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Target layer state estimation in multi-layer complex dynamical networks considering nonlinear node dynamics

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In many engineering networks, only a part of target state variables are required to be estimated. On the other hand, multi-layer complex network exists widely in practical situations. In this paper, the state estimation of target state variables in multi-layer complex dynamical networks with nonlinear node dynamics is studied. A suitable functional state observer is constructed with the limited measurement. The parameters of the designed functional observer are obtained from the algebraic method and the stability of the functional observer is proven by the Lyapunov theorem. Some necessary conditions that need to be satisfied for the design of the functional state observer are obtained. Different from previous studies, in the multi-layer complex dynamical network with nonlinear node dynamics, the proposed method can estimate the state of target variables on some layers directly instead of estimating all the individual states. Thus, it can greatly reduce the placement of observers and computational cost. Numerical simulations with the three-layer complex dynamical network composed of three-dimensional nonlinear dynamical nodes are developed to verify the effectiveness of the method.

Keywords: multi-layer complex dynamical network, nonlinear node dynamics, target state estimation, functional state observer

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1. Introduction

Complex dynamical networks have been selected as powerful tools for studying the characteristics and evolutions of many practical social and engineering networks, such as multi-agent systems, power grids, biological networks, epidemic spreading, and so on (Refs. [1–4]). In order to control the operation and optimize the resource allocation of the network, we need to obtain timely the state information of the nodes (Refs. [5–7]). However, in many real situations, the state information of partial nodes in the complex network is unknown or unpredictable. Therefore, how to achieve the state estimation of various complex networks is important yet challenging.

Over the last decades, the main principle of the network state estimation is to construct an observer network. Each node of the observer network is set with an output feedback controller to measure the difference between the observer network output and the original network output. When the observer network reaches the external synchronization with the original network, the unmeasurable states of the original network can be estimated by states of the observer network. This principle requires that all nodes be set with controllers and can estimate states of all nodes in the original network. Based on the principle, Giorgio *et al.*[8] developed a series of algorithms and considered a collective Lyapunov function to address the consensus-based estimation of a nonlinear dynamical network. Li *et al.*[9] used a family of Bernoulli random variables to depict the switching behaviour of the network when studying the state estimation with the switching topology in complex networks. They[10] also proposed a state estimator to research the estimation with random coupling strength by adopting structures of the Kalman filter in the network. Due to the network congestion or transmission faults such as node failure and link interruption, external disturbance is a common phenomenon in real networks, which can affect the performance of the network seriously. Therefore, packet loss, noise and other external disturbances were taken into consideration by researchers. Wu *et al.*[11] used the Bernoulli distribution to describe the packet loss and applied the stochastic analysis method when studying the state estimation with pocket loss. However, most existing methods on state estimation from observation and control of dynamics are focused on single-layer networks.

In the practical engineering, networks rarely appear with a single-layer structure and the modeling of single-layer networks has some limitations in accurately capturing the functional details. For example, the integration of the power layer and information layer is crucial in smart grid networks to ensure reliable grid operations (Ref. [12]). In the communication network, wired communication and wireless communication cooperate with each other to ensure normal communication requirements. When the wireless communication network malfunctions, the wired communication network can

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still be used to maintain necessary communication functions (Refs. [13,14]). The multi-layer complex dynamical network has gained significant attention as it allows for a more realistic characterization of real-world networks.

The current research on the control problem of multi-layer complex dynamical networks has obtained great attention, mainly focusing on synchronization (Refs. [15, 16]) topology identification (Refs. [17,18]), controllability (Refs. [19]). Li *et al.*[15] proposed a scheme based on an aperiodically intermittent pinning control to achieve cluster exponential synchronization in the two-layer complex network with time-varying delays and deduced some conditions for cluster synchronization. Jin *et al.*[16] developed a multi-layer complex network with delay parameter and studied the influence of parameters on multi-layer explosive synchronization with delay. Wang *et al.*[17] investigated topology identification in a two-layer network considering peer-to-peer unidirectional couplings by constructing simple control inputs and updating laws. Mei *et al.*[18] studied the problem of structure reconstruction for multi-layer complex dynamical networks based on compressive sensing and regularization. Wu *et al.*[19] addressed the state controllability of multi-layer networks with inter-layer couplings and obtained some necessary and sufficient conditions for the network controllability. However, the research on the state estimation of the multi-layer complex dynamical network is limited. Li *et al.*[20,21] investigated the state estimation of a general type of multi-layer complex network with coupling delay and time-varying delay based on establishing a multi-layer observer network. The efficient solution to this problem are consistent with the principle mentioned above for the single-layer complex dynamical network. It is also necessary to apply an observer to each node and can estimate the states of all nodes in the multi-layer complex dynamical network.

Considering either economic costs or physical difficulties, it becomes challenging to place sensors for each node in order to obtain state information as the scale of the complex dynamical network increases. For example, in the public opinion network, it is prohibitive for us to obtain every individual's understanding of public opinions. Meanwhile, in many practical engineering problems, there is often no need to estimate the state information of all nodes in a high-dimensional or large-scale network. Instead, our focus is typically on the state information of some target nodes that are of interest (Ref. [22]). For example, in the human brain, among the thousands of neurons, we only need to grasp the information of the neurons at the patient's site in order to prescribe the appropriate treatment (Ref. [23]). Therefore, it has practical value to estimate the state information of interested target nodes based on limited node measurements in the network.

