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RAPID COMMUNICATION

Dynamical localization in a non-Hermitian Floquet synthetic system

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We investigate the non-Hermitian effects on quantum diffusion in a kicked rotor model where the complex kicking potential is quasi-periodically modulated in the time domain. The synthetic space with arbitrary dimension can be created by incorporating incommensurate frequencies in the quasi-periodical modulation. In the Hermitian case, strong kicking induces the chaotic diffusion in the four-dimension momentum space characterized by linear growth of mean energy. We find that the quantum coherence in deep non-Hermitian regime can effectively suppress the chaotic diffusion and hence result in the emergence of dynamical localization. Moreover, the extent of dynamical localization is dramatically enhanced by increasing the non-Hermitian parameter. Interestingly, the quasi-energies become complex when the non-Hermitian parameter exceeds a certain threshold value. The quantum state will finally evolve to a quasi-eigenstate for which the imaginary part of its quasi-energy is large most. The exponential localization length decreases with the increase of the non-Hermitian parameter, unveiling the underlying mechanism of the enhancement of the dynamical localization by non-Hermiticity.

Keywords: Floquet system, non-Hermitian physics, dynamical localization

PACS: 05.45.–a, 05.45.Mt, 05.45.Pq **DOI:** 10.1088/1674-1056/ad36bb

1. Introduction

Quantum diffusion of chaotic systems with unique quantum coherence effects is a fundamental problem,[1–4] which has potential applications in quantum control[5–8] and quantum communication.[9–13] Understanding the fingerprint of the quantum coherence in chaotic diffusion is critical for the resolution of the elusive issue of the quantum–classical transition.[14–22] For example, the dynamical localization (DL) occurring after the Ehrenfest time demonstrates the lack of quantum–classical correspondence in Floquet driven systems due to the quantum coherence, which has received extensive investigations in the past few decades.[23–26] The mechanism governing chaotic diffusion plays a key role in quantum scrambling and non-equilibrium dynamics. It has been reported that the chaotic dynamics induced by Floquet driven potential dominates the exponentially-fast information scrambling,[27–29] the quantum thermalization process[30,31] and entanglement.[32–34] In addition, the unique features of quantum chaos have been exploited to optimize quantum algorithms and enhance quantum communication, thereby advancing the practical development of quantum technologies.[35–38]

The Floquet engineering technique serves as an effective method for creating high-dimensional synthetic spaces without the need to physically introduce extra spatial degrees of freedom, addressing a challenge encountered in traditional systems.[39,40] The application of Floquet synthetic systems in quantum simulation provides an ideal platform to investigate rich physics, such as topologically protected transport,[41,42] dynamical phase transition[43,44] and time crystals.[45,46] Interestingly, one can realize the synthetic space with controllable dimensions by adding quasi-periodical driven potential with incommensurate frequencies to the kicked rotor model, and observe the Anderson metal–insulator transition in different effective dimensions.[47–49] Recently, the extension of Floquet synthetic systems to non-Hermitian regime enables the investigations on novel physics that might not be easily achievable in Hermitian settings. It is found that the combination of non-Hermiticity and nonlinear interaction leads to super-exponentially fast energy diffusion,[50] the non-Hermitian driven potential enhances the degree of DL[51–54] and induces the quantum criticality of information scrambling.[55] The versatile nature of Floquet synthetic systems opens the opportunity for exploring diverse regimes of quantum behavior by tailoring system parameters and controlling interactions.

In this context, we investigate quantum diffusion in a quasi-periodically driven non-Hermitian kicked rotor (QNKR) model, where one can realize the synthetic momentum space with arbitrary dimension by incorporating incommensurate frequencies in the quasi-periodical modulation of kicking potential.[43,49] We concentrate on the non-Hermitian effects on the quantum diffusion in a four-dimension momentum space. Interestingly, we find that the quantum coherence in deep non-Hermitian regime can effectively suppress the chaotic diffusion and hence lead to the emergence of the DL.

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Correspondingly, the quantum state is exponentially localized in momentum space. Moreover, the localization length decreases with the increase of the imaginary part of the complex kicking potential, indicating the enhancement of DL by the non-Hermitian driven potential. In order to reveal the underlying mechanism, we extend the Floquet theory to this quasi-periodical system, and predict that a quantum state will finally evolve to a quasi-eigenstate with the large most value of the imaginary part of quasi-energy. We employ the average value of the inverse participation ratio (IPR) over an ensemble of quasi-eigenstates to quantify the localization length of quasi-eigenstates. Intrinsically, the average value of the IPR decreases with the increase of the imaginary part of the complex kicking potential, which demonstrates the enhancement of DL by non-Hermiticity. Therefore, the localized feature of quasi-eigenstates dominates the DL of this system.

