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Anomalous time-reversal symmetric non-Hermitian systems

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The conditions for the emergence of the non-Hermitian skin effect, as a unique physical response of non-Hermitian systems, have now become one of the hot research topics. In this paper, we study the novel physical responses of non-Hermitian systems with anomalous time-reversal symmetry, in both one dimension and two dimensions. Specifically, we focus on whether the systems will exhibit a non-Hermitian skin effect. We employ the theory of generalized Brillouin zone and also numerical methods to show that the anomalous time-reversal symmetry can prevent the skin effect in one-dimensional non-Hermitian systems, but is unable to exert the same effectiveness in two-dimensional cases.

Keywords: skin effect, symmetry, non-Hermitian systems

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1. Introduction

Since the non-Hermitian skin effect has been confirmed in a series of classical wave systems,[1–4] this phenomenon has received increasing attention. For a long time, the bulk–boundary correspondence has been fundamental for describing topological properties of the system.[5] Based on the bulk states in the periodic boundary conditions (PBC), we can predict the edge states of the system with open boundary conditions (OBC). The bulk–boundary correspondence has always played an important role in the development of the topological materials, and has been intensively studied in various topological systems.

However, with the in-depth study of non-Hermitian systems, people have realized that in a large class of non-Hermitian systems, the conventional theory of bulk–boundary correspondence has been challenged.[6–18] By studying the exact solution of non-Hermitian systems, it is found that for certain non-Hermitian systems, the energy spectra of the system with PBC are completely different from that with OBC. Meanwhile, all the eigenstates of the system are localized at its boundary (boundaries),[19] which cannot be explained by the conventional Bloch band theory. This phenomenon is named the non-Hermitian skin effect.[10,14,18–30] When the skin effect appears, the traditional bulk–boundary correspondence is invalid. In this case, when OBC is imposed on the system, the edge states cannot be described by a Bloch Hamiltonian.[27,31–34]

The non-Hermitian skin effect is one of the critical points that distinguish non-Hermitian systems from Hermitian ones, and many novel physical responses are derived from the non-Hermitian skin effect, which are crucial for people to understand the topological properties of non-Hermitian systems. Therefore, the origin of the non-Hermitian skin effect and the conditions for its emergence have now become the core issues in this field.[7–9,12,13,29,35–45] Meanwhile, due to the existence of skin effect, the traditional band theory, which is an important cornerstone of condensed matter physics, is no longer applicable. Developing and perfecting a set of non-Hermitian band theory has become one of the hotspots in condensed matter field in recent years.[6–18]

Recently, there has been substantial progress in this research. It is very convenient to introduce the concept of generalized Brillouin zone (BZ) to study the physical response of a one-dimensional (1D) non-Hermitian system. The generalized BZ is a generalization of the conventional BZ in an open-boundary system.[19–21,24,25,31] With this concept, one can accurately predict the appearance or disappearance of the topologically protected edge states. Specifically, if the generalized BZ of a non-Hermitian system roughly coincides with the BZ, it means that the conventional bulk–boundary correspondence still holds, and the system will not have a skin effect when OBC is imposed. If the generalized BZ is inconsistent with the BZ, the conventional bulk–boundary correspondence fails, and the non-Hermitian skin effect appears at the same time.

In addition, when we study the factors that are related to the skin effect, we must take the symmetry of the system into account. As an intrinsic property of the system, it would not be surprising if there is some sort of link between the symmetry of the system and the appearance of non-Hermitian skin effects. In fact, research has shown that the non-Hermitian skin effect would never appear when the system has certain symmetries, e.g., systems with inversion symmetry in both 1D and two-dimensional (2D) cases.[25,46,47] However, for now, a complete theory clarifying the correspondence between symmetries and non-Hermitian skin effect is still missing. In this paper, we focus our discussion on the relationship between anomalous time-reversal symmetry (ATRS) and skin effect in both 1D and 2D systems.

