begin{aligned}&\frac{1}{4}\Big(\omega'\log_2\frac{(1+\omega)^2-\theta^2}{(1-\omega)^2-\theta^2} - \varphi'\log_2\frac{(1+\varphi)^2-\theta^2}{(1-\varphi)^2-\theta^2}\Big) \\ &= \frac{2v-1}{8(1-\gamma)\omega}\Big(\frac{\varphi^2\omega}{\varphi}\log_2\frac{(1+\varphi)^2-\theta^2}{(1-\varphi)^2-\theta^2} \\ &\quad - (\varphi^2+(1-\gamma)^{2v-1}d_2^2)\log_2\frac{(1+\omega)^2-\theta^2}{(1-\omega)^2-\theta^2}\Big) \\ &< \frac{2v-1}{8(1-\gamma)\omega}\Big(\frac{\varphi^2\omega}{\omega} - (\varphi^2+(1-\gamma)^{2v-1}d_2^2)\Big)\log_2\frac{(1+\omega)^2-\theta^2}{(1-\omega)^2-\theta^2} \\ &= -\frac{(2v-1)(1-\gamma)^{2v-2}d_2^2}{8\omega}\log_2\frac{(1+\omega)^2-\theta^2}{(1-\omega)^2-\theta^2} \le 0. \quad (39)\end{aligned}$$

Therefore, we have $D'_{A_1;\ldots;A_{2v}}(\varepsilon(\rho)) < 0$. Similarly, if $\varphi = (1-\gamma)^{\frac{2v-1}{2}}|d_2|$, we will get the same result.

Moreover, for $\theta = 0$, the derivative of $D_{A_1;\ldots;A_{2v}}(\varepsilon(\rho))$ given in Eq. (34) of γ is

$$D'_{A_1;\ldots;A_{2v}}(\varepsilon(\rho)) = \frac{1}{4}\omega' \log_2 \frac{(1+\omega)^2}{(1-\omega)^2} - \frac{1}{4}\varphi' \log_2 \frac{(1+\varphi)^2}{(1-\varphi)^2}. \tag{40}$$

Setting $\theta = 0$ in Eq. (39), we get again $D'_{A_1;\ldots;A_{2v}}(\varepsilon(\rho)) < 0$.

Therefore, the quantum discord $D_{A_1;\ldots;A_{2v}}(\varepsilon(\rho))$ of the $(2v)$-qubit $(v > 1)$ sates are always monotonically decreasing. The dashed lines in Fig. 2 show the dynamic behavior of quantum discord for four-qubit states under the phase damping channel, where (a) for the case of $d_1 = 1/3$, $d_3 = 1/4$, and $d_2 = -d_1 d_3 = -1/12$, and (b) for the case of $d_1 = 1/3$, $d_3 = 1/4$, and $d_2 = d_1 d_3 = 1/12$.

(iii) $N = 4v-1$, $v \in \mathbb{N}^+$.

We consider the case of $c_1 = c_2 = c_3 = 0$ and $d_2 = -d_1 d_3$. Since $-1 \le d_1 \le 1$, we have $(1-\gamma)^{2v-1}|d_2| \le (1-\gamma)^{2v-1}|d_1|$ for any γ. By Eq. (31), we have

$$D_{A_1;\ldots;A_{4v-1}}(\varepsilon(\rho)) = 2(H_0(d_3) + H_0((1-\gamma)^{2v-1}d_1) - H_0(\phi)), \tag{41}$$

where $\phi = \max\{(1-\gamma)^{2v-1}|d_1|, |d_3|\}$.

If $\phi = |d_3|$, we have

$$D_{A_1;\ldots;A_{4v-1}}(\varepsilon(\rho)) = 2H_0((1-\gamma)^{2v-1}d_1), \tag{42}$$

it is easy to check $D'_{A_1;\ldots;A_{4v-1}}(\varepsilon(\rho)) < 0$ with respect to γ, and hence $D_{A_1;\ldots;A_{4v-1}}(\varepsilon(\rho))$ monotonically decreases to zero.

If $\phi = (1-\gamma)^{2v-1}|d_1| > |d_3|$, then $0 \le \gamma < 1 - |\frac{d_3}{d_1}|^{\frac{1}{2v-1}}$, we have

$$D_{A_1;\ldots;A_{4v-1}}(\varepsilon(\rho)) = 2H_0(d_3), \tag{43}$$

which is constant under the phase damping channel during the time interval.

