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Decompositions of the Kadomtsev–Petviashvili equation and their symmetry reductions

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Starting with a decomposition conjecture, we carefully explain the basic decompositions for the Kadomtsev–Petviashvili (KP) equation as well as the necessary calculation procedures, and it is shown that the KP equation allows the Burgers–STO (BSTO) decomposition, two types of reducible coupled BSTO decompositions and the BSTO–KdV decomposition. Furthermore, we concentrate ourselves on pointing out the main idea and result of Bäcklund transformation of the KP equation based on a special superposition principle in the particular context of the BSTO decompositions. Using the framework of standard Lie point symmetry theory, these decompositions are studied and the problem of computing the corresponding symmetry constraints is treated.

Keywords: Kadomtsev–Petviashvili (KP) equation, decomposition, Bäcklund transformation, symmetry reduction

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1. Introduction

Nonlinear systems, fundamental to mathematical physics, often display intricate behaviors due to the interconnectedness of their components. One such behavior is the violation of the linear superposition principle, a foundational concept in linear systems. However, research has demonstrated that many specific nonlinear systems adhere to the superposition principle under certain conditions.[1–10] This balance in nonlinear effects is attributed to the interactions between multiple nonlinear terms present in the system.[11–14]

In the context of a linear system, the presence of linear superposition implies a corresponding decomposition for the system. Furthermore, the nonlinearization[15–17] of eigenvalue problems of an integrable system inherently provides a decomposition for the original system. This observation suggests that, due to the presence of nonlinear effects, it is reasonable to consider decompositions directly for an integrable system, potentially revealing certain superpositions. A recent study[18] adopts this approach, applying it to the B-type Kadomtsev–Petviashvili equation hierarchy, where various linear superpositions can be derived from appropriate decompositions for that nonlinear system.

This paper investigates a fascinating phenomenon where certain types of nonlinear systems, specifically the potential Kadomtsev–Petviashvili (KP) equation ($u_x = v \to$ KP equations for v)

$$u_{xt} + u_{xxxx} + 6u_x u_{xx} + 3\gamma^2 u_{yy} = 0, \tag{1}$$

where γ is a nonzero constant can be decomposed into either a Burgers–Sharma–Tasso–Olver (BSTO) system or a potential KdV system. Further arguments also reveal the relation between those decompositions and the Bäcklund transformation of the KP equation.

The remainder of this paper is divided into two sections. The first section outlines the process of constructing various decompositions of the potential KP equation and presents a Bäcklund transformation theorem by some special decompositions; such a transformation leads to the desired superposition. The second section revisits these specific decompositions and examines the corresponding symmetry reductions.

2. Solution theorems

The recent seminal work[18] introduces a so-called formally variable separation approach (FVSA). This method generally offers a means to decompose a $(2+1)$-dimensional nonlinear system, even in the absence of lax pairs.[19,20]

Specifically, for any $(2+1)$-dimensional nonlinear system with its solution $w(x,y,t)$, it is reasonable to conjecture formally that w possesses some decompositions as follows:

$$\begin{cases} w_y = F\left(w, w_x, w_{xx}, \ldots, w_{mx}\right), \\ w_t = G\left(w, w_x, w_{xx}, \ldots, w_{nx}\right), \end{cases} \tag{2}$$

where the consistency condition

$$w_{yt} - w_{ty} = 0 \tag{3}$$

is satisfied. Here, we use abbreviations $w_x = \partial_x w$, $w_{xx} = \partial_x^2 w, \ldots, w_{mx} = \partial_x^m w$ for positive integer m. To derive the specific forms for such decompositions, we require them to be consistent with the original nonlinear system.

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To construct the decomposition of the potential KP equation, we apply the FVSA to Eq. (1) as described above. The following theorems summarize the results.

Theorem 1 Let v be either a solution of the BSTO equation system

$$\begin{aligned} &\gamma v_y + v_{xx} + (v - C_1) v_x = 0, \\ &v_t + \left[4v_{xx} + 6(v - C_1) v_x + (v - C_1)^3\right]_x = 0, \end{aligned} \tag{4}$$

where C_1 is an arbitrary constant, or a solution of the potential KdV system

$$\begin{aligned} &\gamma v_y - C_1 v_x + C_2 = 0, \\ &v_t + 3v_x^2 + v_{xxx} + 3C_1^2 v_x + C_3 = 0, \end{aligned} \tag{5}$$

where C_1, C_2, and C_3 are arbitrary constants. Then $u = v$ is a solution of the potential KP equation (1).