This article is organized as follows. In Section 2, we in-

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troduce the concept and relevant theories of generalized BZ, and apply them to 1D non-Hermitian systems with ATRS. In this section, we demonstrate that the ATRS can prevent skin effect in 1D non-Hermitian systems. In Section 3, we provide the numerical results of a specific model to verify the theory proposed in In Section 2. In Section 4, we then extend the discussion to 2D systems, and show that in this case, the ATRS is no longer effective for preventing skin effect. In Section 5, we present a concluding discussion concerning the role of ATRS in non-Hermitian systems.

2. The combination of generalized Brillouin zone theory and symmetries of the system

The generalized BZ is an alternative to BZ in large systems with OBC. In fact, the generalized BZ is composed of a series of possible values of $\beta \equiv \mathrm{e}^{\mathrm{i}k}$, where k is the wave number. In 1D systems, the generalized BZ forms a loop on the complex plane. And for Hermitian cases in particular, since $k \in R$, we can easily derive that $|\beta| = 1$. Thus, the generalized BZ becomes a unit circle, indicating the eigenstates extend over the bulk.

For a non-Hermitian system, the wave number k becomes complex and some calculations are needed to obtain the generalized BZ of the system. We need first to rewrite the Bloch Hamiltonian $\mathcal{H}(k)$ in terms of β as $\mathcal{H}(\beta)$ by substituting $\mathrm{e}^{\mathrm{i}k}$ with β.[21,25] Thus, the characteristic equation $\det[\mathcal{H}(\beta) - E] = 0$ gives an algebraic equation of β.

If we sort its solutions β_i by their length as $|\beta_1| \leq |\beta_2| \leq \cdots \leq |\beta_{2M}|$, where $2M$ is the degree of the characteristic equation, then the generalized BZ is given by the trajectory of β_M or β_{M+1} under the condition[21,25]

$$|\beta_M| = |\beta_{M+1}|. \tag{1}$$

In fact, the generalized BZ is always a single loop encircling the origin, but not necessarily a unit circle like in Hermitian cases. Furthermore, the geometric shape of the generalized BZ can serve as a criterion for judging whether the non-Hermitian skin effect will occur.

When the generalized BZ does form a unit circle, the system is comparable to a Hermitian one and the traditional bulk–boundary correspondence still holds. The eigenstates of $\mathcal{H}(\beta)$ extend over the bulk and show an insensibility to boundaries. In this case, the energy spectrum of the system with OBC and that with PBC are almost equivalent. Thus, we can safely deduce the absence of skin effect.

When the generalized BZ is not a unit circle, it means that the wave functions of the system show a feature of localization, i.e., the system exhibits a skin effect.

On this basis, we account for the role of the symmetry. Specifically, if we apply a certain symmetry to a non-Hermitian Hamiltonian $\mathcal{H}(\beta)$, accordingly, we can obtain a constraint equation for β and E.

In conventional Hermitian systems, the time-reversal symmetry (TRS) is defined by

$$\mathcal{T}H^*\mathcal{T}^{-1} = H, \tag{2}$$

where the asterisk denotes complex conjugation. However, the classification of symmetries is more complicated in non-Hermitian systems than in Hermitian ones because transposition ($H \to H^{\mathrm{T}}$) and complex conjugation ($H \to H^*$) are inequivalent operations in former. In particular, the TRS splits into two inequivalent non-Hermitian generalizations, distinguished by whether or not complex conjugation is replaced by transposition in Eq. (2).

In this paper, we focus on the ATRS, denoted as $\bar{\mathcal{T}}$,[12,36] and we limit our discussion to the case where the energy levels are not degenerate.

If a periodic non-Hermitian system possesses the ATRS, then the Bloch Hamiltonian $\mathcal{H}(k)$ must satisfy

$$\bar{\mathcal{T}}\mathcal{H}^{\mathrm{T}}(-k)\bar{\mathcal{T}}^{-1} = \mathcal{H}(k), \tag{3}$$

where $\bar{\mathcal{T}}$ is a anomalous time-reversal operator, and the superscript T represents the transpose operation.