The paper is organized as follows. In Section 2, we describe the QNKR model and show the DL in a non-Hermitian regime. In Section 3, we reveal the mechanism of the DL induced by non-Hermiticity. The conclusion and discussion are presented in Section 4.

2. Model and main results

The dimensionless Hamiltonian of the QNKR model reads

$$H = \frac{p^2}{2} + V_K(\theta,t)\sum_n \delta(t-t_n), \tag{1}$$

with the time-dependent kicking potential

$$V_K(\theta,t) = (K+\mathrm{i}\lambda)\left[1+\eta\prod_{j=1}^{3}\cos(\omega_j t)\right]\cos(\theta), \tag{2}$$

where $p=-\mathrm{i}\hbar_{\rm eff}\partial/\partial\theta$ is the angular momentum operator, θ is the angle coordinate, satisfying the commutation relation $[\theta,p]=\mathrm{i}\hbar_{\rm eff}$ with $\hbar_{\rm eff}$ being the effective Planck constant. Here, the parameters K and λ control the strength of the real and imaginary parts of the kicking potential, respectively. Time t_n is integer, i.e., $t_n=1,2,\ldots$, indicating the kicking number. The ω_j indicates the modulation frequency on the kicking strength, with the modulation strength denoted by η. When $\omega_j s$ are mutually incommensurate, and both are also incommensurate with 2π, the QNKR model is equivalent to a synthetic quantum kicked rotor with a four-dimensional momentum space.[40,43,47,48] We consider the case with $\omega_1=2\pi/\kappa$, $\omega_2=2\pi/\kappa^2$, and $\omega_3=2\pi/\sqrt{2}$, where $\kappa\approx1.3247$ is the real root of the cubic equation.[49]

The eigenequation of angular momentum operator is $p|n\rangle=p_n|n\rangle$ with eigenvalue $p_n=n\hbar_{\rm eff}$ and eigenstate $\langle\theta|n\rangle=\mathrm{e}^{\mathrm{i}n\theta}/\sqrt{2\pi}$. With this complete basis, an arbitrary state can be expanded as $|\psi\rangle=\sum_n\psi_n|n\rangle$. Based on the features of delta-kicking systems, the Floquet operator can be split into two components, namely $U(t)=U_{\rm f}U_{\rm K}(t)$, with the free evolution operator $U_{\rm f}=\exp(-\mathrm{i}p^2/2\hbar_{\rm eff})$ and the kicking term $U_{\rm K}(t)=\exp[-\mathrm{i}V_K(\theta,t)/2\hbar_{\rm eff}]$. The time evolution of a quantum state over one period from t_n to t_{n+1} is governed by $|\psi(t_{n+1})\rangle=U(t_n)|\psi(t_n)\rangle$. We numerically investigate quantum diffusion, characterized by the scaled expectation value $\langle p^2(t)\rangle=\sum_n p_n^2|\psi_n(t)|^2/\mathcal{N}(t)$, with the norm $\mathcal{N}(t)=\sum_n|\psi_n(t)|^2$. Note that this scaled mean energy eliminates the contribution of the norm which exponentially grows with time for sufficiently large value of λ. Without loss of generality, we choose the ground state of the angular momentum operator as the initial state, i.e., $\psi(\theta,t_0)=1/\sqrt{2\pi}$ for numerical simulations.

Figure 1(a) shows the time evolution of $\langle p^2\rangle$ for a wide regime of K with $\lambda=0$. One can see that, for small K the $\langle p^2\rangle$ increases during a short time duration, and eventually fluctuates around a saturation level which increases with the increase of K. This demonstrates the emergence of DL. Correspondingly, the quantum state is exponentially localized in momentum space, i.e., $|\psi(p)|^2\propto\exp(-|p|/\xi)$, serving as a distinctive feature of the DL (see Fig. 1(c)). Interestingly, the $\langle p^2\rangle$ exhibits a linear increase over time, i.e., $\langle p^2\rangle=Dt$ when K is sufficiently large K (e.g., $K=5$ and 10 in Fig. 1(a)), signifying the onset of chaotic diffusion. Meanwhile, the probability density distribution in momentum space displays the well-known Gaussian function $|\psi(p)|^2\propto\exp(-p^2/\zeta)$, characterizing the random walk of fully chaotic systems. Note that such dynamical transition from DL to chaotic diffusion has been observed in ultracold atom experiments.[40] Since the modulation frequencies within the time-dependent kick strength ω_j ($j=1,2,3$) are irrational numbers, this form of temporal modulation effectively introduces random noises ($\eta\ll1$) to the kick strength in the time domain. It is known that random noises can destroy DL and result in classically normal diffusion, i.e., $\langle p^2\rangle\approx K^2t/2$, which is confirmed by our numerical results in Fig. 1(a) (e.g., $K=10$).