(iv) $N = 4v+1$, $v \in \mathbb{N}^+$.

We consider the case of $c_1 = c_2 = c_3 = 0$ and $d_2 = d_1 d_3$. Since $-1 \le d_1 \le 1$, we have $(1-\gamma)^{2v}|d_2| \le (1-\gamma)^{2v}|d_1|$ for any γ. By Eq. (32), we have

$$D_{A_1;\ldots;A_{4v+1}}(\varepsilon(\rho)) = 2(H_0(d_3) + H_0((1-\gamma)^{2v}d_1) - H_0(\psi)), \tag{44}$$

where $\psi = \max\{(1-\gamma)^{2v}|d_1|, |d_3|\}$.

If $\psi = |d_3|$, then we have

$$D_{A_1;\ldots;A_{4v+1}}(\varepsilon(\rho)) = 2H_0((1-\gamma)^{2v}d_1), \tag{45}$$

it is easy to check $D'_{A_1;\ldots;A_{4v+1}}(\varepsilon(\rho)) < 0$ with respect to γ, and hence $D_{A_1;\ldots;A_{4v+1}}(\varepsilon(\rho))$ monotonically decreases to zero.

If $\psi = (1-\gamma)^{2v}|d_1| > |d_3|$, then $0 \le \gamma < 1 - |d_3/d_1|^{\frac{1}{2v}}$, we have

$$D_{A_1;\ldots;A_{4v+1}}(\varepsilon(\rho)) = 2H_0(d_3), \tag{46}$$

which is constant under the phase damping channel during the time interval.

Thus, we have shown that there exists frozen phenomenon of the quantum discord for the $(4v \pm 1)$-qubit states under the phase damping channel, while for the $(2v)$-qubit states $(v > 1)$, this phenomenon does not appear. The solid lines in Fig. 2 show the frozen phenomenon in the two cases: (a) $N = 3$, $d_1 = 1/3$, $d_3 = 1/4$, and $d_2 = -d_1 d_3 = -1/12$; (b) $N = 5$, $d_1 = 1/3$, $d_3 = 1/4$, and $d_2 = d_1 d_3 = 1/12$. A sudden transition of quantum discord happens at $\gamma = 0.25$ in (a) or $\gamma = 0.134$ in (b), respectively.

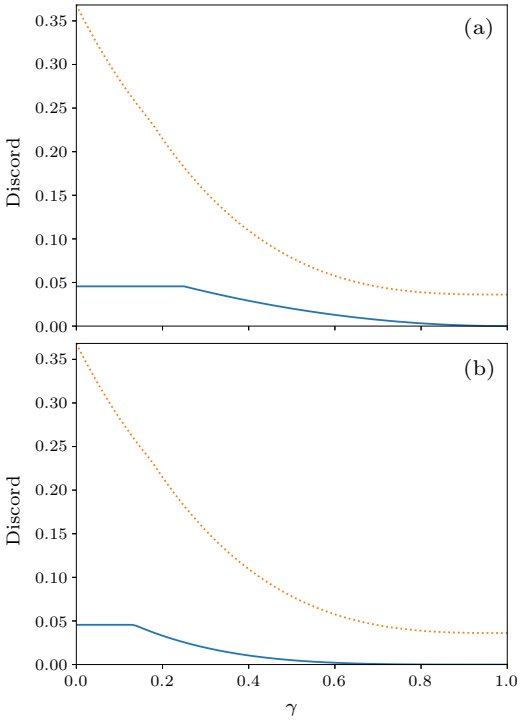


Fig. 2. The dynamic behavior of quantum discord under the phase damping channel: (a) the three-qubit state (solid line) and four-qubit state (dashed line), (b) the five-qubit state (solid line) and four-qubit state (dashed line).

4. Conclusions

We have derived the explicit analytic solutions of the quantum discord for a family of N-qubit states. For a bipartite system, we derive a zero quantum discord which will remain unchanged under the phase damping channel. For multiparitite systems, the quantum discord can be classified into three categories and it is shown that the quantum discord for the odd-qubit states can be freezing under the phase damping channel, but for the even-qubit states, the quantum discord is always decreasing under the phase damping channel. Moreover, the level surfaces of the quantum discord for these N-qubit states are illustrated in Fig. 1. We expect that our results will shed some light on further discussion on multipartite quantum discord and its applications in quantum information processing.

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