By plugging in the above formula to the function $f(k,E) \equiv \det[\mathcal{H}(k) - E]$, we readily have

$$\begin{aligned} f(k,E) &= \det[\mathcal{H}(k) - E] = \det[\bar{\mathcal{T}}\mathcal{H}^{\mathrm{T}}(-k)\bar{\mathcal{T}}^{-1} - E] \\ &= \det[\mathcal{H}^{\mathrm{T}}(-k) - E] = \det[\mathcal{H}(-k) - E] \\ &= f(-k,E). \end{aligned} \tag{4}$$

Now we expand $f(k,E)$ in the form of infinite series

$$f(k,E) = \sum_{l,m,n} c_{l,m,n} E^l (\sin k)^m (\cos k)^n, \tag{5}$$

where l, m, n are all integers. Substitute this expansion into Eq. (4), and compare the coefficients of terms with the same exponent on each side of the equation. Thus, we arrive at

$$c_{l,m,n} = (-1)^m c_{l,m,n}, \tag{6}$$

which implies

$$c_{l,2m_0+1,n} = 0, \tag{7}$$

where m_0 is an arbitrary integer.

Next, we substitute β for $\mathrm{e}^{\mathrm{i}k}$, and rewrite the expansion Eq. (5) in terms of β as $f(\beta,E)$. With Eq. (7) taken into account, we have

$$\begin{aligned} f(\beta,E) &= \sum_{l,m,n} c_{l,m,n} E^l \left(\frac{\beta^2-1}{2\mathrm{i}\beta}\right)^m \left(\frac{\beta^2+1}{2\beta}\right)^n \\ &= \sum_{l,m,n} c_{l,2m,n} E^l \left(\frac{\beta^2-1}{2\mathrm{i}\beta}\right)^{2m} \left(\frac{\beta^2+1}{2\beta}\right)^n \\ &= \sum_{l,m,n} c_{l,2m,n} E^l \left(\frac{1-\beta^2}{2\mathrm{i}\beta}\right)^{2m} \left(\frac{\beta^2+1}{2\beta}\right)^n. \end{aligned} \tag{8}$$

After a straightforward calculation, we finally arrive at

$$f(\beta,E)=f(1/\beta,E). \tag{9}$$

From Eq. (9) we can conclude that for a fixed value of E, if β is a solution to the characteristic equation $f(\beta,E)=0$, then $1/\beta$ must also be a solution to the equation. Thus, We can express the solutions of the characteristic equation $f(\beta,E)=0$ as

$$|\beta_1|\leq|\beta_2|\leq\cdots\leq|\beta_M|\leq 1/|\beta_M|\leq\cdots\leq 1/|\beta_2|\leq 1/|\beta_1|. \tag{10}$$

Now, with the help of Eq. (1) (forming condition of the generalized BZ), we can conclude that for any point on generalized BZ, we must have

$$|\beta_M|=1/|\beta_M|, \tag{11}$$

namely,

$$|\beta_M|=1, \tag{12}$$

signifying that k becomes real. Therefore, the possible values of β must form a unit circle as in Hermitian systems. Consequently, we can conclude that the non-Hermitian skin effect is absent in a 1D system possessing the ATRS.

In the next section, this conclusion will be verified by numerical calculation.

3. Model

In general, the non-Hermitian systems can be roughly divided into two types. First, non-reciprocal coupled systems, such as non-reciprocal optical systems. The non-Hermiticity of the system is derived from the asymmetry of energy exchange within the system. Second, dissipative systems, in this case, the non-Hermiticity of the system is derived from the on-site dissipation at each lattice.[3,10,31,48]

Here, we start with a general four-band non-Hermitian Hamiltonian, which is mathematically complete

$$\begin{aligned}\mathcal{H}(k) \;=\; & h_{00}(k)\boldsymbol{\sigma}_0\times\boldsymbol{\sigma}_0+h_{01}(k)\boldsymbol{\sigma}_0\times\boldsymbol{\sigma}_1\\ &+h_{02}(k)\boldsymbol{\sigma}_0\times\boldsymbol{\sigma}_2+h_{03}(k)\boldsymbol{\sigma}_0\times\boldsymbol{\sigma}_3\\ &+h_{10}(k)\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_0+h_{11}(k)\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_1\\ &+h_{12}(k)\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2+h_{13}(k)\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_3\\ &+h_{20}(k)\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_0+h_{21}(k)\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_1\\ &+h_{22}(k)\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_2+h_{23}(k)\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3\\ &+h_{30}(k)\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_0+h_{31}(k)\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_1\\ &+h_{32}(k)\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_2+h_{33}(k)\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_3, \end{aligned}\tag{13}$$

where $\boldsymbol{\sigma}_0$ is identity matrix; $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3$ are Pauli matrices; $h_*(k)$ are complex algebraic functions of k.

