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Non-Gaussianity detection of single-mode rotationally symmetric quantum states via cumulant method

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The non-Gaussianity of quantum states incarnates an important resource for improving the performance of continuous-variable quantum information protocols. We propose a novel criterion of non-Gaussianity for single-mode rotationally symmetric quantum states via the squared Frobenius norm of higher-order cumulant matrix for the quadrature distribution function. As an application, we study the non-Gaussianities of three classes of single-mode symmetric non-Gaussian states: a mixture of vacuum and Fock states, single-photon added thermal states, and even/odd Schrödinger cat states. It is shown that such a criterion is faithful and effective for revealing non-Gaussianity. We further extend this criterion to two cases of symmetric multi-mode non-Gaussian states and non-symmetric single-mode non-Gaussian states.

Keywords: non-Gaussianity criterion, cumulant matrix, quadrature distribution

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1. Introduction

Continuous-variable (CV) non-Gaussian states have been identified as an important resource since they can be exploited to improve the performance of CV quantum information processing, such as quantum teleportation,[1–4] quantum key distribution,[5] and quantum cloning.[6] Such an improvement is largely attributed to the existence of non-Gaussianity contained in quantum state, which is usually referred to as the departure of a given state from its Gaussian counterpart. Therefore, in the past two decades, several different measures of non-Gaussianity for quantum states have been proposed based on geometric distance,[7–10] entropic distance,[8,11] the negentropy of quadrature distributions,[12] and Wigner–Yanase skew information.[13] A survey on these non-Gaussianity measures was reported in Ref. [12].

In addition to its quantification aspect, another problem of concern in quantum information theory is to identify whether a given quantum state is Gaussian or not. As is well known, in classical statistics, the cumulants are useful quantities for characterizing some important statistical properties, such as independence, Gaussianity, and symmetry of probability distribution. For example, the third-order cumulant can be utilized to measure the degree of distortion from the Gaussian distribution, while the fourth-order cumulant is a measure of whether the probability distribution is heavy-tailed or light-tailed relative to a Gaussian distribution. Therefore, the high-order cumulants have been recently introduced to the field of quantum information as an indictor of the non-Gaussian statistics of quantum states, such as Kerr-squeezed states[14] and non-Gaussian pure states generated by the anharmonic oscillator[15] in phase space. In particular, we have exploited the averaged-cumulant technique to reveal the non-Gaussianity of some typical quantum states.[16–20] Unfortunately, we encountered a failure of detecting the non-Gaussian natures of some quantum states using only a single higher-order cumulant. A typical case is a mixture of vacuum and n photon states of the form $\rho = (1-g)|0\rangle\langle 0| + g|n\rangle\langle n|$ with $|n\rangle$ being a Fock state for n photons and the fractions $g \in [0,1]$. Its fourth-order cumulant is calculated to be $\kappa_4 = -\frac{3}{2}ng(2ng+1-n)$.[20] One sees that the κ_4 may equal zero for the case of $g = \frac{n-1}{2n}$. However, in this case, its Wigner function clearly shows negative values around the origin, i.e., non-Gaussian behaviors. It is thus natural to ask how to detect the non-Gaussianity of quantum states by means of the cumulant method.

In this paper, we continue to focus on and uncover the non-Gaussianity of quantum states based on the higher-order cumulant method. However, unlike our pervious works we focus on the high-order cumulant matrix of the Wigner function for quantum state ρ, and then use its squared Frobenius norm as non-Gaussianity indicator. It is well known that the squared Frobenius norm of a Hermitian matrix is defined as the sum of the squared absolute values of its elements[21] and has been applied to measure the correlation among electronic system[22,23] and quantum coherence and asymmetry.[24] Therefore, our measure is novel and to our knowledge, relatively little research has been carried out to detect the non-Gaussianity of quantum states in such a manner.

The paper is structured as follows: In Section 2, we begin by introducing an alternative to the description of a probability distribution of a quantum state by its cumulants. For single-mode rotationally symmetric quantum states, we introduce the squared Frobenius norm of high-order cumulant matrix for a

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single quadrature variable to characterize its non-Gaussianity. In Section 3, our proposed criterion is applied to check the non-Gaussian natures of some symmetric single-mode quantum states. In Section 4, we extend our method to estimate the non-Gaussianity of symmetric multi-mode quantum states and more general single-mode non-symmetric states. Finally, Section 5 contains conclusion.

2. Non-Gaussian criterion based on cumulant theory

The homodyne rotated quadrature operator for a single-mode electromagnetic field is given by

$$\hat{X}(\theta)=\frac{1}{\sqrt{2}}(\hat{a}\,\mathrm{e}^{-\mathrm{i}\theta}+\hat{a}^{+}\,\mathrm{e}^{\mathrm{i}\theta}), \tag{1}$$

where θ represents the phase of the local oscillator in homodyne detection arrangement $\theta\in[0,2\pi]$. If $|x_\theta,\theta\rangle$ is the eigenstate of the Hermitian operator $\hat{X}(\theta)$ with eigenvalue x_θ, then the optical tomogram of a density matrix ρ being measured is defined as[25]

$$P(x_\theta,\theta)=\langle x_\theta,\theta|\,\hat{\rho}\,|x_\theta,\theta\rangle, \tag{2}$$

satisfying the normalization condition $\int_{-\infty}^{+\infty}\mathrm{d}x_\theta P(x_\theta,\theta)=1$. The optical tomogram is having the symmetry property: $P(x_\theta,\theta+\pi)=P(-x_\theta,\theta)$.

The optical tomogram is related to the Wigner function as follows:

$$P(x,\theta)=\int_{-\infty}^{+\infty}W(x\cos\theta-p\sin\theta,x\sin\theta+p\cos\theta)\,\mathrm{d}p. \tag{3}$$

Thus the Wigner function of the state ρ can be reconstructed by tomographic inversion of a set of measured probability distribution $P(x,\theta)$ of the quadrature amplitude. Moreover, one can obtain the statistical properties and non-Gaussian features of quantum states by means of homodyne detection technique.

Fully characterizing the non-Gaussianity of a quantum state is more challenging from the viewpoint of the geometric configuration of its quasi-probability distribution function, but special symmetries of a state can simplify the problem. In this study we consider a slightly more general class of states: single-mode rotationally symmetric states, whose advantage is that any single-quadrature probability function shares the same statistical properties as the Wigner function of quantum state. Namely, the quadrature distribution is the same for all phase angles θ.[12] Therefore, we would like to reveal the non-Gaussian signatures of single-mode rotationally symmetric quantum states by using the distribution of single quadrature operator (e.g., $\hat{X}(0)=\hat{x}$). Its eigenequation is given by $\hat{x}|x\rangle=x|x\rangle$.

