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Citation: Chin. Phys. B, 2024, 33 (**6**): 060307. DOI: 10.1088/1674-1056/ad2f20
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Wigner function of optical cumulant operator and its dissipation in thermo-entangled state representation

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(Received 24 January 2024; revised manuscript received 22 February 2024; accepted manuscript online 1 March 2024)

To conveniently calculate the Wigner function of the optical cumulant operator and its dissipation evolution in a thermal environment, in this paper, the thermo-entangled state representation is introduced to derive the general evolution formula of the Wigner function, and its relation to Weyl correspondence is also discussed. The method of integration within the ordered product of operators is essential to our discussion.

Keywords: Wigner function, optical cumulant operator, dissipation evolution, thermo-entangled state representation, integration within ordered product of operators

PACS: 03.65.–w, 03.65.Ud, 03.67.–a **DOI:** 10.1088/1674-1056/ad2f20

1. Introduction

The optical cumulant operator $\mathrm{e}^{\lambda a^{\dagger}a}$ can provide the generating function for deriving various statistics, such as mean and mean-square of photon numbers, since

$$\frac{\partial^n}{\partial\lambda^n}\mathrm{e}^{\lambda a^{\dagger}a}|_{\lambda=0}=\left(a^{\dagger}a\right)^n, \tag{1}$$

where $a^{\dagger}$ and a denote the optical creation operator and annihilation operator, respectively. Then, the ensemble average of $\left(a^{\dagger}a\right)^n$ for any density operator ρ is equal to

$$\mathrm{Tr}\left[\rho\left(a^{\dagger}a\right)^n\right]=\frac{\partial^n}{\partial\lambda^n}\mathrm{Tr}\left[\rho\,\mathrm{e}^{\lambda a^{\dagger}a}\right]|_{\lambda=0}\,. \tag{2}$$

Meanwhile, the classical Weyl correspondence function $h(q,p)$ of the density operator ρ is

$$\rho=\iint_{-\infty}^{\infty}\mathrm{d}p\mathrm{d}q h(q,p)\Delta(q,p)\,. \tag{3}$$

Here, $\Delta(q,p)$ is referred to as the Wigner operator. Substituting this expression into $\mathrm{Tr}\left[\rho\left(a^{\dagger}a\right)^n\right]$ yields

$$\begin{aligned}\mathrm{Tr}\left[\rho\left(a^{\dagger}a\right)^n\right]&=\frac{\partial^n}{\partial\lambda^n}\iint_{-\infty}^{\infty}\mathrm{d}p\mathrm{d}q h(q,p)\\&\quad\times\mathrm{Tr}\left[\Delta(q,p)\,\mathrm{e}^{\lambda a^{\dagger}a}\right]\Big|_{\lambda=0},\end{aligned} \tag{4}$$

where $\mathrm{Tr}\left[\Delta(q,p)\,\mathrm{e}^{\lambda a^{\dagger}a}\right]$ is just the Wigner function of the cumulant operator. Once it is known, the various ensemble averages of $\left(a^{\dagger}a\right)^n$ can be directly derived. Since $\left(1-\mathrm{e}^{\lambda}\right)\mathrm{e}^{\lambda a^{\dagger}a}=\rho_{\mathrm{c}}$ and $\mathrm{tr}\,\rho_{\mathrm{c}}=1$, we have

$$\begin{aligned}\mathrm{Tr}\left[a^{\dagger}a\rho_{\mathrm{c}}\right]&=(1-\mathrm{e}^{\lambda})\,\mathrm{Tr}\left[a^{\dagger}a\mathrm{e}^{\lambda a^{\dagger}a}\right]\\&=(1-\mathrm{e}^{\lambda})\frac{\partial}{\partial\lambda}\mathrm{Tr}\left[\mathrm{e}^{\lambda a^{\dagger}a}\right]\\&=(1-\mathrm{e}^{\lambda})\frac{\partial}{\partial\lambda}\frac{1}{1-\mathrm{e}^{\lambda}}\\&=(\mathrm{e}^{-\lambda}-1)^{-1},\end{aligned} \tag{5}$$

where ρ_{c} denotes the density operator of the chaotic light field,[1,2] so it is vital to study the Wigner function[3–5] of the optical cumulant operator and its evolution in dissipation. This paper focuses on this topic: first, the Wigner operator[6–8] is emphasized as a mixed state representation by using the method of integration within ordered product (IWOP) of operators;[9–13] then, the thermo-entangled state representation[14–17] is introduced to derive the general evolution formula of the Wigner function; also, the relation to Weyl correspondence[18–20] is discussed.