Next, we count in the restriction of symmetry on Hamiltonian, and substitute the above expression of $\mathcal{H}(k)$ into Eq. (3). For simplicity, we take the representation of ATRS as $\bar{\mathcal{T}}=\boldsymbol{\sigma}_0\times\boldsymbol{\sigma}_0$ (which will not lead to degeneracy of energy levels).

Now, by comparing each side of the equation, we can conclude that $h_{00}(k)$, $h_{01}(k)$, $h_{03}(k)$, $h_{10}(k)$, $h_{11}(k)$, $h_{13}(k)$, $h_{22}(k)$, $h_{30}(k)$, $h_{31}(k)$, $h_{33}(k)$ are even functions, i.e.,

$$h_*(-k)=h_*(k), \tag{14}$$

and $h_{02}(k)$, $h_{12}(k)$, $h_{20}(k)$, $h_{21}(k)$, $h_{23}(k)$, $h_{32}(k)$ are odd functions, i.e.,

$$h_*(-k)=-h_*(k). \tag{15}$$

In numerical calculations, we take the interaction between nearest neighbors and next nearest neighbors into account. Thus, the expansion of $h_*(k)$ should contain terms of e^{2ik}, e^{ik}, e^{-ik} and e^{-2ik}. We take their coefficients as random numbers while two sets of $h_*(k)$ satisfy Eqs. (14) and (15), respectively.

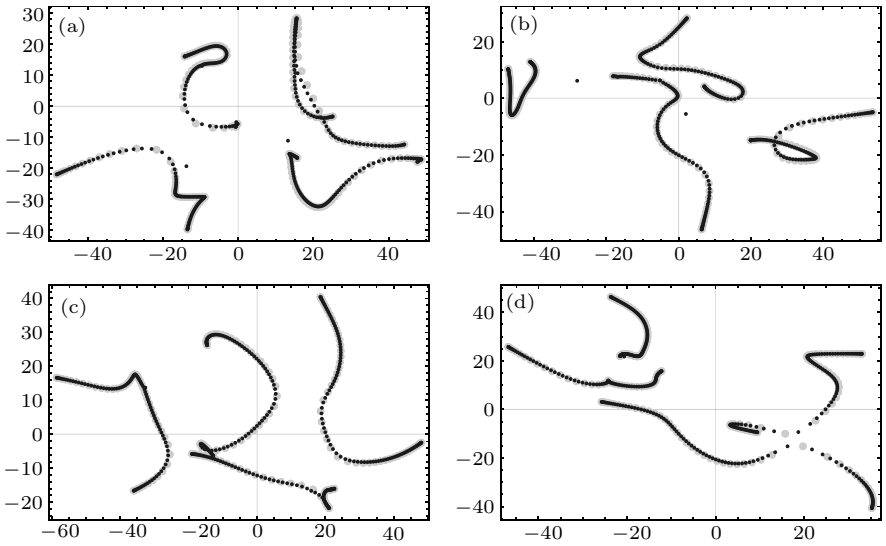


Fig. 1. Energy spectra of 1D four-band non-Hermitian models (13) with ATRS (i.e., model parameters satisfy Eqs. (14) and (15)) under OBC/PBC. Gray dots represent energy levels of periodic systems and black dots represent energy levels of open systems.

Next, by use of Fourier inverse transform, we can obtain the Hamiltonian in real space, denoted as $H_{\rm r}$, on which the OBC is imposed.

Finally, we calculate the eigenvalues of $\mathcal{H}(k)$ and $H_{\rm r}$, respectively, and display the results on the same graph for comparison.