From the mathematical statistical point of view, the logarithm of the quadrature distribution function can be regarded as the cumulant generating function of quadrature variable x, namely,

$$\phi(x)\equiv\ln[P(x)], \tag{4}$$

which implies that the mth-order cumulants $\kappa_m(x)$ for measurement of amplitude quadrature operator $\hat{x}$ can be expressed as

$$\kappa_m(x)=\frac{\partial^m}{\mathrm{i}^m\partial x^m}\phi(x)\bigg|_{x=0}. \tag{5}$$

Furthermore, by virtue of the cumulant theorem, one can recast the distribution function of amplitude quadrature operator $\hat{x}$ in another form

$$P(x)=\mathrm{e}^{\phi(x)}=\exp\Big(\sum_{m=1}^{\infty}\frac{\mathrm{i}^m\kappa_m(x)x^m}{m!}\Big). \tag{6}$$

It is well known that a Gaussian state is entirely specified by the first- and second-order cumulants, while a non-Gaussian state by both the lower-order cumulants and the high-order cumulants. Thus, our important task is how to formulate the higher-cumulant-based tool for revealing non-Gaussian correlation of single-mode rotationally symmetric states.

For any non-Gaussian quantum state ρ having $\kappa_j(\rho)<\infty$, it is impossible for all its cumulants of high order to be simultaneously disappeared and then the combined high-order cumulants can be used to characterize the intrinsic non-Gaussian nature. On the other hand, for a symmetric quantum state, we have $P(-x)=P(x)$ in such a way that all odd cumulants are zero. Thus, Eq. (6) only contains the cumulants of even order and can be expressed as

$$P(x)=\exp[-\boldsymbol{\Lambda}\boldsymbol{M}_{2m}(x)\boldsymbol{\Lambda}^{\mathrm{T}}], \tag{7}$$

where $\boldsymbol{\Lambda}=(x,x^2,\ldots,x^m)$ and the cumulant matrix $\boldsymbol{M}_{2m}(x)$ is given by

$$\boldsymbol{M}_{2m}(x)=\begin{pmatrix}\frac{1}{2!}\kappa_2(x) & 0 & \cdots & 0\\ 0 & -\frac{1}{4!}\kappa_4(x) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{(-1)^{m-1}}{(2m)!}\kappa_{2m}(x)\end{pmatrix}. \tag{8}$$

Introducing the Frobenius norm of higher-order cumulant matrix $\boldsymbol{M}_{m>2}(x)$, we can rewrite Eq. (8) as

$$\|\boldsymbol{M}_{m>2}(x)\|_{\mathrm{F}}=\sum_{l=2}^{\infty}\|\boldsymbol{M}_{2l}(x)\|_{\mathrm{F}}. \tag{9}$$

For an $n\times n$ matrix $\boldsymbol{A}$, its Frobenius norm is defined as[21]

$$\|\boldsymbol{A}\|_{\mathrm{F}}=\sqrt{\mathrm{Tr}(\boldsymbol{A}^{+}\boldsymbol{A})}=\sqrt{\textstyle\sum_{i,j=1}^{n}|a_{ij}|^2}, \tag{10}$$

where a_{ij} is the elements of matrix $\boldsymbol{A}$.

Moreover, we can calculate the squared Frobenius norm of $\boldsymbol{M}_{m>2}(x)$ to be

$$\kappa_{\mathrm{F}}(x)=\sum_{m=2}^{\infty}\Big|\frac{\kappa_{2m}(x)}{(2m)!}\Big|^2. \tag{11}$$

Thus, we propose a criterion for single-mode rotationally symmetric states being non-Gaussian:

$$\kappa_{\rm F}(x) > 0. \tag{12}$$

Sufficient criterion If the condition (12) holds, then we have at least one of the higher-order cumulants of quadrature variable x being not equal to zero, implying that the function $P(x)$ has a non-Gaussian form and then the corresponding Wigner function is also non-Gaussian for single-mode rotationally symmetric states. It shows that a state concerned is a non-Gaussian state.

Necessary criterion If the condition (12) does not hold for the function $P(x)$, i.e., $\kappa_{\rm F}(x) = 0$, then all the higher-order cumulants of quadrature variable x will vanish simultaneously, but its first- and second-order cumulants are not equal to zero. This implies that the function $P(x)$ has a Gaussian form. In statement as mentioned above, there is a one-to-one correspondence between single-quadrature probability function and the Wigner function in capturing the geometric shape of single-mode rotationally symmetric states[12] and then we can say only that a single-mode quantum state under examination is not a rotationally symmetric non-Gaussian one. Therefore, we obtain a necessary and sufficient criterion for revealing non-Gaussianity of single-mode rotationally symmetric states, thus overcoming the drawback of using single higher-order cumulant to judge whether a quantum state is Gaussian or not. However, we should also point out here that the criterion (12) is a sufficient criterion for non-Gaussianity of general single-mode non-Gaussian states. This is due to the fact that for a general non-Gaussian state, if its single-quadrature probability densities are Gaussian, then it may be Gaussian or non-Gaussian.

On the other hand, calculating all the higher-order cumulants in Eq. (11) is very troublesome and unrealistic work. Hence, from an application, we extract the lower-order 3×3 sub-matrix from high-order cumulant matrix $\boldsymbol{M}_{m>2}(x)$ and use its squared Frobenius norm to characterize the non-Gaussianity of quantum state concerned. This strategy is plausible since the probability of the first three cumulants of high order being zero at the same time is very low. If the condition is not meet, then one needs to calculate other higher-order cumulants until the emergence of their non-zero value. For convenience of later reference, we write the relationships among first three cumulants of higher order and their corresponding moments as follows:

$$\kappa_4(x) = \mu_4(x) - 3[\mu_2(x)]^3, \tag{13a}$$

$$\kappa_6(x) = \mu_6(x) - 15\mu_4(x)\mu_2(x) + 30[\mu_2(x)]^3, \tag{13b}$$

$$\kappa_8(x) = \mu_8(x) - 28\mu_6(x)\mu_2(x) - 35[\mu_4(x)]^2 + 420\mu_4(x)[\mu_2(x)]^2 - 630[\mu_2(x)]^4, \tag{13c}$$

where the jth moment is given by $\mu_j(x) = \int x^j P(x)\,{\rm d}x$, and then we propose a truncated criterion

$$\kappa_{\rm F}(x) = \sum_{m=2}^{4}\left|\frac{\kappa_{2m}(x)}{(2m)!}\right|^2 > 0 \tag{14}$$

to quickly detect whether a quantum state is Gaussian or not. Of course, this truncated non-Gaussian criterion is only a sufficient one.

3. Examples

In the preceding section we have presented our central result of non-Gaussian criterion based on the squared Frobenius norm of the high-cumulant matrix for a symmetric single-mode non-Gaussian states. In what follows, we use this criterion to probe non-Gaussianity of some interesting and typical non-Gaussian states, including mixture of vacuum and n photon states, single-photon added thermal state (SPATS), and even/odd cat states as example.

3.1. Mixture of vacuum and n photon states

Going back to a case of mixture of vacuum and n photon states given by in the introduction section, we can obtain its Wigner function as

$$W(x,p) = \frac{2}{\pi}[1-g+g(-1)^n L_n(4(x^2+p^2))]\,{\rm e}^{-2(x^2+p^2)}, \tag{15}$$

and according to Eq. (3), we can obtain the distribution function for quadrature variable x as

$$P(x) = \sqrt{\frac{2}{\pi}}\Big\{1-g+g(-1)^n\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j} \times \frac{(-1)^k 4^{k-j}(2j-1)!!}{k!}x^{2k-2j}\Big\}\,{\rm e}^{-2x^2}, \tag{16}$$

where $L_n(x)$ is the Laguerre polynomial of order n, $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$, and $(2n-1)!! = \frac{(2n)!}{n!\,2^n}$.