2. Wigner operator as mixed state representation discussed via IWOP method

In this section, the IWOP method is adopted to introduce the Wigner operator and Wigner function. When using the IWOP method, the completeness relation of coordinate eigenstate $|q\rangle$ (also momentum eigenstate $|p\rangle$) can be recast into a normal distribution within the normal product symbols as follows:

$$\int_{-\infty}^{\infty}\mathrm{d}q|q\rangle\langle q|=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}:\mathrm{e}^{-(q-Q)^2}:\mathrm{d}q=1, \tag{6}$$

$$\int_{-\infty}^{\infty}\mathrm{d}p|p\rangle\langle p|=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}p:\mathrm{e}^{-(p-P)^2}:=1, \tag{7}$$

where the symbol : : denotes normal ordering,[21–24] $Q=(a+a^{\dagger})/\sqrt{2}$ is the coordinate operator, and $P=(a-a^{\dagger})/(\sqrt{2}\mathrm{i})$ is the momentum operator. Thus, we can directly obtain

$$\frac{1}{\pi}:\mathrm{e}^{-(q-Q)^2-(p-P)^2}:\equiv\Delta(q,p) \tag{8}$$

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and refer to it as the Wigner operator, and $\mathrm{tr}\left[\Delta\left(q,p\right)\rho\right]$ or $\mathrm{tr}\left(\left|\psi\right\rangle\left\langle\psi\right|\Delta\left(q,p\right)\right)$ is also called the Wigner function $W\left(q,p\right)$. Here, ρ is the density operator of a mixed state, while $\left|\psi\right\rangle\left\langle\psi\right|$ is a pure state. Since the marginal integration of $\Delta\left(q,p\right)$ is

$$\int_{-\infty}^{\infty}\mathrm{d}p\Delta\left(q,p\right)=\left|q\right\rangle\left\langle q\right|, \tag{9}$$

$$\int_{-\infty}^{\infty}\mathrm{d}q\Delta\left(q,p\right)=\left|p\right\rangle\left\langle p\right|, \tag{10}$$

the marginal distributions of the Wigner function are the probability density in q-space and p-space, respectively, as follows:

$$\int_{-\infty}^{\infty}\mathrm{d}pW\left(q,p\right)=\mathrm{tr}\left(\left|\psi\right\rangle\left\langle\psi\right|\int\mathrm{d}p\Delta\left(q,p\right)\right)=\left|\psi\left(q\right)\right|^{2}, \tag{11}$$

$$\int_{-\infty}^{\infty}\mathrm{d}qW\left(q,p\right)=\mathrm{tr}\left(\left|\psi\right\rangle\left\langle\psi\right|\int\mathrm{d}q\Delta\left(q,p\right)\right)=\left|\psi\left(p\right)\right|^{2}. \tag{12}$$

Therefore, the Wigner function possesses the meaning of pseudo-probability distribution. By using the IWOP method

$$\begin{aligned}&\iint_{-\infty}^{\infty}\mathrm{d}p\mathrm{d}q\Delta\left(q,p\right)\\&=\frac{1}{\pi}\iint_{-\infty}^{\infty}\mathrm{d}p\mathrm{d}q:\mathrm{e}^{-\left(q-Q\right)^{2}-\left(p-P\right)^{2}}:=1,\end{aligned} \tag{13}$$

which indicates completeness, any operator in the $\left(q,p\right)$ phase space can be expanded as

$$H\left(Q,P\right)=\iint_{-\infty}^{\infty}\mathrm{d}p\mathrm{d}qh\left(q,p\right)\Delta\left(q,p\right), \tag{14}$$

and $h\left(q,p\right)$ is called the Weyl correspondence of H. Further, let $Q=\left(a+a^{\dagger}\right)/\sqrt{2}$, $P=\left(a-a^{\dagger}\right)/\left(\sqrt{2}\mathrm{i}\right)$, and $\left(q+\mathrm{i}p\right)/\sqrt{2}=\alpha$, and using

$$:\mathrm{e}^{-2a^{\dagger}a}:=\left(-1\right)^{a^{\dagger}a},\quad D\left(\alpha\right)=\mathrm{e}^{\alpha a^{\dagger}-\alpha^{*}a}, \tag{15}$$

equation (8) can be rewritten as

$$\begin{aligned}&\Delta\left(q,p\right)\rightarrow\frac{1}{\pi}:\mathrm{e}^{-2\left(a^{\dagger}-\alpha^{*}\right)\left(a-\alpha\right)}:\\&=\frac{1}{\pi}\mathrm{e}^{2\alpha a^{\dagger}}\left(-1\right)^{a^{\dagger}a}\mathrm{e}^{2\alpha^{*}a-2\left|\alpha\right|^{2}}\\&=\frac{1}{\pi}D\left(2\alpha\right)\left(-1\right)^{a^{\dagger}a}\\&\equiv\Delta\left(\alpha\right).\end{aligned} \tag{16}$$

Then, by using the completeness relation of coherent state[25–28]

$$\int\frac{\mathrm{d}^{2}z}{\pi}\left|z\right\rangle\left\langle z\right|=1,\quad\left|z\right\rangle=\mathrm{e}^{za^{\dagger}-\left|z\right|^{2}/2}\left|0\right\rangle, \tag{17}$$