For the study of the state estimation of target nodes in complex dynamical networks with limited measurements, the functional state observer is proposed in linear systems. The design of the functional state observer mainly relies on the rank-based condition (Refs. [24,25]). However, the rank-based method is not suitable for large-scale or multi-dimensional networks due to its high computational requirements. Graph-based algorithms can solve the problem of high computation caused by the rank-based method. Liu *et al.*[26] proposed a graph theory algorithm to determine the minimum sensor node set to ensure the observability of complex networks. Arthur *et al.*[27] developed a graph-theoretic approach of functional observability and related algorithms to select auxiliary nodes, making it effective to estimate the target states of a large-scale dynamical network. The above two papers mainly focused on the single-layer network with linear and one-dimensional node dynamics. Wu *et al.*[28] investigated target layer state estimation in multi-layer complex dynamical networks based on network graph theory and functional observability. However, the paper also applies only to complex networks with linear and one-dimensional node dynamics.

In practical networks, nodes generally have multi-dimensional nonlinear dynamics (Refs. [29–31]). For example, in unmanned aerial vehicle systems, the mechanical equations, kinematic equations and navigation equations are all multi-dimensional nonlinear differential equations (Ref. [29]), such as in the kinematic equation, it usually has multi-dimensional state variables including location, velocity, and attitude angle. In the ecological network, the Lotka–Volterra model used to describe the interactions between predators and prey is also multi-dimensional nonlinear (Ref. [30]). In the communication network, chaos is a multi-dimensional nonlinear dynamic system with high randomness and complexity, which is usually used to encrypt the communication system in order to improve the security of the communication (Ref. [31]). Therefore, complex dynamical networks with multi-dimensional nonlinear node dynamics provide a more accurate and comprehensive modeling approach, enabling us to better understand and explain the behavior of complex systems.

From the above discussion, in this paper, we investigate the state estimation of some components in target nodes in multi-layer complex dynamical networks with nonlinear node dynamics by limited state measurements. First, we establish a corresponding multi-layer complex dynamical network, where the information of some state components in certain nodes on one layer has been obtained by sensors. Our objective is to estimate the state information of some components in target nodes on other layers. We also assume that every node is subject to external disturbances. Second, a functional state

observer is constructed according to the established multi-layer network. We can obtain error systems from the established multi-layer network and the designed functional state observer. Third, the parameters of the functional state observer are solved by the arithmetic method and the stability of the functional state observer is verified based on the Lyapunov stability principle. Finally, some numerical simulations are developed to validate the effectiveness of the proposed method. The results demonstrate that we can achieve the accurate estimation of target state components with fewer observer placements and computational cost.

The theory and method proposed in this paper have good application prospects in the field of social network public opinion spreading prediction and integrated cluster warfare of air-space-sea. The major contributions of this paper are summarized as follows:

(i) We have investigated the problem of the target layer state estimation in multi-layer complex dynamical networks considering nonlinear node dynamics based on the actual needs of the engineering.

(ii) The suitable nonlinear functional state observer has been developed to achieve the state estimation of target nodes in multi-layer complex dynamical networks accurately.

(iii) The proposed method can realize the accurate estimation with less placement of observers and fewer computational costs in the multi-layer complex dynamical network.

Notation: I_n donetes the identity matrix of size n, $\mathrm{diag}\{a_1,a_2,\ldots,a_n\}$ is a diagonal matrix formed by matrices $a_1,a_2,\ldots,a_n$, $\otimes$ is the Kronecker product, $\mathrm{rank}(a)$ represents the rank of a matrix a.

2. Problem formulation

We consider a multi-layer complex dynamical network consisting of L layers, with each layer containing $N^{(l)}$ nodes. The dynamics of the network can be described by the following equation:

$$\begin{cases} \dot{x}_i^l(t) = A_i^l(x_i^l(t)) + f_i^l(x_i^l(t)) + \sum\limits_{j=1}^{N^{(l)}} d_{ij}^l \Gamma^l x_j^l(t) \\ \qquad\qquad + \sum\limits_{R=1,R\neq l}^{L} \sum\limits_{j=1}^{N^{(l)}} d_{ij}^{lR} \Gamma^{lR} x_j^R(t) + B_i^l u_i^l(t) + W_i^l m_i^l(t), \\ y_i^l(t) = C_i^l x_i^l(t), \end{cases} \tag{1}$$

where $i=1,2,\ldots,N^{(l)}$, $l=1,2,\ldots,L$. $x_i^l \in R^n$ and $u_i^l \in R^m$ are the n-dimensional state vector and the m-dimensional control input of the i-th node on the layer l respectively. $m_i^l \in R^n$ is the external disturbance. $y_i^l \in R^p$ represents the output, which is the direct measurement from sensors in the network. $A_i^l \in R^{n\times n}$ is the linear part of the node dynamics, $f_i^l(\cdot)$ is the nonlinear part of the node dynamics. $B_i^l \in R^{n\times m}$ and $C_i^l \in R^{p\times n}$ are the input matrix and output matrix respectively. $W_i^l \in R^{n\times n}$ is the weight matrix of the disturbance. The intralayer configuration matrix of the network on layer l, denoted as $D^l = [d_{ij}^l]_{N^{(l)}\times N^{(l)}}$, represents the connections between nodes within the same layer l. If there is a link from node j to node i, $d_{ij}^l > 0$, otherwise, $d_{ij}^l = 0$. The interlayer configuration matrix of the network, denoted as $D^{lR} = [d_{ij}^{lR}]_{N^{(l)}\times N^{(l)}}$, represents the connections between nodes in layer R and layer l. If there is a link from node j in layer R to node i in layer l, $d_{ij}^{lR} > 0$, otherwise, $d_{ij}^{lR} = 0$. The matrices Γ^l and Γ^{lR} represent the inner coupling matrix of the intralayer network and the interlayer network respectively. Let N be the total number of the network.