Using the identity

$$\int_{-\infty}^{+\infty} x^{2n}\,{\rm e}^{-bx^2}\,{\rm d}x = \frac{(2n-1)!!}{(2b)^n}\sqrt{\frac{\pi}{b}},\quad b>0,\quad n\in\mathbb{N}, \tag{17}$$

we obtain the even-order moments as

$$\mu_{2m}(x) = \frac{(1-g)(2m-1)!!}{4^m} + g\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j} \times \frac{(-1)^{n+k}(2j-1)!!(2k+2m-2j-1)!!}{k!4^m}, \tag{18}$$

and all the odd-order cumulants are zero.

Figure 1 shows the plots of the first three cumulants of higher order of the quadrature variable x for mixture of vacuum and n photon states with several values of photon numbers n. One sees clearly from Fig. 1 that these high-order cumulants are equal to zero for the case of $g = 0$, as expected. However, we observe that these cumulants could be positive, zero to negative values with increasing n and g, which

shows again that a single higher-order cumulant is insufficient to reveal the non-Gaussian natures of quantum states. In other words, the cumulants of higher order are a sufficient but not necessary condition for non-Gaussianity of quantum state. Fortunately, we see from Fig. 1 that all these higher-order cumulants are not equal to zero simultaneously for a given n and g. Therefore, we can use the squared Frobenius norm in Eq. (14) to capture the non-Gaussian character of quantum states, as shown in Fig. 2. We see clearly that non-Gaussianities of mixture of vacuum and n photon states are always greater than zero as soon as $ng \neq 0$, and increase monotonically with increasing g. Namely, we successfully overcome the difficulty appearing in the introduction section. In addition, we see that for the case of $g = 1$, i.e., pure Fock state, it has the most non-Gaussianity, which is in good agrement with the result obtained in Refs. [26,27], where non-Gaussianity measures of quantum states were based on quantum fidelity and the normalized Hilbert–Schmidt distance, respectively. Therefore, our proposed criterion is powerful enough to characterize the non-Gaussian natures of non-Gaussian quantum states.

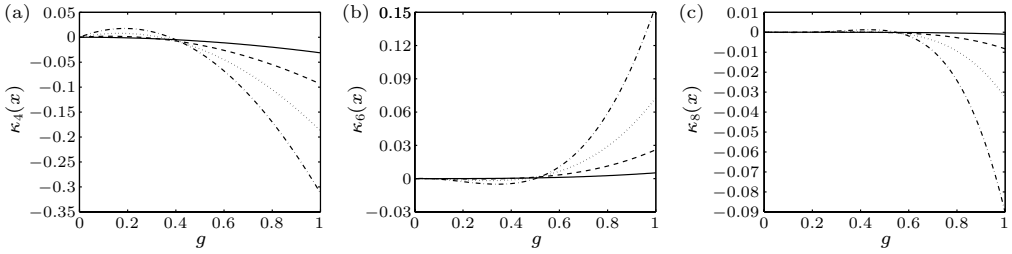


Fig. 1. First three higher-order cumulants of quadrature variable x versus parameter g when the field is in a mixture of vacuum and n photon states. In (a)–(c), solid curve: $n = 1$, dashed curve: $n = 2$, dotted curve: $n = 3$, dot-dashed curve: $n = 4$.

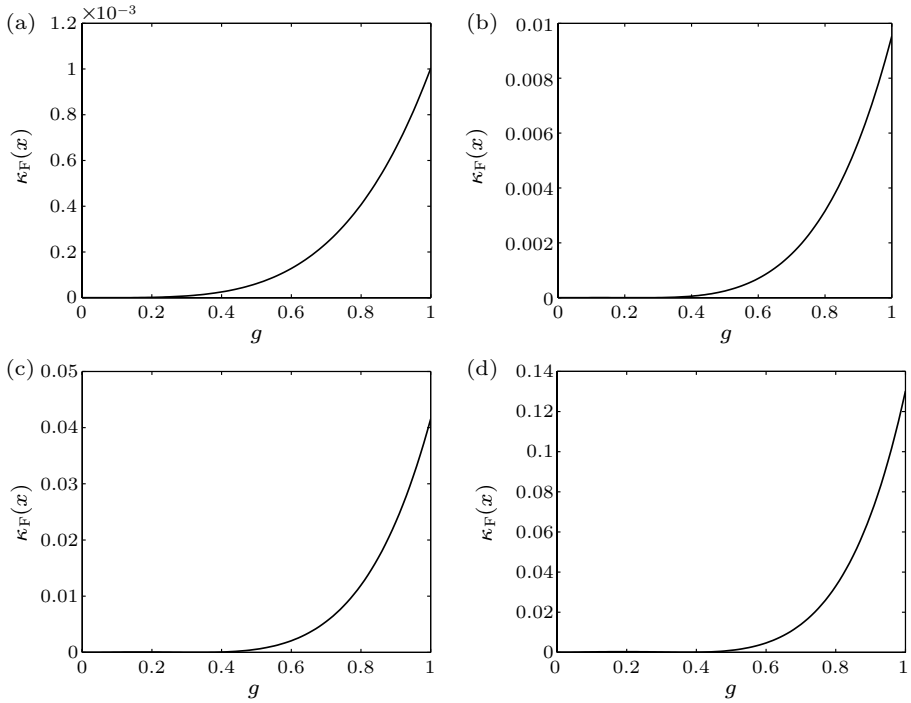


Fig. 2. Non-Gaussian criterion $\kappa_F(x)$ for mixture of vacuum and n photon states versus parameter g: (a) $n = 1$, (b) $n = 2$, (c) $n = 3$, (d) $n = 4$.

3.2. Single-photon-added thermal state

As a next example of non-Gaussian states, we consider a single-photon-added thermal state (SPATS), which was obtained by exciting a classical state thermal filed by a single photon creation,[28]

$$\rho_T^{+1} = N\hat{a}^+\rho_{th}\hat{a}, \tag{19}$$

where $N = \mathrm{Tr}[\hat{a}^+\rho_{th}\hat{a}]$ is a normalization constant and ρ_{th} denotes a single-mode thermal state.

In the Fock-state representation, its density operator can be written as[29]

$$\rho_T^{+1} = \frac{1}{\bar{n}(\bar{n}+1)}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n n|n\rangle\langle n|, \tag{20}$$

where $\bar{n}$ denotes the average photon number of the zero photon-added thermal state, $\rho_T^{+0} \equiv \sum_n[\frac{\bar{n}^n}{(1+\bar{n})^{n+1}}]$.