we have

$$\begin{aligned}&\mathrm{Tr}\left[\Delta\left(\alpha\right)\Delta\left(\alpha'\right)\right]\\&=\frac{1}{\pi^{2}}\int\frac{\mathrm{d}^{2}z}{\pi}\left\langle z\right|\mathrm{e}^{2\alpha a^{\dagger}}\left(-1\right)^{a^{\dagger}a}\mathrm{e}^{2\alpha^{*}a}\mathrm{e}^{2\alpha'a^{\dagger}}\left(-1\right)^{a^{\dagger}a}\\&\quad\times\mathrm{e}^{2\alpha'^{*}a}\left|z\right\rangle\mathrm{e}^{-2\left|\alpha\right|^{2}-2\left|\alpha'\right|^{2}}\\&=\frac{1}{\pi^{2}}\int\frac{\mathrm{d}^{2}z}{\pi}\left\langle z\right|\mathrm{e}^{2\alpha z^{*}}\mathrm{e}^{-2\alpha^{*}a}\mathrm{e}^{-2\alpha'a^{\dagger}}\mathrm{e}^{2\alpha'^{*}z}\left|z\right\rangle\mathrm{e}^{-2\left|\alpha\right|^{2}-2\left|\alpha'\right|^{2}}\\&=\frac{1}{\pi^{2}}\int\frac{\mathrm{d}^{2}z}{\pi}\mathrm{e}^{2\left(\alpha-\alpha'\right)z^{*}}\mathrm{e}^{2\left(\alpha'^{*}-\alpha^{*}\right)z+4\alpha^{*}\alpha'-2\left|\alpha\right|^{2}-2\left|\alpha'\right|^{2}}\\&=\frac{1}{4\pi}\delta\left(\alpha-\alpha'\right)\delta\left(\alpha^{*}-\alpha'^{*}\right)\\&=\frac{1}{2\pi}\delta\left(q-q'\right)\delta\left(p-p'\right),\end{aligned} \tag{18}$$

which indicates the orthogonality of the Wigner operator. Therefore, it can be concluded that the Wigner operator provides a new mixed-state representation. Then, we have

$$\begin{aligned}W\left(q,p\right)&\equiv\mathrm{Tr}\left[\Delta\left(q,p\right)H\left(P,Q\right)\right]\\&=\mathrm{Tr}\left[\Delta\left(q,p\right)\iint_{-\infty}^{\infty}\mathrm{d}p'\mathrm{d}q'h\left(q',p'\right)\Delta\left(q',p'\right)\right]\\&=\iint_{-\infty}^{\infty}\mathrm{d}p'\mathrm{d}q'h\left(q',p'\right)\frac{1}{2\pi}\delta\left(q-q'\right)\delta\left(p-p'\right)\\&=\frac{1}{2\pi}h\left(q,p\right),\end{aligned} \tag{19}$$

which is the relation between the Wigner function $W\left(q,p\right)$ and its Weyl correspondence $h\left(q,p\right)$.

3. Calculating Wigner function in thermo-entangled state representation

To conveniently calculate various Wigner functions of the density operator's evolution in a thermal environment, the following shows how to express it in entangled state representation. To this end, the two-mode entangled state[29–31] is introduced as follows:

$$\left|\eta\right\rangle=\exp\left(-\frac{1}{2}\left|\eta\right|^{2}+\eta a^{\dagger}-\eta^{*}\tilde{a}^{\dagger}+a^{\dagger}\tilde{a}^{\dagger}\right)\left|0,\tilde{0}\right\rangle, \tag{20}$$

where $\tilde{a}^{\dagger}$ denotes the thermo-mode creation operator (fictitious node) accompanying $a^{\dagger}$ of real mode, $\left|0,\tilde{0}\right\rangle\equiv\left|0\right\rangle\left|\tilde{0}\right\rangle$, $\tilde{a}\left|\tilde{0}\right\rangle=0$, and $\left[\tilde{a},\tilde{a}^{\dagger}\right]=1$. The $\left(a^{\dagger},a\right)$ is independent of $\left(\tilde{a}^{\dagger},\tilde{a}\right)$. Respectively operating a and $\tilde{a}$ on $\left|\eta\right\rangle$ yields

$$\begin{aligned}&\left(a-\tilde{a}^{\dagger}\right)\left|\eta\right\rangle=\eta\left|\eta\right\rangle,\quad\left(a^{\dagger}-\tilde{a}\right)\left|\eta\right\rangle=\eta^{*}\left|\eta\right\rangle,\\&\left\langle\eta\right|\left(a^{\dagger}-\tilde{a}\right)=\eta^{*}\left\langle\eta\right|,\quad\left\langle\eta\right|\left(a-\tilde{a}^{\dagger}\right)=\eta\left\langle\eta\right|.\end{aligned} \tag{21}$$

Since $\left[\left(a-\tilde{a}^{\dagger}\right),\left(a^{\dagger}-\tilde{a}\right)\right]=0$, $\left|\eta\right\rangle$ is just the common eigenvector of $\left(a-\tilde{a}^{\dagger}\right)$ and $\left(\tilde{a}-a^{\dagger}\right)$. By using $\left|0,\tilde{0}\right\rangle\left\langle 0,\tilde{0}\right|=:\exp\left(-a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a}\right):$ and the IWOP method, we have

$$\int\frac{\mathrm{d}^{2}\eta}{\pi}\left|\eta\right\rangle\left\langle\eta\right|=1,\ \left\langle\eta'\right|\left.\eta\right\rangle=\pi\delta\left(\eta'-\eta\right)\delta\left(\eta'^{*}-\eta^{*}\right). \tag{22}$$