To reveal the influence of the non-Hermitian driven potential on quantum diffusion, we numerically investigate the time evolution of $\langle p^2(t)\rangle$ for different λ. For very small λ (e.g., $\lambda=0.1$ in Fig. 1(b)), the mean energy follows well the linear growth, i.e., $\langle p^2(t)\rangle\approx K^2t/2$, which demonstrates the existence of the chaotic diffusion in the non-Hermitian regime. Interestingly, for large values of λ (e.g., $\lambda=2.3$ and 2.5), the growth of the $\langle p^2(t)\rangle$ tends to saturate over time, with the saturation values decreasing as λ increases. The mean energy remains almost constant in the case of a sufficiently strong non-Hermitian driven potential (e.g., $\lambda=5$). Along with the saturation of mean energy, the quantum state displays the exponentially localized shape in momentum space,

i.e., $|\psi(p)|^2 \propto \exp(-|p|/\xi)$, where the localization length ξ decreases as the λ increases (see Fig. 1(c)). Our numerical results clearly demonstrate the emergence of DL induced by non-Hermiticity in a Floquet synthetic system with a four-dimensional momentum space which is generated by quasi-periodical modulation.

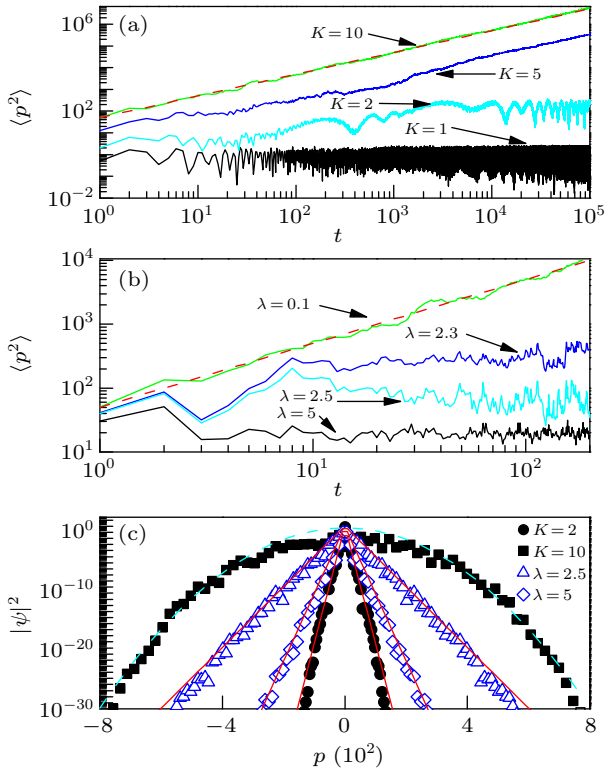


Fig. 1. (a) The $\langle p^2\rangle$ versus time with $\lambda = 0$ for $K = 1$ (black line), $K = 2$ (cyan line), $K = 5$ (blue line), and $K = 10$ (green line). (b) The $\langle p^2\rangle$ versus time with $K = 10$ for $\lambda = 0.1$ (green line), $\lambda = 2.3$ (blue line), $\lambda = 2.5$ (cyan line), and $\lambda = 5$ (black line). Red-dashed lines in panels (a) and (b) indicate the chaotic diffusion $\langle p^2\rangle \approx K^2 t/2$. (c) Momentum distributions at time $t = 100$. In Hermitian case, i.e., $\lambda = 0$ (solid symbols): $K = 2$ (circles) and $K = 10$ (squares). In the non-Hermitian case (empty symbols): $\lambda = 2.5$ (triangles) and $\lambda = 5$ (diamonds) with $K = 10$. Red-solid and cyan-dashed lines indicate the exponentially-localized wavepackets, i.e., $|\psi(p)|^2 \propto \exp(-|p|/\xi)$ and the Gaussian wavepackets, i.e., $|\psi(p)|^2 \propto \exp(-p^2/\zeta)$, respectively. The parameters are $\eta = 0.75$ and $\hbar_{\rm eff} = 2.89$.