The Wigner function of single-photon-added thermal

state is given by

$$W(x,p)=\frac{2}{\pi}\Big[\frac{-1}{(1+2\bar{n})^2}+\frac{4(1+\bar{n})}{(1+2\bar{n})^3}(x^2+p^2)\Big]\times\exp\Big[-\frac{2}{1+2\bar{n}}(x^2+p^2)\Big]. \tag{21}$$

Obviously, the Wigner function of single-photon-added thermal state is rotationally symmetric and is reduced to that of single-photon Fock state for the case of $\bar{n}=0$. The Wigner negativity will appear if the condition $(x^2+p^2)<\frac{1+2\bar{n}}{4(1+\bar{n})}$ in phase space is satisfied and its maximal value is given by $-\frac{2}{(1+2\bar{n})^2\pi}$ at the origin. This feature ensures their non-classicality.

By integrating the Wigner function (21) over p, we can obtain the marginal function of of quadrature variable x as

$$W(x)=\sqrt{\frac{2(1+2\bar{n})}{\pi}}\Big[A+\frac{B}{2b}+Bx^2\Big]\,\mathrm{e}^{-bx^2}, \tag{22}$$

where $A=-\sqrt{\frac{2}{\pi}}\frac{1}{1+2\bar{n}}$, $B=\sqrt{\frac{2}{\pi}}\frac{4(1+\bar{n})}{(1+2\bar{n})^3}$, and $b=\frac{2}{1+2\bar{n}}$. Furthermore, the $2m$th-order moments are calculated to be

$$\mu_{2m}(x)=\frac{1+2\bar{n}}{(2b)^m}\Big[(2m-1)!!\Big(A+\frac{B}{2b}\Big)+\frac{(2m+1)!!B}{2b}\Big]. \tag{23}$$

Thus, we can investigate the first three higher-order cumulants of quadrature variable x and the non-Gaussianity of single-photon-added thermal state by means of Eqs. (13), (14), and (23).

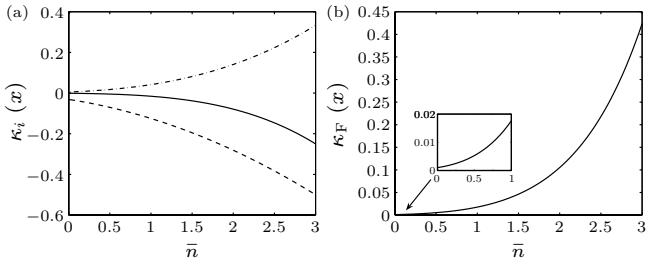


Fig. 3. First three higher-order cumulants $\kappa_j(x)$ of quadrature variable x (a) and non-Gaussian criterion $\kappa_F(x)$ (b) of SPATS for different values of $\bar{n}$. In (a), dashed curve: $\kappa_4(x)$, dot-dashed curve: $\kappa_6(x)$, solid curve: $\kappa_8(x)$. The inset in (b) shows $\kappa_F(x)$ in the region of $\bar{n}=(0,1)$ in more detail.

Figure 3 shows the plots of the fourth-, sixth-, eighth-order of quadrature variable x, and non-Gaussianity of single-photon-added thermal state for several values of thermal average photon numbers $\bar{n}$. It is clearly seen from Fig. 3 that for single-photon-added thermal state with the extreme limit of $\bar{n}=0$, its first three higher-order cumulants are not equal to zero, in accordance to the intuition from Eq. (22). When the $\bar{n}$ are increased, these higher-order cumulants are distinctly different from zero, showing that the $P(x)$ are non-Gaussian distributed. At the same time, one sees that the $\kappa_F(x)$ are strictly monotone increasing function of $\bar{n}$, and for infinite value of $\bar{n}$, it diverges, i.e. $\lim_{\bar{n}\to\infty}\kappa_F(x)=\infty$. Our analysis demonstrates that these cumulants of order greater than two and our proposed non-Gaussian criterion can reveal the intrinsic non-Gaussian natures of single-photon-added thermal state.

3.3. Even and odd Schrödinger cat states

As the final example of non-Gaussian states, we focus on the even and odd Schrödinger cat states, which are represented in the Fock-state representation as[30,31]

$$|\Psi\rangle_+=\frac{1}{\cosh^{1/2}\alpha^2}\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{\sqrt{(2n)!}}|2n\rangle, \tag{24a}$$

$$|\Psi\rangle_-=\frac{1}{\sinh^{1/2}\alpha^2}\sum_{n=0}^{\infty}\frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}}|2n+1\rangle, \tag{24b}$$

where the subscripts $+$ and $-$ denote the even Schrödinger cat state (ESCS) and odd Schrödinger cat state (OSCS), respectively. It has been shown in Refs. [30,31] that the ESCS exhibits squeezing but not sub-Poissionian statistics whereas the OSCS has sub-Poissionian statistics but not squeezing. In particular, they can be exploited for implementing some tasks such as quantum teleportation,[32,33] quantum cryptography,[34] and entanglement preparation.[35,36]

The Wigner functions of these states in Eq. (24) are given by

$$W_{\rm scs}(x,p)=\frac{2}{N_\pm^2\pi}\,\mathrm{e}^{-2(x^2+p^2+\alpha^2)}[\cosh(4\alpha x)\pm\mathrm{e}^{2\alpha^2}\cos(4\alpha p)], \tag{25}$$

where the normalization factor $N_\pm=\sqrt{1\pm\mathrm{e}^{-2\alpha^2}}$ and we have assumed α to be real parameter for simplicity.

By repeating the line above, we can obtain the marginal function $P(x)$ of quadrature variable x as

$$P(x)=\sqrt{\frac{2}{\pi}}\frac{1}{N_\pm^2}\,\mathrm{e}^{-2(x^2+\alpha^2)}[\cosh(4\alpha x)\pm1]. \tag{26}$$

Thus, the corresponding $2m$th-order moments are calculated to be

$$\mu_{2m}(x)=\frac{\mathrm{e}^{-2\alpha^2}}{N_\pm^2}\Big[\frac{\partial^{2m}}{16^m\partial\alpha^{2m}}\mathrm{e}^{2\alpha^2}\pm\frac{(2m-1)!!}{4^m}\Big]. \tag{27}$$

In Fig. 4 the behaviors of the first three higher-order cumulants and non-Gaussianity of even and odd schrödinger cat states are illustrated. According to Eq. (24), when $\alpha\to0$, even schrödinger cat states become vacuum state while odd Schrödinger cat states recover to single-photon Fock state. In this case, the higher-order cumulants of the former are zero and those of the latter are not equal to zero, as clearly confirmed by the plots in Figs. 4(a) and 4(c), respectively. However, when α is increased, these higher-order cumulants are different zero and exhibit extremely strong non-Gaussian correlations. Accordingly, these states are always non-Gaussian. On the other hand, it is clearly seen from Figs. 4(b) and 4(d) that the squared Frobenius norm of marginal cumulant matrix

is also a strictly monotonically increasing function of α, indicating that our criterion can successfully capture the weak non-Gaussian correlations even when the amplitude is enough small. This is in accordance with the fact that the Wigner function for single-mode Schrödinger cat state consists of two Gaussian Bells corresponding to statistical mixture of individual composite states and interference fringes originating from the superposition among different components of the states. The three-peaked structure of Schrödinger cat state becomes more pronounced as soon as values of α is not equal to zero, giving rise to serious deviation from Gaussian shape. Our result is consistent with Ref. [37], in which the non-Gaussianity of even/odd Schrödinger cat states can be detected by the information-theoretic quantifier for non-classicality.