Transform the network (1) into a general system state space expression, which is shown in system (2),

$$\begin{cases} \dot{X}(t) = \varphi X(t) + \tilde{f}(X(t)) + \psi U(t) + WM(t), \\ Y(t) = \phi X(t). \end{cases} \tag{2}$$

In system (2),

$$X = [(x^1)^{\mathrm{T}}, (x^2)^{\mathrm{T}}, \ldots, (x^L)^{\mathrm{T}}]^{\mathrm{T}} \in R^{Nn},$$

where

$$x^l = [(x_1^l)^{\mathrm{T}}, (x_2^l)^{\mathrm{T}}, \ldots, (x_{N^{(l)}}^l)^{\mathrm{T}}]^{\mathrm{T}} \in R^{N^{(l)}n}, \quad l=1,2,\ldots,L;$$
$$U = [(u^1)^{\mathrm{T}}, (u^2)^{\mathrm{T}}, \ldots, (u^L)^{\mathrm{T}}]^{\mathrm{T}} \in R^{Nm},$$

where

$$u^l = [(u_1^l)^{\mathrm{T}}, (u_2^l)^{\mathrm{T}}, \ldots, (u_{N^{(l)}}^l)^{\mathrm{T}}]^{\mathrm{T}} \in R^{N^{(l)}m}, \quad l=1,2,\ldots,L;$$
$$M = [(m^1)^{\mathrm{T}}, (m^2)^{\mathrm{T}}, \ldots, (m^L)^{\mathrm{T}}]^{\mathrm{T}} \in R^{Nn},$$

where

$$m^l = [(m_1^l)^{\mathrm{T}}, (m_2^l)^{\mathrm{T}}, \ldots, (m_{N^{(l)}}^l)^{\mathrm{T}}]^{\mathrm{T}} \in R^{N^{(l)}n}, \quad l=1,2,\ldots,L;$$
$$\tilde{f}(X(t)) = [(\tilde{f}(x^1))^{\mathrm{T}}, \tilde{f}(x^2))^{\mathrm{T}}, \ldots, \tilde{f}(x^L))^{\mathrm{T}} \in R^{Nn},$$

where

$$\tilde{f}(x^l) = [(\tilde{f}(x_1^l))^{\mathrm{T}}, (\tilde{f}(x_2^l))^{\mathrm{T}}, \ldots, (\tilde{f}(x_{N^{(l)}}^l))^{\mathrm{T}}]^{\mathrm{T}} \in R^{N^{(l)}n},$$
$$l=1,2,\ldots,L.$$

$$\varphi = \begin{bmatrix} A^1 + D^1\otimes\Gamma^1 + D^{12}\otimes\Gamma^{12} + \cdots + D^{1L}\otimes\Gamma^{1L} \\ A^2 + D^2\otimes\Gamma^2 + D^{21}\otimes\Gamma^{21} + \cdots + D^{2L}\otimes\Gamma^{2L} \\ \vdots \\ A^L + D^L\otimes\Gamma^L + D^{L1}\otimes\Gamma^{L1} + \cdots + D^{L(L-1)}\otimes\Gamma^{L(L-1)} \end{bmatrix} \in R^{Nn\times Nn},$$

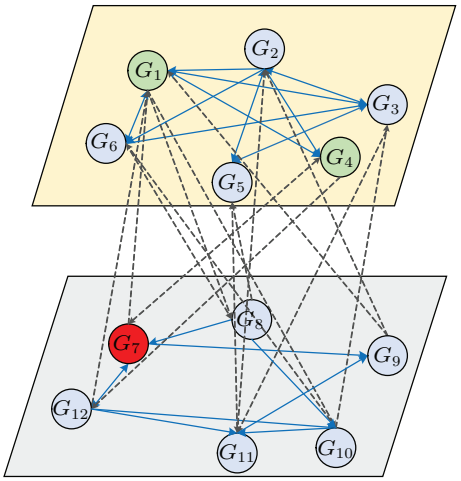


Fig. 1. The two-layer complex dynamical network to study the target layer state estimation.

where

$$A^l = \mathrm{diag}\{A_1^l, A_2^l, \ldots, A_{N^{(l)}}^l\}, \quad l = 1,2,\ldots,L,$$
$$\psi = \mathrm{diag}\{B^1, B^2, \ldots, B^L\},$$

where

$$B^l = \mathrm{diag}\{\sigma_1^l B_1^l, \sigma_2^l B_2^l, \ldots, \sigma_{N^{(l)}}^l B_{N^{(l)}}^l\}, \quad l = 1,2,\ldots,L.$$

σ_i^l represents the connection relationship between external control inputs and nodes of node i on layer l.

$$W = \mathrm{diag}\{w^1, w^2, \ldots, w^L\},$$

where

$$w^l = \mathrm{diag}\{\varsigma_1^l w_1^l, \varsigma_2^l w_2^l, \ldots, \varsigma_{N^{(l)}}^l w_{N^{(l)}}^l\}, \quad l = 1,2,\ldots,L.$$

ς_i^l represents the connection relationship between external disturbance and nodes of node i on layer l.