3. Mechanism of the dynamical localization in non-Hermitian regime

3.1. Emergence of the complex instantaneous quasi-energies

The enhancement of DL by a non-Hermitian driven potential has been reported in Ref. [54], with the underlying mechanism unveiled by the extension of Floquet theory to the non-Hermitian regime. Moreover, we have extended the Floquet theory to the QNKR model with two incommensurate frequencies and successfully predicted the freezing of quantum scrambling in deep non-Hermitian regions. Interestingly, this extension can also reveal the mechanism of the emergence of DL in the synthetic system with four-dimensional momentum space.

Based on the Floquet theory, we define the eigenequation of the instantaneous Floquet operator as $U(t+1,t)|\varphi_\varepsilon(t)\rangle = \mathrm{e}^{-\mathrm{i}\varepsilon(t)}|\varphi_\varepsilon(t)\rangle$, where the quasi-energies are complex, i.e., $\varepsilon = \varepsilon_{\rm r} + \mathrm{i}\varepsilon_{\rm i}$ under non-Hermitian condition. In order to verify the appearance of the complex quasi-energies, we numerically calculate the average value of the imaginary part of the instantaneous quasi-energies, i.e., $|\overline{\varepsilon_{\rm i}}(t)| = \sum_{j=1}^{N} |\varepsilon_{\rm i}^j(t)|/N$ for different λ with N indicating the dimension of the Floquet operator matrix. Figure 2(a) shows that the value of $|\overline{\varepsilon_{\rm i}}|$ is zero for $\lambda = 0$. Interestingly, the $|\overline{\varepsilon_{\rm i}}(t)|$ oscillates around a saturation level for sufficiently large λ (e.g., $\lambda = 2.5$). We further define the average of $|\overline{\varepsilon_{\rm i}}(t)|$ over time, i.e., $\mathcal{R} = \sum_{j=0}^{n} |\overline{\varepsilon_{\rm i}}(t_j)|/n$, with n being the total number of kicks, to quantify the saturation level of $|\overline{\varepsilon_{\rm i}}(t)|$. Our results show that the $\mathcal{R}$ increases linearly with λ (see the inset in Fig. 2(a)), which evidences the emergence of complex quasi-energies.