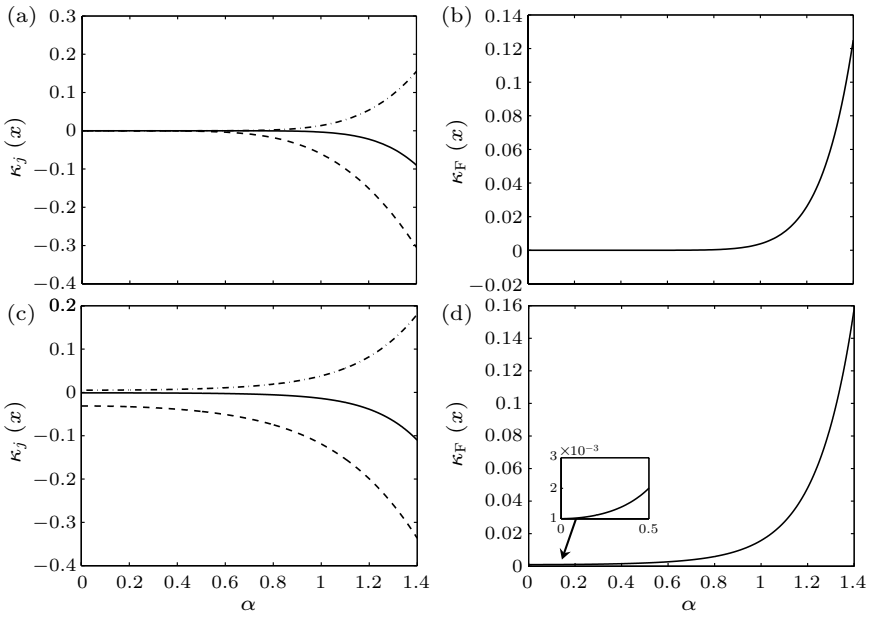


Fig. 4. First three higher-order cumulants $\kappa_j(x)$ of quadrature variable x and non-Gaussian criterion $\kappa_{\rm F}(x)$ of ESCSs (top planar) and OSCSs (bottom planar) as a function of the amplitude α. In (a) and (c), dashed curve: $\kappa_4(x)$, dot-dashed curve: $\kappa_6(x)$, solid curve: $\kappa_8(x)$. The inset in (d) shows $\kappa_{\rm F}(x)$ in the region of $\bar{n}=(0,0.5)$ in more detail.

4. Extension to more general cases

In the above section, we focus on a special class of single-mode non-Gaussian states. Without loss of generality, we apply our criterion to two cases. First, we consider a case of multi-mode symmetric non-Gaussian states. In general, the Wigner function of a multi-mode quantum state ρ is defined as

$$W(\{x_j,p_j\})=\frac{1}{(2\pi)^{2N}}\underbrace{\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}}_{N}\mathrm{d}^N\{x_j'\}\prod_{j=1}^{N}\mathrm{e}^{\mathrm{i}p_jx_j}\times\left\langle\left\{x_j-\frac{x_j'}{2}\right\}\middle|\hat{\rho}\middle|\left\{x_j+\frac{x_j'}{2}\right\}\right\rangle, \tag{28}$$

where $\{x_j,p_j\}$ is an abbreviation of the notation $x_1,p_1,x_2,p_2,\ldots,x_N,p_N$ and $\mathrm{d}^N\{x_j'\}=\mathrm{d}x_1'\,\mathrm{d}x_2'\cdots\mathrm{d}x_N'$.

Thus, we can obtain the marginal distribution of one quadrature variable (say x_j) by integration of the Wigner function $W(\{x_j,p_j\})$ over the remaining $2N-1$ variables $\{x_j\}'$ and $\{p_j\}$:

$$P(x_j)=\underbrace{\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}}_{2N-1}\mathrm{d}^{N-1}\{x_j\}'\mathrm{d}^N\{p_j\}W(\{x_j,p_j\}), \tag{29}$$

where $\{x_j\}'$ denotes the remainders excluding the variable x_j. Along the same route as above, we can analyze the non-Gaussianity of multi-mode symmetric non-Gaussian states using the squared Frobenius norm of high-order cumulant matrix of a single quadrature variable x_j given by the formula (11).

Second, we consider a case of a single-mode non-symmetric quantum state. For any single-mode state, its Wigner function provides complete information about two variables in phase space[38] and in principle, we require the bivariable cumulants to reveal the non-Gaussianity of quantum state. By virtue of the cumulants, we can express the Wigner function of any single-mode quantum state as

$$W(\boldsymbol{\xi})=\exp\Big\{\sum_{k=0}^{\infty}\frac{(-1)^k\mathrm{i}}{(2k+1)!}(\otimes^k\boldsymbol{\xi})^{\mathrm{T}}\boldsymbol{M}_{2k+1}(\boldsymbol{\xi})(\otimes^{k+1}\boldsymbol{\xi})+\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}(\otimes^k\boldsymbol{\xi})^{\mathrm{T}}\boldsymbol{M}_{2k}(\boldsymbol{\xi})(\otimes^k\boldsymbol{\xi})\Big\}, \tag{30}$$

where $\otimes$ is the Kronecker product, $\boldsymbol{M}_{2k+1}(\boldsymbol{\xi}):=\boldsymbol{\kappa}_{2k+1}(\otimes^k\boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}\otimes\boldsymbol{\xi}^{\mathrm{T}})$ is a $2^k\times2^{k+1}$ rectangular cumulant matrix of odd order $2k+1$, $\boldsymbol{M}_{2k}(\boldsymbol{\xi}):=\boldsymbol{\kappa}_{2k}(\otimes^k\boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}})$ is a $2^k\times2^k$ square cumulant matrix of even order $2k$, and $\otimes^l\boldsymbol{\xi}=\underbrace{\boldsymbol{\xi}\otimes\cdots\otimes\boldsymbol{\xi}}_{l}$ with $\boldsymbol{\xi}=(x,p)$. We see from Eq. (30) that a quantum state can also be equivalently described by the entire hierarchy of cumulant matrices. For a Gaussian state, it is entirely specified by the first- and second-order cumulant matrices, while

a non-Gaussian state by both lower-order cumulant matrices and higher-order cumulant matrices. However, the non-Gaussianity of quantum state is solely determined by the high-order cumulant matrices, which can be viewed as a direct sum of the cumulant matrices of odd order and even order greater than two, namely,

$$\boldsymbol{M}_{\rm h}(x,p)=\boldsymbol{M}_{\rm ho}(x,p)\oplus\boldsymbol{M}_{\rm he}(x,p), \tag{31}$$

where the cumulant matrix $\boldsymbol{M}_{\rm ho}(x,p)$ of odd order greater than two is a $2^{k-1}\times 2^k$ matrix and can be expressed as

$$\boldsymbol{M}_{\rm ho}(x,p)=\begin{pmatrix}\boldsymbol{0} & \boldsymbol{M}_3(x,p) & \boldsymbol{0} & \boldsymbol{0} & \dots\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_5(x,p) & \boldsymbol{0} & \dots\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_7(x,p) & \dots\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dots\\ \vdots & \vdots & \vdots & \vdots & \dots\end{pmatrix}, \tag{32}$$

and the cumulant matrix of even order greater than two is a $2^{k-1}\times 2^{k-1}$ matrix and can be expressed as $\boldsymbol{M}_{\rm he}(x,p)=\bigoplus_{k=2}^{\infty}\boldsymbol{M}_{2k}(x,p)$.