Without loss of generality, we take a two-layer complex dynamical network with two-dimensional node dynamics as an illustrative example which is shown in Fig. 1. Assume that on the first layer of the network, the first-dimensional and second-dimensional state components of node 1 and the second-dimensional state component of node 4 can be measured by sensors. Then one has

$$\xi = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}. \tag{3}$$

In large-scale networks, it is not necessary for us to obtain the state estimation of all nodes. Only a series of state components in target nodes will be focused on. Defined a target matrix $F \in R^{r\times Nn}$, the partial state components in target nodes $z(t) = FX(t) \in R^r$, where r can be much smaller than Nn. In Fig. 1, we suppose that the first-dimensional of node 7 is of interest. Then we have

$$F = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Our objective is to estimate the state information of state components in target nodes $z(t)$ from the limited number of measured nodes.

Remark 1 If the target nodes are distributed in each layer of the network, the proposed method is still working. For example, assume that the second-dimensional of node 6 on the first layer and the first-dimensional of node 7 on the second layer are target nodes. Then the matrix F will be set as follows:

$$F = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

3. Design functional state observer

To reconstruct the target nodes $z(t) = FX(t)$, we develop the following functional state observer which is described in Eq. (4).

$$\begin{cases} \dot{q}(t) = Sq(t) + JY(t) + K\left(\tilde{f}'\begin{bmatrix} Y(t) \\ \hat{z}(t) \end{bmatrix}\right) + K\psi U(t), \\ \hat{z}(t) = Dq(t) + EY(t), \end{cases} \tag{4}$$

where $q(t)$ is the observer state vector which has a low order, S, J, D, E, and $\tilde{f}'$ are unknown and need to be determined such that $\hat{z}(t)$ converges asymptotically to $z(t)$, K is the observer gain matrix.

Define the error between $q(t)$ and $KX(t)$ as $e_1(t) = q(t) - KX(t)$, then

$$\begin{aligned} \dot{e}_1(t) &= \dot{q}(t) - K\dot{X}(t) \\ &= Se_1(t) + (SK + J\phi - K\varphi)X(t) \\ &\quad + K\left(\tilde{f}'\begin{bmatrix} Y(t) \\ \hat{z}(t) \end{bmatrix} - \tilde{f}\begin{bmatrix} Y(t) \\ \hat{z}(t) \end{bmatrix}\right) - KWM(t). \end{aligned} \tag{5}$$

Define the error between $\hat{z}(t)$ and $FX(t)$ as $e_2(t) = \hat{z}(t) - FX(t)$, then

$$e_2(t) = \hat{z}(t) - FX(t)$$

$$= De_1(t) + (DK + E\phi - F)X(t). \tag{6}$$

Theorem 1 According to $e_1(t)$ and $e_2(t)$, the estimate $\hat{z}(t)$ will converge asymptotically to $z(t)$ if and only if the following conditions hold.

Condition 1 The following matrix equations in Eq. (7) hold:

$$\begin{cases} S \text{ is Hurwitz}, \\ SK + J\phi - K\varphi = 0, \\ KW = 0, \\ F - DK - E\phi = 0. \end{cases} \tag{7}$$

Condition 2 Let $e(t) = e_1(t)$, The error $e(t)$ determined by the following observer error in system (8):

$$\dot{e}(t) = Se(t) + K\left(\tilde{f}'\begin{bmatrix} Y(t) \\ \hat{z}(t) \end{bmatrix} - \tilde{f}\begin{bmatrix} Y(t) \\ \hat{z}(t) \end{bmatrix}\right) \tag{8}$$

converges asymptotically to zero.

4. Solvability and stability of the functional state observer

In this section, We analyze the solvability and stability of the functional state observer referring to the method in (Ref. [25]).

4.1. Solvability of the functional state observer

In this subsection, we calculate the parameters S, J, D, E, K from Eq. (7).

Define the invertible matrix $P = [\phi^+,\ \phi^\perp]$, ϕ^+ is the Moore–Penrose inverse of ϕ ($\phi\phi^+ = I$) and $\phi^\perp$ is the orthogonal complement of the row space of ϕ ($\phi\phi^\perp = 0$).

$$\begin{aligned} &P^{-1}\varphi P = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}, \quad P\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \\ &P^{-1}W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \\ &FP = \begin{bmatrix} F_1 & F_2 \end{bmatrix}, \quad KP = \begin{bmatrix} K_1 & K_2 \end{bmatrix}. \end{aligned} \tag{9}$$

Combine with Eq. (9), equation (7) can be simplified to the following expressions

$$SK_2 - K_1\varphi_{12} - K_2\varphi_{22} = 0, \tag{10a}$$
$$J = K_1\varphi_{11} + K_2\varphi_{21} - SK_1, \tag{10b}$$
$$K_1W_1 = -K_2W_2, \tag{10c}$$
$$E = F_1 - DK_1, \tag{10d}$$
$$F_2 = DK_2, \tag{10e}$$
$$D = I. \tag{10f}$$

Combining with Eqs. (10c), (10e), and (10f), we have

$$K_1W_1 = -F_2W_2. \tag{11}$$

Lemma 1 $K_1W_1 = -F_2W_2$ in Eq. (11) has a solution for K_1 if and only if

$$\operatorname{rank}\begin{bmatrix} W_1 \\ F_2W_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} W_1 \end{bmatrix}.$$

Remark 2 Lemma 1 is a necessary condition for the existence of the functional state observer. Since F_2, W_1, and W_2 are known matrices, Lemma 1 can be tested easily. If Lemma 1 is not satisfied, one can declare that there does not exist any solution to the parameters S, J, D, E, K. Therefore, the weight matrix of external disturbance W cannot be randomly selected and its selection needs to meet Lemma 1.