It is remarkable that the two-mode squeezing operator[32,33] has its natural representation in the $\left|\eta\right\rangle$ representation as follows:

$$\exp\left[\lambda\left(a^{\dagger}\tilde{a}^{\dagger}-a\tilde{a}\right)\right]=\int\frac{\mathrm{d}^{2}\eta}{\pi\mu}\left|\eta/\mu\right\rangle\left\langle\eta\right|,\quad\mu=\mathrm{e}^{\lambda}, \tag{23}$$

where μ is a squeezing parameter. Obviously,

$$\left|\eta=0\right\rangle=\mathrm{e}^{a^{\dagger}\tilde{a}^{\dagger}}\left|0,\tilde{0}\right\rangle=\sum_{n=0}^{\infty}\left|n,\tilde{n}\right\rangle, \tag{24}$$

which satisfies the following identity:

$$\begin{aligned} &a\left|\eta=0\right\rangle=\tilde{a}^{\dagger}\left|\eta=0\right\rangle,\\ &a^{\dagger}\left|\eta=0\right\rangle=\tilde{a}\left|\eta=0\right\rangle,\\ &\left(a^{\dagger}a\right)^{n}\left|\eta=0\right\rangle=\left(\tilde{a}^{\dagger}\tilde{a}\right)^{n}\left|\eta=0\right\rangle. \end{aligned} \tag{25}$$

Note that

$$\left|\eta\right\rangle=D\left(\eta\right)\left|\eta=0\right\rangle,\quad D\left(\eta\right)=\exp\left[\eta a^{\dagger}-\eta^{*}a\right]. \tag{26}$$

By using the orthonormal property of number states in the virtual Fock space $\langle\tilde{n}|\,\tilde{m}\rangle=\delta_{n,m}$, and letting $\left|\rho\right\rangle\equiv\rho\left|\eta=0\right\rangle$, we have

$$\begin{aligned} W\left(\alpha\right) &= \mathrm{Tr}\left[\Delta\left(\alpha\right)\rho\right]=\sum_{m=0}^{\infty}\left\langle m\right|\Delta\left(\alpha\right)\rho\left|m\right\rangle\\ &= \sum_{m,n=0}^{\infty}\left\langle n\right|\Delta\left(\alpha\right)\rho\left|m\right\rangle\delta_{n,m}\\ &= \sum_{m,n=0}^{\infty}\left\langle n,\tilde{n}\right|\Delta\left(\alpha\right)\rho\left|m,\tilde{m}\right\rangle=\left\langle\eta=0\right|\Delta\left(\alpha\right)\rho\left|\rho=0\right\rangle\\ &= \frac{1}{\pi}\left\langle\eta=0\right|D\left(2\alpha\right)\left(-1\right)^{a^{\dagger}a}\left|\rho\right\rangle\\ &= \frac{1}{\pi}\left\langle\eta=0\right|D^{\dagger}\left(-2\alpha\right)\left(-1\right)^{a^{\dagger}a}\left|\rho\right\rangle\\ &= \frac{1}{\pi}\left\langle\eta=-2\alpha\right|\left(-1\right)^{a^{\dagger}a}\left|\rho\right\rangle=\frac{1}{\pi}\left\langle\xi=2\alpha\right|\rho\left\rangle, \right. \end{aligned} \tag{27}$$

where $\left|\xi\right\rangle$ is the conjugate state of $\left|\eta\right\rangle$,

$$\begin{aligned} \left|\xi\right\rangle &= D\left(\xi\right)\mathrm{e}^{-a^{\dagger}\tilde{a}^{\dagger}}\left|0\tilde{0}\right\rangle\\ &= \exp\left(-\frac{1}{2}\left|\xi\right|^{2}+\xi a^{\dagger}+\xi^{*}\tilde{a}^{\dagger}-a^{\dagger}\tilde{a}^{\dagger}\right)\left|0\tilde{0}\right\rangle, \end{aligned} \tag{28}$$

which is related to $\left|\eta\right\rangle$ by

$$\left|\eta\right\rangle=\left(-1\right)^{\tilde{a}^{\dagger}\tilde{a}}\left|\xi\right\rangle_{\xi=\eta},\quad \left(-1\right)^{a^{\dagger}a}\left|\eta\right\rangle=\left|\xi\right\rangle_{\xi=-\eta}, \tag{29}$$

$$\left\langle\eta\right|\left.\xi\right\rangle=\frac{1}{2}\exp\left(\frac{\xi\eta^{*}-\xi^{*}\eta}{2}\right). \tag{30}$$

Equation (27) provides a new formalism to calculate the Wigner function. Further, using Eq. (22), equation (27) can be rewritten as

$$\begin{aligned} W\left(\alpha\right) &= \int\frac{\mathrm{d}^{2}\eta}{\pi^{2}}\left\langle\xi=2\alpha\right|\left.\eta\right\rangle\left\langle\eta\right|\left.\rho\right\rangle\\ &= \int\frac{\mathrm{d}^{2}\eta}{2\pi^{2}}\,\mathrm{e}^{\alpha^{*}\eta-\alpha\eta^{*}}\left\langle\eta\right|\left.\rho\right\rangle. \end{aligned} \tag{31}$$

Once the overlap $\left\langle\eta\right|\left.\rho\right\rangle$ is known, its Fourier integration is performed according to Eq. (31), leading to the Wigner function of ρ.