Thus, we can calculate the squared Frobenius norm of matrix (31) to be

$$\begin{aligned}&\|\boldsymbol{M}_{\rm h}(x,p)\|_{\rm F}^2\\ &={\rm Tr}\{[\boldsymbol{M}_{\rm ho}(x,p)\oplus\boldsymbol{M}_{\rm he}(x,p)]^{+}[\boldsymbol{M}_{\rm ho}(x,p)\oplus\boldsymbol{M}_{\rm he}(x,p)]\}\\ &={\rm Tr}\{[\boldsymbol{M}_{\rm ho}^{+}(x,p)\oplus\boldsymbol{M}_{\rm he}^{+}(x,p)][\boldsymbol{M}_{\rm ho}(x,p)\oplus\boldsymbol{M}_{\rm he}(x,p)]\}\\ &={\rm Tr}\{[\boldsymbol{M}_{\rm ho}^{+}(x,p)\boldsymbol{M}_{\rm ho}(x,p)]\oplus[\boldsymbol{M}_{\rm he}^{+}(x.p)\boldsymbol{M}_{\rm he}(x,p)]\}\\ &={\rm Tr}\{[\boldsymbol{M}_{\rm ho}^{+}(x,p)\boldsymbol{M}_{\rm ho}(x,p)]+{\rm Tr}[\boldsymbol{M}_{\rm he}^{+}(x.p)\boldsymbol{M}_{\rm he}(x,p)]\}\\ &=\|\boldsymbol{M}_{\rm ho}(x,p)\|_{\rm F}^2+\|\boldsymbol{M}_{\rm he}(x,p)\|_{\rm F}^2,\end{aligned} \tag{33}$$

where we have used ${\rm Tr}[(\boldsymbol{A}\oplus\boldsymbol{B})(\boldsymbol{C}\oplus\boldsymbol{D})]={\rm Tr}[(\boldsymbol{AC})\oplus(\boldsymbol{BD})]$ and ${\rm Tr}[\boldsymbol{A}\oplus\boldsymbol{B}]={\rm Tr}[\boldsymbol{A}]+{\rm Tr}[\boldsymbol{B}]$.

One immediately sees that the squared Frobenius norm of matrix for single-mode quantum state is equal to a sum of that of its odd- and even-order cumulant matrices. Therefore, we can obtain the cumulant-based criterion of non-Gaussianity for a single-mode non-symmetric quantum state,

$$\kappa_{\rm F}(x,p)=\sum_{m=3}^{\infty}\left|\frac{\kappa_m(x,p)}{m!}\right|^2, \tag{34}$$

where the mth-order joint cumulants $\kappa_m(x,p)$ is defined as

$$\kappa_m(x,p)=(-{\rm i})^m\frac{\partial^m\ln W(x,p)}{\partial x^{m_1}\partial p^{m_2}}\bigg|_{x=p=0}, \tag{35}$$

where $m=m_1+m_2$ and the Wigner function $W(x,p)$ is given by Eq. (30).

To illustrate the power of our non-Gaussian criterion (34), we will consider a typical non-symmetric quantum state: a Kerr state, which can be obtained using a photon coherent light pulse propagating through a lossy Kerr medium with the third-order nonlinearity $\chi^{(3)}$. The Kerr interaction Hamiltonian is given by[39]

$$\hat{H}=\chi^{(3)}\hat{a}^{+2}\hat{a}^2, \tag{36}$$

in which we have set $\hbar=1$.

The evolution of the system is given by

$$|\psi(\gamma)\rangle=\hat{U}(\gamma)|\psi(0)\rangle, \tag{37}$$

where $\hat{U}(\gamma)=\exp(-{\rm i}\gamma\hat{a}^{+2}\hat{a}^2)$ is a unitary evolution operator with $\gamma=\chi^{(3)}t$ being the unitless interaction time.

Using the Baker–Campbell–Hausdorff formula one can easily obtain the time-dependent annihilation operator $\hat{a}(\gamma)$ as

$$\hat{a}(\gamma)=\hat{U}^{+}(\gamma)\hat{a}\hat{U}(\gamma)={\rm e}^{-2{\rm i}\gamma\hat{a}^{+}\hat{a}}\hat{a}. \tag{38}$$

Furthermore, we obtain the action of $\hat{a}^m(\gamma)$ on a coherent state $|\alpha\rangle$ as

$$\hat{a}^m(\gamma)|\alpha\rangle=\alpha^m{\rm e}^{-{\rm i}m(m-1)\gamma}|\alpha\,{\rm e}^{-2{\rm i}m\gamma}\rangle. \tag{39}$$

Using Eq. (39), we can calculate the expectation value of $\hat{a}_\theta^{+m}(\gamma)\hat{a}_\theta^n(\gamma)$ as

$$\langle\hat{a}_\theta^{+m}(\gamma)\hat{a}_\theta^n(\gamma)\rangle=\frac{\alpha^{*m}\alpha^n\exp\{|\alpha|^2[{\rm e}^{-2{\rm i}(n-m)\gamma}-1]\}}{\exp\{{\rm i}(n-m)[\theta+(n+m-1)\gamma]\}}, \tag{40}$$

where $\hat{a}_\theta=\hat{a}\,{\rm e}^{-{\rm i}\theta}$.

Assuming the input pulse to be in coherent state $|\alpha\rangle$, the output state undergoing Kerr interaction can be written as

$$|\alpha_\gamma\rangle=\hat{U}(t)|\alpha\rangle={\rm e}^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}{\rm e}^{-{\rm i}\gamma n(n-1)}|n\rangle, \tag{41}$$

which is a Kerr state.[39] In general, this is a highly nonclassical state and after a certain time of evolution γ in Kerr medium its Wigner function would take negative values in the phase space.