If Lemma 1 is satisfied, one can proceed as follows. Let F^+ be the Moore–Penrose inverse of F and $F^\perp$ is the orthogonal complement of the row space of F. It is clear that $[F^+,\ F^\perp]$ is a square and invertible matrix. Let Ω, Φ, S_1, S_2, K_1, K_2 be defined according to the following equations:

$$S_1 = (\Phi\Omega^+\varphi_{12} + F_2\varphi_{22})F_2^+, \tag{12a}$$
$$S_2 = (\Omega\Omega^+ - I)\varphi_{12}F_2^+, \tag{12b}$$
$$K_1 = \Phi\Omega^+ + Z(I - \Omega\Omega^+), \tag{12c}$$
$$K_2 = F_2, \tag{12d}$$
$$\Omega = [\varphi_{12}F_2^+\ W_1], \tag{12e}$$
$$\Phi = -[F_2\varphi_{22}F_2^\perp\ F_2W_2], \tag{12f}$$
$$S = S_1 - ZS_2, \tag{12g}$$

where Z is an arbitrary matrix in Eq. (12g) which can be determined in Condition 2.

Combining with Eqs. (12c) and (12d), the observer gain K is as follows:

$$K = [K_1,\ K_2]P^{-1}. \tag{13}$$

Lemma 2 $S = S_1 - ZS_2$ must be Hurwitz, otherwise, the functional observer does not exist.

Remark 3 An appropriate functional state observer can be designed if and only if Lemma 2 is satisfied. As we can see from Eqs. (12a)–(12g). The solutions of S_1 and S_2 are mainly related to the system matrix φ, output matrix ϕ, and target matrix F. Therefore, the matrix φ, ϕ, and F need to satisfy certain numerical relations for the existence of the functional state observer. For the networks with linear node dynamics, the research in (Ref. [27]) developed a graph-theoretic method to select auxiliary nodes to place observers, making it effective to design a functional state observer. However, the graph-theoretic method is not suitable for nonlinear systems. For further research, we hope to study a method to find the appropriate matrix φ, ϕ, and F for designing the functional state observer in the network with nonlinear node dynamics.

From the above discussion, S, J, D, E, K can be calculated from Eqs. (12g), (10b), (10d), (10f), and (13).

4.2. Stability of the functional state observer

We then analyze the stability of the error $e(t)$ in system (8).

Let

$$P^{-1} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.$$

From Eq. (13), we have

$$K = K_1 P_1 + K_2 P_2. \tag{14}$$

By substituting Eqs. (12c) and (12d) into Eq. (14). Then

$$K = \overline{K}_1 - Z\overline{K}_2, \tag{15}$$

where $\overline{K}_1 = \Phi\Omega^+ P_1 + K_2 P_2$, $\overline{K}_2 = (\Omega\Omega^+ - I)P_1$.

Substituting Eqs. (12g) and (15) into the error $e(t)$ in system (8), one can express $e(t)$ as follows:

$$\begin{aligned} e(t) \;=\; & (S_1 - ZS_2)e(t) \\ & + (\overline{K}_1 - Z\overline{K}_2)K\left(\tilde{f}'\begin{bmatrix} Y(t) \\ \hat{z}(t)\end{bmatrix} - \tilde{f}\begin{bmatrix} Y(t) \\ \hat{z}(t)\end{bmatrix}\right). \end{aligned} \tag{16}$$

Assumption 1 We assume that for functions $\tilde{f}'(\xi)$, there exists positive constant γ satisfying

$$\|\tilde{f}'(\xi) - \tilde{f}'(\hat{\xi})\| \le \gamma\|\xi - \hat{\xi}\|,$$

where γ is a positive real number and $\xi(t) = \begin{bmatrix} Y(t) \\ z(t)\end{bmatrix}$.

Lemma 3 For any vectors δ and η, the inequality $\pm(\delta^{\mathrm{T}}\eta + \eta^{\mathrm{T}}\delta \le \beta\eta\eta^{\mathrm{T}} + (1/\beta)\delta\delta^{\mathrm{T}})$ holds for any β.