4. Wigner function of optical cumulant operator

The optical cumulant operator is defined as

$$\mathrm{e}^{\lambda a^{\dagger}a}=\sum_{n=0}^{\infty}\frac{\left(\lambda a^{\dagger}a\right)^{n}}{n!}. \tag{32}$$

Let $\rho\equiv\mathrm{e}^{\lambda a^{\dagger}a}$. Operating it on $\left|\eta=0\right\rangle$ yields

$$\begin{aligned} \rho\left|\eta=0\right\rangle &= \mathrm{e}^{\lambda a^{\dagger}a}\left|\eta=0\right\rangle=\mathrm{e}^{\lambda a^{\dagger}a}\mathrm{e}^{a^{\dagger}\tilde{a}^{\dagger}}\left|0\tilde{0}\right\rangle\\ &= \mathrm{e}^{\mathrm{e}^{\lambda}a^{\dagger}\tilde{a}^{\dagger}}\left|0\tilde{0}\right\rangle\equiv\left|\rho\right\rangle. \end{aligned} \tag{33}$$

Using the completeness relation of the two-mode coherent state[34–36] (real mode and fictitious mode)

$$\int\frac{\mathrm{d}^{2}z\mathrm{d}^{2}z'}{\pi^{2}}\left|z,\tilde{z}'\right\rangle\left\langle z,\tilde{z}'\right|=1 \tag{34}$$

with

$$a\left|z,\tilde{z}'\right\rangle=z\left|z,\tilde{z}'\right\rangle,\quad \tilde{a}\left|z,\tilde{z}'\right\rangle=z'\left|z,\tilde{z}'\right\rangle, \tag{35}$$

we have

$$\begin{aligned} \left\langle\eta\right|\left.\rho\right\rangle &= \exp\left(-\frac{1}{2}\left|\eta\right|^{2}\right)\left\langle 0,\tilde{0}\right|\\ &\quad\times\exp\left(\eta^{*}a-\eta\tilde{a}+a\tilde{a}\right)\mathrm{e}^{\mathrm{e}^{\lambda}a^{\dagger}\tilde{a}^{\dagger}}\left|0,\tilde{0}\right\rangle\\ &= \exp\left(-\frac{1}{2}\left|\eta\right|^{2}\right)\left\langle 0,\tilde{0}\right|\int\frac{\mathrm{d}^{2}z\mathrm{d}^{2}z'}{\pi^{2}}\\ &\quad\times\exp\left(\left(\eta^{*}a-\eta\tilde{a}+a\tilde{a}\right)\right)\left|z,\tilde{z}'\right\rangle\left\langle z,\tilde{z}'\right|\mathrm{e}^{\mathrm{e}^{\lambda}a^{\dagger}\tilde{a}^{\dagger}}\left|0,\tilde{0}\right\rangle\\ &= \exp\left(-\frac{1}{2}\left|\eta\right|^{2}\right)\int\frac{\mathrm{d}^{2}z\mathrm{d}^{2}z'}{\pi^{2}}\\ &\quad\times\exp(-\left|z\right|^{2}-\left|z'\right|^{2}+\eta^{*}z-\eta z'+zz'+\mathrm{e}^{\lambda}z^{*}z'^{*})\\ &= \exp\left(-\frac{1}{2}\left|\eta\right|^{2}\right)\int\frac{\mathrm{d}^{2}z}{\pi}\\ &\quad\times\exp[-\left|z\right|^{2}(1-\mathrm{e}^{\lambda})+\eta^{*}z-\eta\,\mathrm{e}^{\lambda}z^{*}]\\ &= \frac{1}{1-\mathrm{e}^{\lambda}}\exp\left[-\left|\eta\right|^{2}\frac{1+\mathrm{e}^{\lambda}}{2\left(1-\mathrm{e}^{\lambda}\right)}\right]. \end{aligned} \tag{36}$$

Then, according to Eq. (31), we have

$$\begin{aligned} W\left(\alpha\right) &= \int\frac{\mathrm{d}^{2}\eta}{2\pi^{2}}\exp\left(\alpha^{*}\eta-\alpha\eta^{*}\right)\left\langle\eta\right|\left.\rho\right\rangle\\ &= \frac{1}{1-\mathrm{e}^{\lambda}}\int\frac{\mathrm{d}^{2}\eta}{2\pi^{2}}\exp\left(\alpha^{*}\eta-\alpha\eta^{*}\right)\\ &\quad\times\exp\left[-\frac{1+\mathrm{e}^{\lambda}}{2\left(1-\mathrm{e}^{\lambda}\right)}\left|\eta\right|^{2}\right]\\ &= \frac{2}{\left(1+\mathrm{e}^{\lambda}\right)\pi}\exp\left[-\frac{2\left(1-\mathrm{e}^{\lambda}\right)}{1+\mathrm{e}^{\lambda}}\left|\alpha\right|^{2}\right], \end{aligned} \tag{37}$$

where $\alpha=\left(q+\mathrm{i}p\right)/\sqrt{2}$. This result can be validated by reforming the Weyl correspondence relation in Eq. (14) as

$$\rho=\int\mathrm{d}^{2}\alpha h\left(\alpha\right)\Delta\left(\alpha\right),\quad h\left(\alpha\right)=2\pi W\left(\alpha\right). \tag{38}$$