As is well known, for the case of many variables, the cumulants can be divided into two categories: one category is auto-cumulants having all variables identical, and the other is cross-cumulants with different variables. Calculating these cumulants is very difficult by means of Eq. (35). Here we consider directly the relation between moments and cumulants. Following the work of Ould-Baba *et al.*,[40] all higher-order cumulants can be expressed in terms of moments as

$$\kappa_{1,2,\dots,l}=\sum_{\boldsymbol{j}_1,\boldsymbol{j}_2,\dots,\boldsymbol{j}_i\in\boldsymbol{B}(1,2,\dots,l)}(-1)^{i-1}(i-1)!\mu_{\boldsymbol{j}_1}\mu_{\boldsymbol{j}_2}\dots\mu_{\boldsymbol{j}_i}, \tag{42}$$

where $\boldsymbol{B}(1,2,\dots,l)$ is the set of all non-crossing partition of the $1,2,\dots,l$ indices. These subsets are denoted as $\boldsymbol{j}_1,\boldsymbol{j}_2,\dots,\boldsymbol{j}_i$. For $l=3$, we have $\boldsymbol{j}_1=\{1,2,3\}$, $\boldsymbol{j}_1,\boldsymbol{j}_2=\{1\},\{2,3\}$, $\boldsymbol{j}_1,\boldsymbol{j}_2=\{2\},\{1,3\}$, $\boldsymbol{j}_1,\boldsymbol{j}_2=\{3\},\{1,2\}$, and $\boldsymbol{j}_1,\boldsymbol{j}_2,\boldsymbol{j}_3=\{1\},\{2\},\{3\}$. We emphasize that in classical statistics classical random variables commute, so cumulants are fully symmetric in the permutation of observables, e.g.,

$\kappa(X_1X_2X_3)$ is the same as $\kappa(X_2X_1X_3)$. However, the non-commuting quantum mechanical operators do not.

For convenience of later reference, we consider a case of single-mode quantum state described by the $\hat{x}$ and $\hat{p}$ quadrature operators. Let $\hat{\mathrm{X}} = (\hat{X}_1, \hat{X}_2) = (\hat{x}, \hat{p})$, according to Eq. (42), we can write the third-, fourth- and fifth-order cumulants of $\hat{\mathrm{X}}$ as

$$\begin{aligned}\kappa_3(\hat{X}_j\hat{X}_k\hat{X}_l) &= \mu(\hat{X}_j\hat{X}_k\hat{X}_l) - \mu(\hat{X}_j)\mu(\hat{X}_k\hat{X}_l)[3] \\ &\quad + 2\mu(\hat{X}_j)\mu(\hat{X}_k)\mu(\hat{X}_l), \qquad (43a)\\ \kappa_4(\hat{X}_j\hat{X}_k\hat{X}_l\hat{X}_m) &= \mu(\hat{X}_j\hat{X}_k\hat{X}_l\hat{X}_m) - \mu(\hat{X}_j)\mu(\hat{X}_k\hat{X}_l\hat{X}_m)[4] \\ &\quad - \mu(\hat{X}_j\hat{X}_k)\mu(\hat{X}_l\hat{X}_m)[3] \\ &\quad + \mu(\hat{X}_j)\mu(\hat{X}_k)\mu(\hat{X}_l\hat{X}_m)[12] \\ &\quad - 6\mu(\hat{X}_j)\mu(\hat{X}_k)\mu(\hat{X}_l)\mu(\hat{X}_m), \qquad (43b)\\ \kappa_5(\hat{X}_j\hat{X}_k\hat{X}_l\hat{X}_m\hat{X}_n) &= \mu(\hat{X}_j\hat{X}_k\hat{X}_l\hat{X}_m\hat{X}_n) - \mu(\hat{X}_j)\mu(\hat{X}_k\hat{X}_l\hat{X}_m\hat{X}_n)[5] \\ &\quad - \mu(\hat{X}_j\hat{X}_k)\mu(\hat{X}_l\hat{X}_m\hat{X}_n)[10] \\ &\quad + \mu(\hat{X}_j)\mu(\hat{X}_k)\mu(\hat{X}_l\hat{X}_m\hat{X}_n)[20] \\ &\quad + \mu(\hat{X}_j\hat{X}_k)\mu(\hat{X}_l\hat{X}_m)\mu(\hat{X}_n)[30] \\ &\quad - \mu(\hat{X}_j\hat{X}_k)\mu(\hat{X}_l)\mu(\hat{X}_m)\mu(\hat{X}_n)[60] \\ &\quad + 24\mu(\hat{X}_j)\mu(\hat{X}_k)\mu(\hat{X}_l)\mu(\hat{X}_m)\mu(\hat{X}_n), \qquad (43c)\end{aligned}$$

where the number in the square bracket represents the number of terms in the combination as the indices rotate, e.g., $j \to k$, $k \to l$, etc. We see from Eq. (42) that the pth-order cumulants of $\hat{\mathrm{X}}$ have two auto-cumulants and 2^{p-1} cross-cumulants. It is a tedious job to calculate these higher-order cumulants. Thus, we propose a truncated criterion for Kerr state based on the first six higher-order auto-cumulants, which is defined as

$$\kappa_{\mathrm{F}} = \sum_{n=3}^{5} \frac{1}{n!}[|\kappa_n(\hat{x}^n)|^2 + |\kappa_n(\hat{p}^n)|^2]. \qquad (44)$$

To analytically calculate these cumulants, we use the following relationships between the expectation values of $\hat{X}(\theta)$ and that of the combinations of $\hat{a}_\theta^+$ and $\hat{a}_\theta$:

$$\begin{aligned}\langle\hat{X}_\theta^3\rangle &= \left\langle \hat{a}_\theta^3 + 3\hat{a}_\theta^+\hat{a}_\theta^2 + 3\hat{a}_\theta^{+2}\hat{a}_\theta + 3\hat{a}_\theta + 3\hat{a}_\theta^+ + \hat{a}_\theta^{+3}\right\rangle, \qquad (45a)\\ \langle\hat{X}_\theta^4\rangle &= \Big\langle \hat{a}_\theta^4 + 4\hat{a}_\theta^+\hat{a}_\theta^3 + 6\hat{a}_\theta^2 + 4\hat{a}_\theta^{+3}\hat{a}_\theta \\ &\quad + 6\hat{a}_\theta^{+2} + \hat{a}_\theta^{+4} + 6\hat{a}_\theta^{+2}\hat{a}_\theta^2 + 12\hat{a}_\theta^+\hat{a}_\theta + 3\Big\rangle, \qquad (45b)\\ \langle\hat{X}_\theta^5\rangle &= \Big\langle \hat{a}_\theta^5 + 5\hat{a}_\theta^+\hat{a}_\theta^4 + 10\hat{a}_\theta^{+2}\hat{a}_\theta^3 + 10\hat{a}_\theta^{+3}\hat{a}_\theta^2 + 5\hat{a}_\theta^{+4}\hat{a}_\theta \\ &\quad + 30\hat{a}_\theta^{+2}\hat{a}_\theta + 30\hat{a}_\theta^+\hat{a}_\theta^2 + 15\hat{a}_\theta^+ + 15\hat{a}_\theta + 10\hat{a}_\theta^3 \\ &\quad + 10\hat{a}_\theta^{+3} + \hat{a}_\theta^{+5}\Big\rangle, \qquad (45c)\end{aligned}$$

where $\langle\hat{X}_\theta^m\rangle = \mu(\hat{X}_\theta^m)$.