We develop the following Lyapunov function:

$$V(t) = e^{\mathrm{T}}(t)Qe(t),$$

where $Q = Q^{\mathrm{T}} > 0$, the derivative of $V(t)$ can be obtained as follows:

$$\begin{aligned} \dot{V}(t) \;=\; & e^{\mathrm{T}}(t)[Q(S_1 - ZS_2) + (S_1 - ZS_2)^{\mathrm{T}}Q]e(t) \\ & + e^{\mathrm{T}}(t)Q(\overline{K}_1 - Z\overline{K}_2)\left(\tilde{f}'\begin{bmatrix} Y(t) \\ \hat{z}(t)\end{bmatrix} - \tilde{f}\begin{bmatrix} Y(t) \\ \hat{z}(t)\end{bmatrix}\right) \\ & + \left(\tilde{f}'\begin{bmatrix} Y(t) \\ \hat{z}(t)\end{bmatrix} - \tilde{f}\begin{bmatrix} Y(t) \\ \hat{z}(t)\end{bmatrix}\right)^{\mathrm{T}}(\overline{K}_1 - Z\overline{K}_2)^{\mathrm{T}}Qe(t). \end{aligned} \tag{17}$$

According to Assumption 1, equation (17) can be expressed as

$$\begin{aligned} \dot{V}(t) \;\le\; & e^{\mathrm{T}}(t)[Q(S_1 - ZS_2) + (S_1 - ZS_2)^{\mathrm{T}}Q]e(t) \\ & + e^{\mathrm{T}}(t)Q(\overline{K}_1 - Z\overline{K}_2)\gamma e(t) \\ & + e^{\mathrm{T}}(t)\gamma^{\mathrm{T}}(\overline{K}_1 - Z\overline{K}_2)^{\mathrm{T}}Qe(t) \\ \le\; & e^{\mathrm{T}}(t)[Q(S_1 - ZS_2) + (S_1 - ZS_2)^{\mathrm{T}}Q]e(t) \\ & + e^{\mathrm{T}}(t)Q\overline{K}_1\gamma e(t) - e^{\mathrm{T}}(t)QZ\overline{K}_2\gamma e(t) \\ & + e^{\mathrm{T}}(t)\gamma^{\mathrm{T}}(\overline{K}_1)^{\mathrm{T}}Qe(t) - e^{\mathrm{T}}(t)\gamma^{\mathrm{T}}(Z\overline{K}_2)^{\mathrm{T}}Qe(t) \\ =\; & e^{\mathrm{T}}(t)[Q(S_1 - ZS_2) + (S_1 - ZS_2)^{\mathrm{T}}Q]e(t) \\ & + e^{\mathrm{T}}(t)Q\overline{K}_1\gamma e(t) + e^{\mathrm{T}}(t)\gamma^{\mathrm{T}}(Q\overline{K}_1)^{\mathrm{T}}e(t) \\ & - (e^{\mathrm{T}}(t)QZ\overline{K}_2\gamma e(t) + e^{\mathrm{T}}(t)\gamma^{\mathrm{T}}(QZ\overline{K}_2)^{\mathrm{T}}e(t)). \end{aligned}$$

According to Lemma 3, then we have

$$\dot{V}(t) \le e^{\mathrm{T}}(t)\varepsilon e(t),$$

where

$$\begin{aligned} \varepsilon \;=\; & Q(S_1 - ZS_2) + (S_1 - ZS_2)^{\mathrm{T}}Q + \frac{1}{\beta_1}Q\overline{K}_1\overline{K}_1^{\mathrm{T}}Q \\ & + \frac{1}{\beta_2}QZ\overline{K}_2\overline{K}_2^{\mathrm{T}}Z^{\mathrm{T}}Q + \gamma^2(\beta_1 + \beta_2)I. \end{aligned}$$

By utilizing the Schur decomposition[32] and defining $G = QZ$, we can derive the following linear matrix inequality (LMI):

$$\begin{bmatrix} \Delta & Q\overline{K}_1 & G\overline{K}_2 \\ \overline{K}_1^{\mathrm{T}}Q & -\beta_1 I & 0 \\ \overline{K}_2^{\mathrm{T}}G^{\mathrm{T}} & 0 & -\beta_2 I \end{bmatrix} < 0, \tag{18}$$

where $\Delta = QS_1 + S_1^{\mathrm{T}}Q - GS_2 - S_2^{\mathrm{T}}G^{\mathrm{T}} + \gamma^2(\beta_1 + \beta_2)I$, $Z = Q^{-1}G$.

Theorem 2 The error $e(t)$ in Eq. (8) converges asymptotically to zero if and only if there exists a positive definite matrix $Q = Q^{\mathrm{T}} > 0$, a matrix G and two positive numbers β_1 and β_2 such that the LMI in system (8) holds.

If Theorem 2 is satisfied, we can prove the stability of the designed functional state observer.

5. Numerical simulation

In this section, we take a three-layer complex dynamical network with three-dimensional nonlinear node dynamics whose topology is shown in Fig. 2 as an example to verify the accuracy of the proposed method. The configuration matrix of the inter-layer network is shown in Eqs. (21)–(23). The input matrix of each layer $B_i^l = [1,\ 2,\ 3]^{\mathrm{T}}$ and the weight matrix of the disturbance on each layer can be described as $w_i^l = [2,\ 0,\ 0.5]^{\mathrm{T}}$. The external disturbance $m_i^l = 0.5(x_{1,2} + x_{3,1})$. The linear part of the node dynamics on each layer can be described as

$$A_i^l = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 0 \\ -1 & 0 & -3 \end{bmatrix}. \tag{19}$$