The numerical results are shown in Fig. 1 for four sets of random values of parameters, from which we can see that the energy spectra of PBC and OBC indeed coincide with each other, signifying that the skin effect is absent.

4. Two-dimensional case

Now we turn to 2D non-Hermitian systems.[4,17,30,33,49–54] In this paper, we discuss simple square lattices, and accordingly, we need to introduce two parameters, k_x and k_y, in the Bloch Hamiltonian $\mathcal{H}(k_x,k_y)$, and the corresponding characteristic equation $\det[\mathcal{H}(k_x,k_y)-E]=0$ gives a bivariate algebraic equation for k_x and k_y.

The concept of the generalized BZ can be extended to higher dimensions as well. Nevertheless, thus far, how to obtain the generalized BZ has been known only for 1D systems. A clear image of generalized BZ in systems of two or higher dimensions is still missing.

Here, we obtain the energy spectra and the distributions of wave functions directly by numerical calculations.

As in the 1D case, we focus on systems with ATRS, and the Bloch Hamiltonian $\mathcal{H}(k_x,k_y)$ must satisfy

$$\bar{\mathcal{T}}\mathcal{H}^{\rm T}(-k_x,-k_y)\bar{\mathcal{T}}^{-1}=\mathcal{H}(k_x,k_y), \tag{16}$$

where $\bar{\mathcal{T}}$ is an anomalous time-reversal operator.

We expand the Bloch Hamiltonian $\mathcal{H}(k_x,k_y)$ into the same form as Eq. (13).

Here, $h_{00}(k_x,k_y)$, $h_{01}(k_x,k_y)$, …, $h_{33}(k_x,k_y)$ are all binary functions of k_x and k_y, which satisfy (when $\bar{\mathcal{T}}=\boldsymbol{\sigma}_0\times\boldsymbol{\sigma}_0$)

$$h_*(-k_x,-k_y)=h_*(k_x,k_y), \tag{17}$$

for $h_{00}(k_x,k_y)$, $h_{01}(k_x,k_y)$, $h_{03}(k_x,k_y)$, $h_{10}(k_x,k_y)$, $h_{11}(k_x,k_y)$, $h_{13}(k_x,k_y)$, $h_{22}(k_x,k_y)$, $h_{30}(k_x,k_y)$, $h_{31}(k_x,k_y)$, $h_{33}(k_x,k_y)$, and

$$h_*(-k_x,-k_y)=-h_*(k_x,k_y), \tag{18}$$

for $h_{02}(k_x,k_y)$, $h_{12}(k_x,k_y)$, $h_{20}(k)$, $h_{21}(k_x,k_y)$, $h_{23}(k_x,k_y)$, $h_{32}(k_x,k_y)$.

Since we only take into account the interactions between the nearest neighbor lattices, the expansion of $h_*(k_x,k_y)$ should contain terms of $\mathrm{e}^{\mathrm{i}k_x}$, $\mathrm{e}^{\mathrm{i}k_y}$, $\mathrm{e}^{-\mathrm{i}k_x}$ and $\mathrm{e}^{-\mathrm{i}k_y}$. We take their coefficients as random numbers while Eqs. (17) and (18) are satisfied.

The energy spectrum of a periodical 2D system is shown in Fig. 2. By comparing it with Fig. 1, we can easily pick up on the first difference between 1D system and 2D system, i.e., the energy spectrum of 1D systems forms a loop on the complex plane, while in 2D systems, the energy spectrum is generically a limited continuous area.[30]

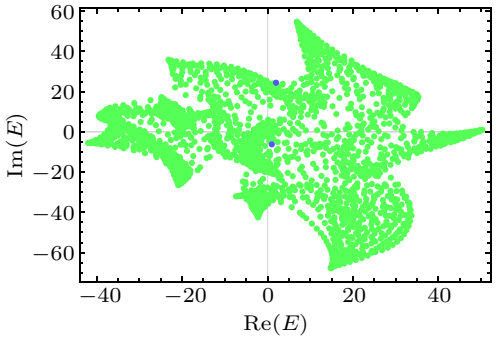


Fig. 2. Energy spectrum of a 2D system with ATRS (i.e., model parameters satisfy Eqs. (17) and (18)) under triangular OBC.