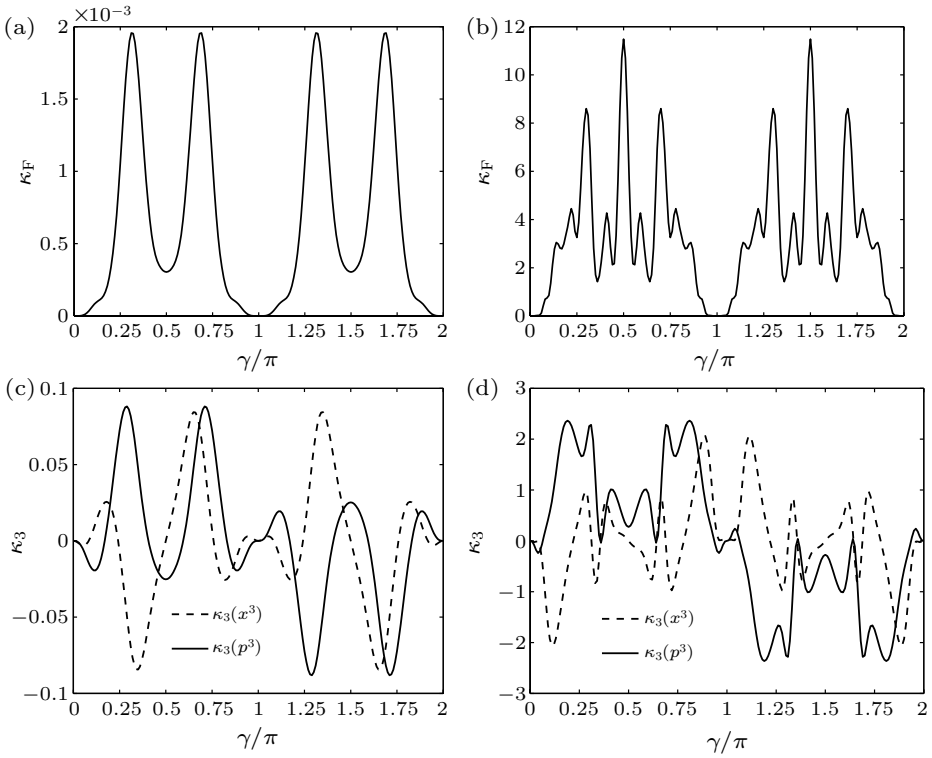


Fig. 5. Time dependence of (top) κ_{F} and (bottom) third-order auto-cumulants for several values of the input amplitudes α. Left: $\alpha = 0.3$; right: $\alpha = 1.2$.

Figure 5 shows the κ_{F} as a function of γ for different values of the input amplitudes α. As it is apparent from the plots, our proposed non-Gaussian criterion for Kerr state may be greater than or equal to zero during the evolutions, showing that the evolved state may be Gaussian or non-Gaussian state. The features can be understood from Eqs. (37) and (41). It is

easy to see that the time evolution is exactly periodic with the period π since

$$|\psi(\gamma+\pi)\rangle = \hat{U}(\gamma+\pi)|\psi(0)\rangle = \mathrm{e}^{-\mathrm{i}\gamma\hat{a}^{+2}\hat{a}^2}\,\mathrm{e}^{-\mathrm{i}\pi\hat{a}^{+2}\hat{a}^2}|\psi(0)\rangle$$
$$= \mathrm{e}^{-\mathrm{i}\gamma\hat{a}^{+2}\hat{a}^2}\sum_n c_n\,\mathrm{e}^{-\mathrm{i}\pi n(n-1)}|n\rangle = |\psi(\gamma)\rangle, \quad (46)$$

where we have used $|\psi(0)\rangle = \sum_n c_n|n\rangle$ and $\mathrm{e}^{-\mathrm{i}\pi n(n-1)} = 1$ for integer n. It is obvious that when the system is initially in a Gaussian state, it does come back to the initial state after a period, which would result in zero value for our non-Gaussian criterion. However, we can see that at other times, the input coherent state can be transformed into a superposition of two or more well-localized coherent states. For example, when $\gamma = \pi/2$, the system evolves to $|\psi(\gamma)\rangle = \frac{1}{\sqrt{2}}[\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}|\mathrm{i}\alpha\rangle + \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|-\mathrm{i}\alpha\rangle]$, which is the known to be Yurke–Stoler state. When $\gamma = \frac{\pi}{3}$, the system evolves to $|\psi(\gamma)\rangle = \sum_{j=0}^{2} f_j|\alpha\,\mathrm{e}^{\frac{2\pi j\mathrm{i}}{3}}\rangle$ with $f_j = \frac{1}{3}\sum_{k=0}^{2}\exp[\frac{\mathrm{i}k\pi(2j-k+1)}{3}]$. Clearly, these states are non-Gaussian, which is well reflected in our non-Gaussian criterion. Finally, it can be seen from Figs. 5(c) and 5(d) that the third-order cumulants of Kerr state may be equal to zero during the evolution. However, such zero values do not lead to the disappearance of non-Gaussianity. For example, for the case of $\gamma = 0.36\pi$ and $\alpha = 1.2$, we have $\kappa_3(p^3) = 0$, but $\kappa_{\mathrm{F}} = 1.4382$. It is shown again that our non-Gaussian criterion is powerful.

5. Discussion and summary

The non-Gaussianity and non-classicality are two important features exhibited by continuous variable quantum states and play an important role in quantum information science. It is believed that non-Gaussian states may be classical states or quantum states, while for classical states, they may be Gaussian states or non-Gaussian states. The connection between them has drawn considerable attention in recent years. We can see that the intrinsic relation between non-classicality and non-Gaussianity of quantum states were established by the information-theoretic means.[37] In addition, an entanglement criterion with the third- and fourth-order cumulants was used to detect the entanglement of non-Gaussian states.[41] Therefore, in the future work we will try to link the higher-order cumulants or cumulant matrix of the state analyzed with its quantum correlations, thereby formulating a methodology for characterizing quantum correlations in terms of cumulant theory.

In the experimental aspect, a method to obtain information on continuous variable quantum states is to measure quadrature distribution along many different axes in phase space by using quantum state tomography.[42] However, such a technique has a drawback, that is, the measured distributions do not yield a physical state when directly used due to finite data and finite binning size. To overcome the difficulty, Park *et al.* proposed a method that transforms the one-dimension quadrature distribution into a factorized two-dimension Wigner distribution by multiplying the measured distribution or the vacuum-state distribution along an orthogonal axis.[43] At the same time, non-Gaussian states present in our examples have been experimentally realized.[44–47] Hence, the results reported here seem to be experimentally realizable with current techniques of measurements on quadrature operators of optical field.

In conclusion, we have proposed a novel criterion for capturing non-Gaussian features of single-mode rotationally symmetric continuous-variable quantum states based on the Frobenius norm of higher-order cumulant matrix from single quadrature function. We have applied our criterion to various non-Gaussian states, such as a mixture of vacuum and Fock states, single-photon added thermal states, and even/odd schrodinger cat states. It has been shown that our criterion can detect faithfully the non-Gaussianity of these quantum states. Excitingly, it can perfectly overcome the problem that the high-order cumulants of only single quadrature variable fails to reveal the non-Gaussian nature of mixture of vacuum and Fock states. Finally, in order to facilitate widespread applications, we extend our approach to two cases: symmetric multi-mode non-Gaussian states and non-symmetric single-mode non-Gaussian states, providing the interesting perspective for quantum source theory.

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