Substituting Eq. (37) into Eq. (38) yields

$$\begin{aligned} \rho &= \frac{4}{1+\mathrm{e}^{\lambda}}\int\mathrm{d}^{2}\alpha\exp\left[-\frac{2\left(1-\mathrm{e}^{\lambda}\right)}{1+\mathrm{e}^{\lambda}}\left|\alpha\right|^{2}\right]\frac{1}{\pi}\\ &\quad\times:\mathrm{e}^{-2(\alpha-a)\left(\alpha^{*}-a^{\dagger}\right)}:\\ &= \frac{4}{1+\mathrm{e}^{\lambda}}\int\frac{\mathrm{d}^{2}\alpha}{\pi} \end{aligned}$$

$$\times \exp\left[-\frac{4}{1+\mathrm{e}^{\lambda}}|\alpha|^2+2\alpha^* a+2\alpha a^\dagger-2a^\dagger a\right]$$
$$=\frac{1}{1+\mathrm{e}^{\lambda}}\frac{1+\mathrm{e}^{\lambda}}{4}:\exp[(1+\mathrm{e}^{\lambda})a^\dagger a-2a^\dagger a]:$$
$$=\exp\left(\lambda a^\dagger a\right). \tag{39}$$

5. Wigner operator's evolution in thermal environment

Now, consider the damping master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=\kappa\left(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a\right), \tag{40}$$

where κ denotes the decay rate. To derive the law of how Wigner function $W(\alpha,0)$ evolves into $W(\alpha,t)$ at time t, operating the two sides of equation (40) on $|\eta=0\rangle$ and letting $|\rho\rangle=\rho|\eta=0\rangle$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle=\kappa\left(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a\right)|\eta=0\rangle. \tag{41}$$

Using Eq. (21) and considering the commutativity between $\left(a^\dagger,a\right)$ and $\left(\tilde{a}^\dagger,\tilde{a}\right)$, equation (40) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle=\kappa\left(2a\tilde{a}-a^\dagger a-\tilde{a}^\dagger\tilde{a}\right)|\rho\rangle \tag{42}$$

and its standard solution is

$$|\rho(t)\rangle=\exp\{\kappa t(2a\tilde{a}-a^\dagger a-\tilde{a}^\dagger\tilde{a})\}|\rho_0\rangle$$
$$=\exp\{\kappa t[-(a^\dagger-\tilde{a})(a-\tilde{a}^\dagger)+a\tilde{a}-\tilde{a}^\dagger a^\dagger+1]\}|\rho_0\rangle, \tag{43}$$

where $|\rho_0\rangle\equiv\rho|\eta=0\rangle$.

Using

$$\left[a\tilde{a}-\tilde{a}^\dagger a^\dagger,-\left(a^\dagger-\tilde{a}\right)\left(a-\tilde{a}^\dagger\right)\right]$$
$$=-2\left[-\left(a^\dagger-\tilde{a}\right)\left(a-\tilde{a}^\dagger\right)\right] \tag{44}$$

and the operator identity

$$\mathrm{e}^{\lambda(A+\sigma B)}=\mathrm{e}^{\lambda A}\exp\left[\sigma\left(1-\mathrm{e}^{-\lambda\tau}\right)B/\tau\right] \tag{45}$$

for $[A,B]=\tau B$, the right-hand side of Eq. (43) can be decomposed as

$$|\rho(t)\rangle=\mathrm{e}^{\kappa t(a\tilde{a}-\tilde{a}^\dagger a^\dagger+1)}$$
$$\times\exp[(1-\mathrm{e}^{-2\kappa t})(a^\dagger-\tilde{a})(a-\tilde{a}^\dagger)/2]|\rho_0\rangle, \tag{46}$$

where $\mathrm{e}^{\kappa t\left(a\tilde{a}-\tilde{a}^\dagger a^\dagger+1\right)}$ represents a two-mode squeezing operator, and κt is considered to be a squeezing parameter. Therefore,

$$\langle\eta|\exp\left[\kappa t\left(a\tilde{a}-\tilde{a}^\dagger a^\dagger\right)\right]=\mathrm{e}^{-\kappa t}\left\langle\eta\,\mathrm{e}^{-\kappa t}\right|. \tag{47}$$

Then, using Eqs. (21) and (45), we have

$$\langle\eta|\,\rho(t)\rangle=\exp\left(-\frac{1}{2}T|\eta|^2\right)\left\langle\eta\,\mathrm{e}^{-\kappa t}\right|\rho_0\rangle, \tag{48}$$

where $T=1-\mathrm{e}^{-2\kappa t}$. Substituting Eq. (48) into Eq. (31) yields the following time-dependent Wigner function formula

$$W(\alpha,t)=\int\frac{\mathrm{d}^2\eta}{2\pi^2}\exp(\alpha^*\eta-\alpha\eta^*)$$
$$\times\exp\left(-\frac{1}{2}T|\eta|^2\right)\left\langle\eta\,\mathrm{e}^{-\kappa t}\right|\rho_0\rangle. \tag{49}$$

Further, using

$$\int\frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi|=1, \tag{50}$$
$$\int\frac{\mathrm{d}^2z}{2\pi^2}\exp\left(\xi|z|^2+\xi z+\eta z^*\right)=-\frac{1}{\xi}\exp\left(-\frac{\xi\eta}{\xi}\right),$$
$$\mathrm{Re}(\xi)<0 \tag{51}$$