The nonlinear part of the node dynamics on each layer is characterized as

$$\begin{cases} \dot{x}_{i,1} = 0, \\ \dot{x}_{i,2} = 0.5\sin(x_{i,2}), \\ \dot{x}_{i,3} = 2\cos(x_{i,1} + x_{i,3}). \end{cases} \tag{20}$$

$$d_{ij}^{12}=\begin{bmatrix}0&0&0&1&0&0&1&0\\0&0&1&0&0&0&0&0\\0&1&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&1&0&0&1&0&0&1\\0&0&0&0&0&0&1&0\\0&1&0&0&1&0&0&0\\0&0&0&1&0&0&0&1\end{bmatrix},\quad d_{ij}^{13}=\begin{bmatrix}0&0&1&0&0&0&0&1\\0&0&0&1&0&0&0&0\\0&0&0&1&1&0&0&0\\0&1&0&0&0&0&0&1\\0&0&0&0&0&1&0&1\\0&0&0&0&0&0&1&1\\0&0&1&0&0&0&0&0\\0&1&0&0&0&0&0&0\end{bmatrix},\tag{21}$$

$$d_{ij}^{21}=\begin{bmatrix}0&0&0&0&1&0&1&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\end{bmatrix},\quad d_{ij}^{23}=\begin{bmatrix}0&0&1&0&0&0&0&0\\1&0&0&0&0&0&1&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\end{bmatrix},\tag{22}$$

$$d_{ij}^{31}=\begin{bmatrix}0&1&0&0&0&0&0&1\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&1&0\\0&0&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\end{bmatrix},\quad d_{ij}^{32}=\begin{bmatrix}1&0&0&0&0&0&1&0\\1&0&0&1&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&1&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&1&0&0&0&0&0&0\end{bmatrix}.\tag{23}$$

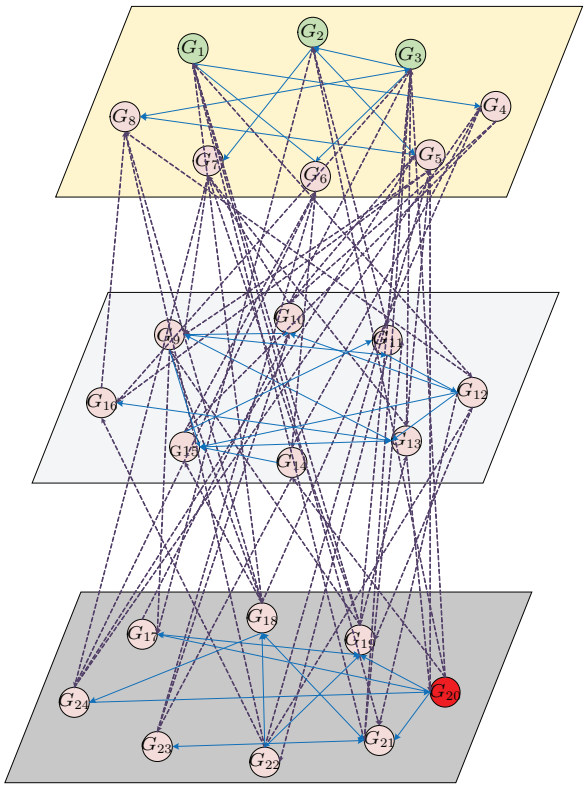


Fig. 2. The three-layer complex dynamical network to study the target layer state estimation.

The first-dimensional, second-dimensional, and third-dimensional state components of node 1, the first-dimensional and third-dimensional state components of node 2 and node 3 on the first layer marked in green are sensor components, their state information can be obtained directly. The third-dimensional state component of node 20 marked in red on the third layer is the target component to be estimated.

We obtain the parameters S, J, D, E, K of the designed functional observer from the method proposed in Section 3. Then we have $S=-7.6797$, $J=[0.381,\ 0,\ 0.122,\ 0.4981,\ 0.7497,\ -0.3021,\ -0.3691]$, $D=1$, $E=[0,\ 0,\ 0.381,\ -0.0886,\ -0.1139,\ 0.4444,\ 0.2353]$, $K=[0,0,-0.381,0.1139,0,-0.4444,0.0886,0,-0.2353,0,\ldots,0]_{1\times 72}$.

The estimation performance of the component in the target node from the method is shown in Fig. 3. It displays clearly that by adopting the proposed method, the state component in the target node on the third layer can be estimated successfully. The target state estimation error also converges to zero in Fig. 4. The method only need to place observers on the state component in the target node instead of all unknown components, therefore, we can reduce the number of required observers and computational costs.

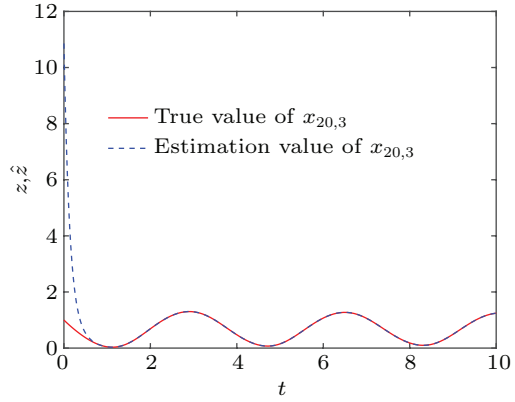


Fig. 3. The estimation performance of the target node from the proposed method.

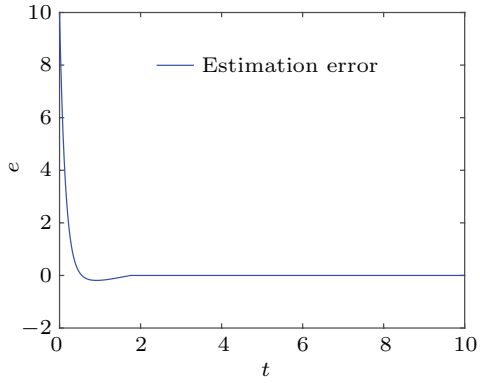


Fig. 4. The estimation error of the target node from the proposed method.