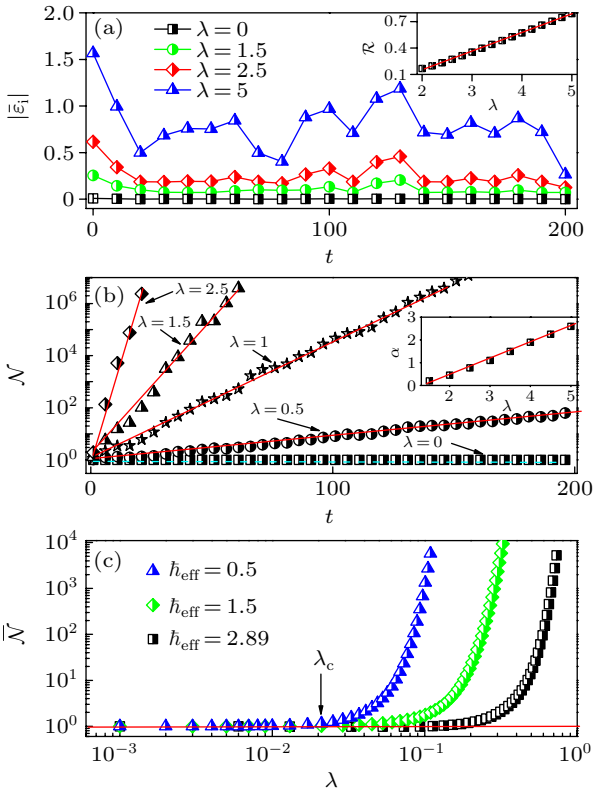


Fig. 2. (a) Time dependence of the average value of the imaginary parts of the quasi-energies $|\overline{\varepsilon_{\rm i}}|$ for $\lambda = 0$ (squares), 1.5 (circles), 2.5 (diamonds), and 5 (triangles). Inset: the $\mathcal{R}$ versus λ. Solid line indicates the linear growth $\mathcal{R} \propto \lambda$. (b) Norm $\mathcal{N}$ versus time with $K = 10$ for $\lambda = 0$ (squares), 0.5 (circles), 1 (pentagrams), 1.5 (triangles), and 2.5 (diamonds). Red-solid lines indicate the exponential increase, i.e., $\mathcal{N} = \mathrm{e}^{\alpha t}$. Cyan-dashed line denotes $\mathcal{N} = 1$. Inset: the growth rate α of the norm versus λ. The solid line indicates the linear growth $\alpha \propto \lambda$. (c) Time averaged value $\overline{\mathcal{N}}$ versus λ for $K = 10$ with $\hbar_{\rm eff} = 0.5$ (triangles), 1.5 (diamonds), and 2.89 (squares). The arrow marks the critical value $\lambda_{\rm c}$. Solid line denotes $\overline{\mathcal{N}} = 1$. Other parameters are the same as in Fig. 1(a).

A dynamical character of the emergence of the complex quasi-energies is the exponential growth of the norm. To confirm this, we numerically investigate the time evolution of the norm $\mathcal{N}$ for various values of λ. Figure 2(b) shows that, for

$\lambda = 0$, the norm $\mathcal{N}$ equals to unity with time evolution. For sufficiently large values of λ (e.g., $\lambda = 0.5$), the norm $\mathcal{N}$ displays exponential growth over time, i.e., $\mathcal{N} = e^{\alpha t}$, where the growth rate α linearly increases with the increase of λ (see the inset in Fig. 2(b)). We further investigate the time averaged value of the norm $\overline{\mathcal{N}} = \sum_{j=1}^{M} \mathcal{N}(t_j)/M$ for different $\hbar_{\rm eff}$. Figure 2(c) shows that, for a specific $\hbar_{\rm eff}$, the $\overline{\mathcal{N}}$ remains at unity for λ smaller than a threshold value $\lambda_{\rm c}$, and increases rapidly with the increase of λ for $\lambda > \lambda_{\rm c}$. Our results demonstrate the existence of a non-Hermitian boundary $\lambda_{\rm c}$ which separates the real quasi-energies (for $\lambda < \lambda_{\rm c}$) and complex quasi-energies (for $\lambda > \lambda_{\rm c}$). Interestingly, the critical value $\lambda_{\rm c}$ increases with $\hbar_{\rm eff}$.

3.2. The localized feature of quasi-eigenstates with complex instantaneous quasi-energies

We have observed that the quantum state of non-Hermitian Floquet systems tends to evolve towards a quasi-eigenstate with the maximum imaginary part of the quasienergy.[54] Thus, the feature of the quasi-eigenstate influences the dynamical behaviors, such as DL and the freezing of quantum scrambling, in these systems.[56] We further numerically investigate the fidelity between the time-evolved quantum state and the quasi-eigenstates, i.e., $\mathcal{F} = |\langle\psi(t)|\varphi_\varepsilon(t)\rangle|^2$ for $\lambda \gg \lambda_{\rm c}$. Figure 3(a) shows that for $\lambda = 5$, the fidelity $\mathcal{F}$ at time $t = 1000$ is nearly equal to unity at the maximum $\varepsilon_{\rm i} = 1.6321$, indicating a high overlap between the quantum state and the quasi-eigenstate associated with the maximum $\varepsilon_{\rm i}$. To verify this issue, we compare the probability density distribution between the quantum state $|\psi(t = 1000)\rangle$ and quasi-eigenstate $|\varphi_\varepsilon(t = 1000)\rangle$ with $\varepsilon_{\rm i} = 1.6321$ in Fig. 3(b). One can see that the two states exhibit good consistency with each other, both of which are exponentially localized in momentum space, i.e., $|\psi(p)|^2 \propto \exp(-|p|/\xi)$.

A commonly used measurement of the localization of quantum states is IPR, i.e., $I = (\sum_n |\psi_n|^2)^2/\sum_n |\psi_n|^4$, which is proportional to the localization length ξ for the exponentially localized state. We numerically calculate the average value of the IPR, i.e., $\langle I(t)\rangle = \sum_{j=1}^{N} I_j(t)/N$, over the ensemble of quasi-eigenstates, where $I_j(t)$ is the IPR of the j-th quasi-eigenstate, and N is the total number of quasi-eigenstates. Figure 3(c) shows that for $\lambda = 0$, the $\langle I(t)\rangle$ fluctuates around a saturation level with time evolution. Interestingly, the saturation level decreases with the increase of λ. We further define the average of $\langle I(t)\rangle$ over time, i.e., $\langle\bar{I}\rangle = \sum_{j=1}^{n} \langle I(t_j)\rangle/n$, where n is the total number of kicks, to quantify the saturation level. Our results show that $\langle\bar{I}\rangle$ exhibits a plateau for λ smaller than a critical value $\lambda_{\rm c}$, beyond which it monotonically decreases with the increase of λ (see the inset in Fig. 3(c)). This demonstrates the decrease of the localization length of quasi-eigenstates with the increase of strength of the non-Hermitian driven potential, which leads to the DL in this system.