as well as Eq. (30) we have

$$W(\alpha,t)=\int\frac{\mathrm{d}^2\xi}{\pi}\int\frac{\mathrm{d}^2\eta}{2\pi^2}\exp\left(\alpha^*\eta-\alpha\eta^*-\frac{1}{2}T|\eta|^2\right)$$
$$\times\left\langle\eta\,\mathrm{e}^{-\kappa t}\right|\xi=2\beta\rangle\langle\xi=2\beta|\,\rho_0\rangle$$
$$=\int\frac{\mathrm{d}^2\beta}{\pi}\int\frac{\mathrm{d}^2\eta}{\pi}\exp\left[-\frac{1}{2}T|\eta|^2+\eta\left(\alpha^*-\beta^*\mathrm{e}^{-\kappa t}\right)\right.$$
$$\left.+\eta^*\left(\beta\,\mathrm{e}^{-\kappa t}-\alpha\right)\right]W(\beta,0)$$
$$=\frac{2}{T}\int\frac{\mathrm{d}^2\beta}{\pi}\exp\left(-\frac{2}{T}\left|\alpha-\beta\,\mathrm{e}^{-\kappa t}\right|^2\right)W(\beta,0), \tag{52}$$

where $W(\beta,0)$ represents the Wigner function of the initial state (or operator).

6. Evolution of cumulant operator's Wigner function in dissipation channel

According to Eq. (37), the Wigner function of the initial cumulant operator is

$$W(\beta,0)=\frac{2}{\pi\left(1+\mathrm{e}^{\lambda}\right)}\exp\left(-\frac{2\left(1-\mathrm{e}^{\lambda}\right)}{1+\mathrm{e}^{\lambda}}|\beta|^2\right). \tag{53}$$

Substituting it into Eq. (52) yields

$$W(\alpha,t)=\frac{4}{\pi T\left(1+\mathrm{e}^{\lambda}\right)}\int\frac{\mathrm{d}^2\beta}{\pi}\exp\left(-\frac{2}{T}\left|\alpha-\beta\,\mathrm{e}^{-\kappa t}\right|^2\right)$$
$$\times\exp\left(-\frac{2\left(1-\mathrm{e}^{\lambda}\right)}{1+\mathrm{e}^{\lambda}}|\beta|^2\right)$$
$$=\frac{4\exp\left(-\frac{2}{T}|\alpha|^2\right)}{\pi T\left(1+\mathrm{e}^{\lambda}\right)}\int\frac{\mathrm{d}^2\beta}{\pi}$$
$$\times\exp\left[\frac{2}{T}\mathrm{e}^{-\kappa t}\left(\alpha\beta^*+\alpha^*\beta\right)\right.$$
$$\left.-2|\beta|^2\left(\frac{1-\mathrm{e}^{\lambda}}{1+\mathrm{e}^{\lambda}}+\frac{\mathrm{e}^{-2\kappa t}}{T}\right)\right]$$
$$=\frac{4}{\pi T\left(1+\mathrm{e}^{\lambda}\right)}\frac{1}{2\left(\frac{1-\mathrm{e}^{\lambda}}{1+\mathrm{e}^{\lambda}}+\frac{\mathrm{e}^{-2\kappa t}}{T}\right)}$$

$$\times \exp\left\{\left[\frac{\frac{4}{T^2}\mathrm{e}^{-2\kappa t}}{2\left(\frac{1-\mathrm{e}^{\lambda}}{1+\mathrm{e}^{\lambda}}+\frac{\mathrm{e}^{-2\kappa t}}{T}\right)}-\frac{2}{T}\right]|\alpha|^2\right\}. \tag{54}$$

From $T=1-\mathrm{e}^{-2\kappa t}$, we have

$$\begin{aligned} & T\left(1+\mathrm{e}^{\lambda}\right)\left(\frac{1-\mathrm{e}^{\lambda}}{1+\mathrm{e}^{\lambda}}+\frac{\mathrm{e}^{-2\kappa t}}{T}\right) \\ =\ & T\left(1+\mathrm{e}^{\lambda}\right)\left(\frac{1}{T}-\frac{2\mathrm{e}^{\lambda}}{1+\mathrm{e}^{\lambda}}\right) \\ =\ & 1+\mathrm{e}^{\lambda}-2T\mathrm{e}^{\lambda}, \end{aligned} \tag{55}$$

$$\begin{aligned} & \frac{1-T}{T^2\left(\frac{1-\mathrm{e}^{\lambda}}{1+\mathrm{e}^{\lambda}}+\frac{\mathrm{e}^{-2\kappa t}}{T}\right)}-\frac{1}{T}=\frac{1-T}{T\left(1-T\frac{2\mathrm{e}^{\lambda}}{1+\mathrm{e}^{\lambda}}\right)}-\frac{1}{T} \\ =\ & \frac{\mathrm{e}^{\lambda}-1}{1+\mathrm{e}^{\lambda}-2T\mathrm{e}^{\lambda}}. \end{aligned} \tag{56}$$

Therefore,

$$\begin{aligned} W(\alpha,t) =\ & \frac{2}{\pi\left(1+\mathrm{e}^{\lambda}-2T\mathrm{e}^{\lambda}\right)} \\ & \times \exp\left\{\left[\frac{2\left(\mathrm{e}^{\lambda}-1\right)}{1+\mathrm{e}^{\lambda}-2T\mathrm{e}^{\lambda}}\right]|\alpha|^2\right\}, \end{aligned} \tag{57}$$

and when $t=0$, $T=0$, equation (57) reduces to Eq. (53) as expected. Using Weyl correspondence, the final cumulant operator can be further calculated as