Now let us focus on Fig. 3, the distributions of the wave functions of 2D systems with different geometries. As we mentioned at the beginning, the non-Hermitian skin effect manifests itself in two ways: (i) the wave functions in large systems with OBC are localized at the boundaries of the systems; and (ii) the energy spectra of the system with PBC are significantly different from that with OBC.[19] In the discussion of 1D cases, we employ the second as a criterion to judge whether the skin effect occurs. Here, in 2D systems, we adopt the first.

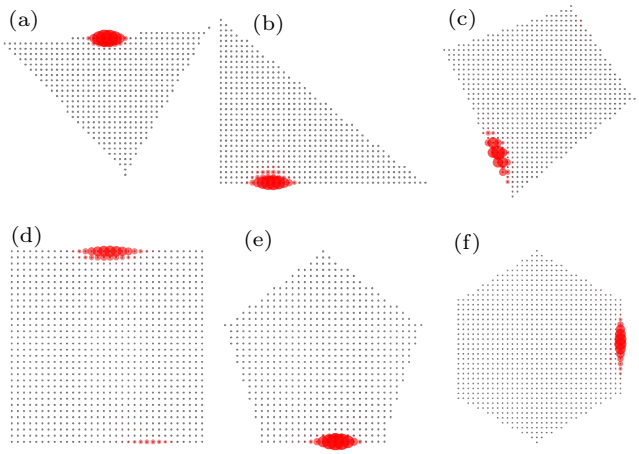


Fig. 3. Distributions of the wave functions of 2D systems with ATRS (i.e., model parameters satisfy Eqs. (17) and (18)) under various geometries of OBC.

From Fig. 3 we can see that, in each case, the wave functions are concentrated at the boundary of the system, indicating that the system has skin effect.

From the previous discussion, we have come to the conclusion that ATRS can eliminate the skin effect in 1D systems. However, based on the numerical results, this conclusion obviously cannot be generalized to 2D systems.

This is a rather surprising result given that we construct Hamiltonians in the same way as we did in 1D. We suspect that this incongruity is due to the diversity of boundary conditions in 2D systems. Here, we note that another major difference between the 1D system and the 2D system is the diversity of

OBC. There is only one geometry for an open system in 1D, that is, an open chain. By contrast, there is an infinite number of geometric shapes in 2D systems, such as triangle, quadrangle and so on. As a result, the 2D systems tend to be more susceptible to boundaries. Since the non-Hermitian skin effect is, in essence, a kind of high sensitivity for boundaries, it seems to be more prevalent in higher dimensions.

5. Discussion and conclusion

In this paper, we study the connections between the anomalous time-reversal symmetric systems and non-Hermitian skin effect. We have learned that this anomalous localization behavior of the bulk states (skin effect) always appears when the conventional bulk–boundary correspondence is broken. For 1D systems, the ATRS ensures all the wave numbers k are real, which leads to the validity of the conventional bulk–boundary correspondence and the absence of skin effect. This conclusion is confirmed by numerical results. From Fig. 1 one can see that the PBC and OBC spectra are not dramatically different from each other, signifying that the non-Hermitian skin effect is absent. This serves as a clear proof that as an intrinsic property of the system, the symmetry has a direct impact on whether the system will exhibit a skin effect. However, this connection is weakened in higher dimensions. According to the numerical results we obtained, in 2D systems, the wave functions are concentrated at the boundary of the system, indicating that the skin effect appears even if the system has the ATRS. This obvious difference in the physical response of ID and 2D systems implies that the NH skin effect becomes more prevalent in higher dimensions.

In this paper, we are only able to determine whether skin effect occurs in 2D non-Hermitian systems based on numerical results, a rigorous theory for us to predict whether the skin effect will occur in 2D systems with arbitrary symmetries is still missing. In fact, the theory of generalized BZ employed in Section 2 can only apply to 1D non-Hermitian systems, a theory applicable to 2D systems remains to be developed, and we hope this paper can inspire new ideas in this respect.

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