6. Conclusion

In this paper, we studied the state estimation of target nodes on some layers by using some known state information of nodes on another layer in the multi-layer complex dynamical network with nonlinear node dynamics. We assume that

some state components can be measured on one layer. The information of target state components on other layers needs to be estimated in the multi-layer complex network. Then a functional state observer is constructed according to the original model, we calculated the parameters of the observer by using algebraic methods and verified its stability from the Lyapunov theorem. The conditions that need to be satisfied for the design of the functional state observer have been obtained. From numerical simulations, we find that the proposed method has an excellent performance. The designed nonlinear functional observers estimate the state of target components on some layers directly without needing to place observers on all the individual state components, which can greatly reduce the complexity and computational cost of the designed observers. Our further research, includes: (i) We hope to find an algorithm to select the appropriate system matrix φ, output matrix ϕ, and target matrix F for designing the functional state observer in the network with nonlinear node dynamics. (ii) We hope to apply the proposed method to practical networks, such as multi-agent systems, power grids, and other real networks.

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References

[1] Wang X, Wang X F, Su H and Jam L 2022 *Automatica* **135** 109989
[2] Kebina M, Cao X, Hu F and Liu Y 2014 *IEEE Trans. Control Netw. Syst.* **1** 370
[3] Li R, Song Y R and Jiang G P 2021 *Chin. Phys. B* **30** 120202
[4] Fan D, Jiang G P, Song Y and Zhang X 2019 *Physica A* **533** 122028
[5] Cai Y, Cao Y, Li Y, Huang T and Zhou B 2016 *IEEE Trans. Smart Grid* **7** 530
[6] Lai Q and Zhang H 2022 *Chin. Phys. B* **31** 068905
[7] Li N, Sun H, Jing X and Chen Z 2021 *Chin. Phys. B* **30** 090507
[8] Giorgio B and Luigi C 2016 *Automatica* **68** 169
[9] Li W, Jia Y and Du J 2017 *IEEE Trans. Automat. Contr.* **62** 6377
[10] Li W, Jia Y and Du J 2017 *Neurocomputing* **219** 1
[11] Wu X, Jiang G P and Wang X 2018 *IEEE Trans. Circuits Syst. II* **65** 1753
[12] Zhang M, Wang X J, Jin L, Song M and Wang X 2020 *Chin. Phys. B* **29** 096401
[13] Zhang Y, Hong Y, Mohsen G, Wu S, Zhang P and Liu R 2023 *IEEE Trans. Veh. Technol.* **10** 1
[14] Xiao Y, Sun Z, Shi G and Dusit N 2023 *IEEE J. Area. Comm.* **41** 639
[15] Li W, Zhou J, Li J, Xie T and Lu J A 2021 *IEEE Trans. Circuits Syst. II* **68** 1338
[16] Jin Y L, Han Q Y, Guo R Z, Gao Y and Shen L Q 2023 *Chin. Phys. B* **32** 100507
[17] Wang Y, Wu X, Lv J H, Lu J A and Raissa M D S 2020 *IEEE Trans. Netw. Sci. Eng.* **7** 538
[18] Mei G, Wu X, Wang Y, Hu M, Lu J A and Chen G R 2018 *IEEE Trans. Cybern.* **48** 754
[19] Wu J, Li Y and Chen G R 2020 *IEEE Trans. Circuits Syst. I* **67** 5211
[20] Jiang G P, Li K and Wang X 2021 *33rd Chinese Control and Decision Conference*, May 22–24, 2021, Kunming, China
[21] Li K, Wang X and Jiang G P 2021 *40th Chinese Control Conference*, July 26–28, 2021, Shanghai, China
[22] Xiong F, Liu Y and Cheng J 2017 *Commun. Nonlinear Sci.* **44** 513
[23] Motter A E 2015 *Chaos* **25** 097621
[24] Darouach M 2000 *IEEE Trans. Automat. Contr.* **45** 940
[25] Trinh H and Fernando T 2012 *Functional Observers for Dynamical Systems* (Berlin, Heidelberg: Springer Verlag)
[26] Liu Y Y, Slotine J J and Barabasi A L 2011 *Nature* **473** 167
[27] Arthur N M, Duan C, Aguirre L A and Motter A E 2021 *Proc. Natl. Acad, Sci. USA* **119** e2113750119
[28] Wu Y, Wang X, Jiang G P and Gu M 2023 *J. Franklin I.* **360** 8178
[29] Wang L, Lu D, Zhang Y and Wang X 2018 *Sensors* **18** 3434
[30] Matthew P A, Scott A S, Kate J H, Christopher M B, Matthew H H, Michaela P, Jacinta H, Kerrie L M and Eve M M 2020 *Ecol. Lett.* **23** 607
[31] Yang H, Tang W K S, Chen G R and Jiang G P 2017 *IEEE Trans. Circuits Syst. I* **64** 2182
[32] Stephen B, Laurent El G, Eric F and Venkataramanan B 1994 *Linear Matrix Inequalities in System and Control Theory* (Philadelphia: Society for Industrial and Applied Mathematics)