$$\begin{aligned} \rho(t) =\ & 2\pi\int \mathrm{d}^2\alpha W(\alpha,t)\Delta(\alpha) \\ =\ & \frac{4}{1+\mathrm{e}^{\lambda}-2T\mathrm{e}^{\lambda}}\int \mathrm{d}^2\alpha \exp\left\{\left[\frac{2\left(\mathrm{e}^{\lambda}-1\right)}{1+\mathrm{e}^{\lambda}-2T\mathrm{e}^{\lambda}}\right]|\alpha|^2\right\}\frac{1}{\pi}:\mathrm{e}^{-2(\alpha-a)\left(\alpha^*-a^{\dagger}\right)}: \\ =\ & \frac{4}{1+\mathrm{e}^{\lambda}-2T\mathrm{e}^{\lambda}}\int \frac{\mathrm{d}^2\alpha}{\pi}:\exp\left[-4\frac{1-T\mathrm{e}^{\lambda}}{1+\mathrm{e}^{\lambda}-2T\mathrm{e}^{\lambda}}|\alpha|^2+2\alpha^* a+2\alpha a^{\dagger}-2a^{\dagger}a\right]: \\ =\ & \frac{1}{1-T\mathrm{e}^{\lambda}}:\exp\left[\left(\frac{1+\mathrm{e}^{\lambda}-2T\mathrm{e}^{\lambda}}{1-T\mathrm{e}^{\lambda}}-2\right)a^{\dagger}a\right]: \\ =\ & \frac{1}{1-T\mathrm{e}^{\lambda}}:\exp\left[\left(\frac{\mathrm{e}^{\lambda}-1}{1-T\mathrm{e}^{\lambda}}a^{\dagger}a\right)\right]:=\frac{1}{1-T\mathrm{e}^{\lambda}}\exp\left[a^{\dagger}a\ln\frac{\mathrm{e}^{\lambda}(1-T)}{1-T\mathrm{e}^{\lambda}}\right], \end{aligned} \tag{58}$$

when $t=0$, $\mathrm{e}^{-2\kappa t}=1$, $T=0$, and $\rho(t)\to\exp\left(\lambda a^{\dagger}a\right)$.

From Eq. (58), the photon number distribution $\rho_{\mathrm{c}}(t)$ of dissipated chaotic light can be calculated as

$$\rho_{\mathrm{c}}(t)=\frac{1-\mathrm{e}^{\lambda}}{1-T\mathrm{e}^{\lambda}}\exp\left[a^{\dagger}a\ln\frac{\mathrm{e}^{\lambda}(1-T)}{1-T\mathrm{e}^{\lambda}}\right], \tag{59}$$

and then its initial density operator is $\left(1-\mathrm{e}^{\lambda}\right)\mathrm{e}^{\lambda a^{\dagger}a}=\rho_{\mathrm{c}}$, where $\left(1-\mathrm{e}^{\lambda}\right)$ is used to make up normalization. Let

$$\begin{aligned} & \lambda'=\ln\frac{\mathrm{e}^{\lambda}(1-T)}{1-T\mathrm{e}^{\lambda}}, \\ & 1-\mathrm{e}^{\lambda'}=1-\frac{\mathrm{e}^{\lambda}(1-T)}{1-T\mathrm{e}^{\lambda}}=\frac{1-\mathrm{e}^{\lambda}}{1-T\mathrm{e}^{\lambda}}, \end{aligned} \tag{60}$$

then we will have

$$\rho_{\mathrm{c}}(t)=\left(1-\mathrm{e}^{\lambda'}\right)\exp\left(\lambda' a^{\dagger}a\right). \tag{61}$$

In view of Eq. (5), the photon number of dissipated chaotic light can be derived as

$$\begin{aligned} \mathrm{Tr}\left[a^{\dagger}a\rho_{\mathrm{c}}(t)\right] =\ & \left(\mathrm{e}^{-\lambda'}-1\right)^{-1}=\left[\frac{1-T\mathrm{e}^{\lambda}}{\mathrm{e}^{\lambda}(1-T)}-1\right]^{-1} \\ =\ & \frac{\mathrm{e}^{\lambda}(1-T)}{1-\mathrm{e}^{\lambda}}=\frac{\mathrm{e}^{\lambda}\mathrm{e}^{-2\kappa t}}{1-\mathrm{e}^{\lambda}}, \end{aligned} \tag{62}$$

which is just

$$\begin{aligned} \mathrm{Tr}\left[a^{\dagger}a\rho_{\mathrm{c}}(t)\right] =\ & \left(\mathrm{e}^{-\lambda}-1\right)^{-1}\mathrm{e}^{-2\kappa t} \\ =\ & \mathrm{Tr}\left[a^{\dagger}a\rho_{\mathrm{c}}(0)\right]\mathrm{e}^{-2\kappa t}, \end{aligned} \tag{63}$$

and it can be seen that the photon number decreases with $\mathrm{e}^{-2\kappa t}$.

7. Conclusions

In this work, it is convenient to calculate the Wigner functions of the optical cumulant operator and its dissipation evolution in a thermal environment by using the IWOP method. Then, the thermos entangled state representation is introduced to derive the general evolution formula of the Wigner function, and its relation to Weyl correspondence is also discussed.

Acknowledgements

Project supported by the Foundation for Young Talents in College of Anhui Province, China (Grant Nos. gxyq2021210

and gxyq2019077) and the Natural Science Foundation of the Anhui Higher Education Institutions, China (Grant Nos. 2022AH051580 and 2022AH051586).

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