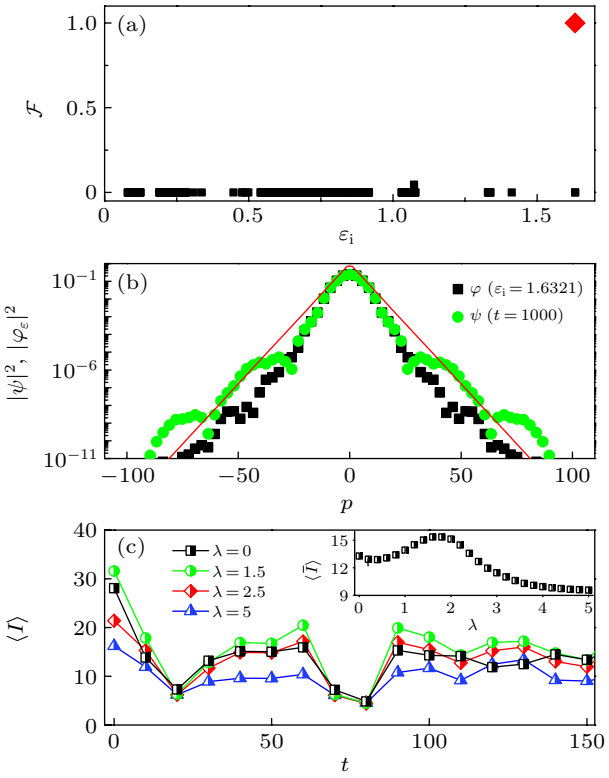


Fig. 3. (a) Fidelity $\mathcal{F}$ at the time $t = 1000$ versus the imaginary part of the quasi-energy $\varepsilon_{\rm i}$ for $K = 10$ and $\lambda = 5$. Red diamond indicates $\mathcal{F} \approx 1$ with $\varepsilon_{\rm i} = 1.6321$. (b) Comparison of the probability density distribution between the state $|\psi(t = 1000)\rangle$ (circles) and the quasi-eigenstate $|\varphi_\varepsilon(t = 1000)\rangle$ (squares) with $\varepsilon_{\rm i} = 1.6321$. Solid line indicates the exponentially-localized distribution of the form $|\psi(p)|^2 \propto \exp(-|p|/\xi)$ with $\xi \approx 3.2$. (c) The averaged value of IPR $\langle I\rangle$ versus time for $\lambda = 0$ (squares), 1.5 (circles), 2.5 (diamonds), and 5 (triangles). Inset: the $\langle\bar{I}\rangle$ versus λ. Other parameters are the same as in Fig. 1(a).

4. Discussion and conclusion

We investigate the dynamics of quantum diffusion, quantified by the scaled mean energy, i.e., $\langle p^2\rangle$, in a QNKR model with the complex kicking potential being modulated quasi-periodically in time. We find the suppression of linear growth of $\langle p^2\rangle$, namely chaotic diffusion, as the increase of the non-Hermitian parameter λ, indicating the emergence of DL. The enhancement of DL by non-Hermiticity is also reflected by the decrease of the localization length of a quantum state with the increase of λ. We numerically investigate the quasi-energies of the QNKR model, and find the emergence of complex quasi-energies when the non-Hermitian parameter is larger than a threshold value, i.e., $\lambda > \lambda_{\rm c}$. Interestingly, a quantum state will finally evolve to a quasi-eigenstate with the large most value of the imaginary part of quasi-energy, which is verified by our numerical results of the fidelity between the time-evolved state and quasi-eigenstates. We numerically investigate the IPR of the quasi-eigenstates and find the decrease of the IPR with the increase of λ, which demonstrates the

enhancement of localization of the quasi-eigenstates by non-Hermitian kicking potential. It is reasonable to believe that the feature of exponential localization of quasi-eigenstates determines the DL of the QNKR model. Our findings shed light on the quantum diffusion in non-Hermitian chaotic systems.

Acknowledgments

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