Proof First we directly substitute the formal decomposition (2) into the potential KP equation (1) and the additional consistent condition (3). After some calculations, it is found that m and n are related with $n = 2m - 1$ and m has to be fixed as $m = 2$. Thus, the formal decompositions of the potential KP equation now become

$$\begin{cases} u_y = F(u, u_x, u_{x2}), \\ u_t = G(u, u_x, u_{x2}, u_{x3}). \end{cases} \tag{6}$$

Substituting the reduced decomposition (6) into Eq. (1), it is found that

$$\xi_4 \left(3\gamma^2 F_{\xi_2}^2 + G_{\xi_3} + 1\right) + W(\xi_0, \xi_1, \xi_2, \xi_3) = 0, \tag{7}$$

where $W \equiv W(\xi_0, \xi_1, \xi_2, \xi_3)$ is a complicated ξ_4-independent function. Here $\xi_m \equiv u_{mx}$ for convenience. Obviously, the coefficient of ξ_4 must be vanished, and the only possible form of G reads

$$G = -3\gamma^2 F_{\xi_2}^2 \xi_3 - \xi_3 + G_1(\xi_0, \xi_1, \xi_2), \tag{8}$$

where $G_1 \equiv G_1(\xi_0, \xi_1, \xi_2)$ is an arbitrary function independent of ξ_3 and ξ_4. With this condition (8), Eq. (7) now becomes

$$\begin{aligned} &-(\xi_3)^2 3\gamma^2 F_{\xi_2} F_{\xi_2\xi_2} + \xi_3 \left(6\gamma^2 F_{\xi_1} F_{\xi_2}\right) \\ &+\xi_3 G_{1_{\xi_2}} + W_1(\xi_0, \xi_1, \xi_2) = 0, \end{aligned} \tag{9}$$

where $W_1 \equiv W_1(\xi_0, \xi_1, \xi_2)$ is a ξ_3-independent function. Vanishing the coefficients of ξ_3^2 and ξ_3, respectively, yields

$$F(\xi_0, \xi_1, \xi_2) = F_1(\xi_0, \xi_1)\xi_2 + H(\xi_0, \xi_1), \tag{10}$$

$$\begin{aligned} G_1(\xi_0, \xi_1, \xi_2) = &-(\xi_2)^2 3\gamma^2 F_{1_{\xi_1}} F_1 - \xi_2 6\gamma^2 H_{\xi_1} F_1 \\ &+ G_2(\xi_0, \xi_1). \end{aligned} \tag{11}$$

Here the functions $F_1 \equiv F_1(\xi_0, \xi_1)$, $H \equiv H(\xi_0, \xi_1)$, and $G_2 \equiv G_2(\xi_0, \xi_1)$ are all dependent at most on ξ_0 and ξ_1.

Furthermore, by substituting Eq. (6) and the relations mentioned in preceding paragraph into the consistent condition Eq. (3), we obtain that

$$\begin{aligned} &3\xi_4 \left(1 - \gamma^2 F_1^2\right) \left(\xi_1 F_{1_{\xi_0}} + \xi_2 F_{1_{\xi_1}}\right) \\ &+W_2(\xi_0, \xi_1, \xi_2, \xi_3) = 0. \end{aligned} \tag{12}$$

Observe that F_1 is independent of ξ_2, it follows by the elimination of the coefficients of $\xi_4\xi_2$ and $\xi_4\xi_1$ for Eq. (12) that

$$F_1(\xi_0, \xi_1) = C_1, \tag{13}$$

for an arbitrary constant C_1.

We now turn back to Eq. (12) with a substitution of Eq. (13). The result can be expressed by nine determined equations as follows:

$$\begin{aligned} &H_{\xi_1\xi_1\xi_1} = 0, \\ &-3C_1\gamma^2 H_{\xi_1\xi_1} = 0, \\ &3\gamma^2 C_1^2 H_{\xi_1\xi_1} + 3H_{\xi_1\xi_1} = 0, \\ &6\gamma^2 C_1 H_{\xi_0} + 3\gamma^2 H_{\xi_1}^2 + G_{2_{\xi_1}} + 6\xi_1 = 0, \\ &3\xi_1\gamma^2 C_1 H_{\xi_0\xi_0} + 3\xi_1\gamma^2 H_{\xi_0\xi_1} + 3\gamma^2 H_{\xi_0} H + \xi_1 G_{2_{\xi_0}} = 0, \\ &3\xi_1 H_{\xi_0\xi_1\xi_1} + 3\gamma^2 C_1 H_{\xi_1\xi_1} H_{\xi_1} + 3\gamma^2 C_1^2 H_{\xi_0\xi_1} + 3H_{\xi_0\xi_1} - 6C_1 = 0, \\ &3\xi_1\gamma^2 C_1^2 H_{\xi_0\xi_1} - 3\xi_1 H_{\xi_0\xi_1} + 3\gamma^2 C_1 H_{\xi_1}^2 \\ &+6\gamma^2 C_1^2 H_{\xi_0} + \xi_1 G_{2_{\xi_1}} + 6\xi_1 C_1 = 0, \\ &\xi_1^3 H_{\xi_0\xi_0\xi_0} - 3\xi_1^2\gamma^2 C_1 H_{\xi_0\xi_1} H_{\xi_0} - 3\xi_1\gamma^2 H_{\xi_1}^2 H_{\xi_0} \\ &+3\xi_1\gamma^2 C_1 H_{\xi_0\xi_0} H - 3\xi_1\gamma^2 C_1 H_{\xi_0}^2 - 3\gamma^2 H_{\xi_1} H_{\xi_0} H \\ &-\xi_1 H_{\xi_0} G_{2_{\xi_1}} + H_{\xi_0} G_2 - H G_{2_{\xi_0}} = 0, \\ &3\xi_1^2 H_{\xi_0\xi_0\xi_1} + 3\xi_1\gamma^2 C_1 H_{\xi_1\xi_1} H_{\xi_0} - 3\xi_1\gamma^2 C_1 H_{\xi_0\xi_1} H_{\xi_0} \\ &-3\xi_1\gamma^2 C_1^2 H_{\xi_0\xi_0} + 3\xi_1 H_{\xi_0\xi_0} - 6\xi_1 H_{\xi_1} - 3\gamma^2 H_{\xi_1}^3 \\ &+3\gamma^2 C_1 H_{\xi_0\xi_1} H - 12\gamma^2 C_1 H_{\xi_1} H_{\xi_0} - H_{\xi_1} G_{2_{\xi_1}} - C_1 G_{2_{\xi_0}} = 0. \end{aligned}$$

The calculations would be trivial and yield two distinct alternatives in general, where the specific forms of H and G_2 read

$$\begin{cases} H(\xi_0, \xi_1) &= \frac{1}{\gamma}(C_1 - \xi_0)\xi_1, \\ G_2(\xi_0, \xi_1) &= -3\left[(C_1 - \xi_0)^2 + 2\xi_1\right]\xi_1, \end{cases} \tag{14}$$

$$\begin{cases} H(\xi_0, \xi_1) &= \frac{1}{\gamma}(C_1\xi_1 + C_2), \\ G_2(\xi_0, \xi_1) &= -3\xi_1^2 - 3C_1^2\xi_1 + C_3. \end{cases} \tag{15}$$

BSTO system (4) is produced by Eq. (14), and potential KdV system (5) is produced by Eq. (15), with all necessary equations we obtained before, respectively, as desired.

The former part Eq. (4) of this theorem aligns with the result obtained by Liu for the KP equation.[21] In fact, these two equations correspond to the well-known Burgers equation and the Sharma–Tasso–Olver (STO) equation,[22–25] respectively. The latter part Eq. (5) of this theorem is associated with a $(2+1)$-dimensional potential KdV system.[26,27]

For certain valid reasons, we may recast the FSVA method in a more general form that also considers some undetermined Bäcklund transformations for the potential KP equation (1). Let u and v both be solutions of the potential KP equation (1) and suppose that u and v satisfy the following two decompositions:

$$\begin{cases} u_y = F\left(u, u_x, u_{2x}, v, v_x, v_{3x}\right), \\ u_t = G\left(u, u_x, u_{2x}, u_{3x}, v, v_x, v_{2x}, v_{3x}\right), \end{cases} \tag{16}$$

$$\begin{cases} v_y = M\left(u, u_x, u_{2x}, v, v_x, v_{2x}\right), \\ v_t = N\left(u, u_x, u_{2x}, u_{3x}, v, v_x, v_{2x}, v_{3x}\right), \end{cases} \tag{17}$$

with the consistent condition

$$u_{yt} - u_{ty} = v_{yt} - v_{ty} = 0. \tag{18}$$

Such decompositions of u and v provide us with the following two theorems.

Theorem 2 Let v be a solution of the potential KP equation (1) with the BSTO decomposition (4) in Theorem 1. If u meets either the variable coefficient BSTO system

$$\begin{aligned} &\gamma u_y - u_{xx} - (u - C_2)\, u_x + \left[(u - C_2)\,(v - C_1)\right]_x = 0, \\ &u_t + \Big[4u_{xx} + 6\,(u - v + C_1 - C_2)\, u_x \\ &+ (u - v + C_1 - C_2)^3 + (v - C_1)^3\Big]_x = 0, \end{aligned} \tag{19}$$

where C_2 is an arbitrary constant, or the variable coefficient BSTO system

$$\begin{aligned} &\gamma u_y + u_{xx} + (u + C_2)\, u_x - \left[(u + C_2)\,(v + C_1)\right]_x \\ &+ 2v_{xx} + 2vv_x = 0, \\ &u_t + \Big[4u_{xx} + 6\,(u - v - C_1 + C_2)\, u_x + (u - v - C_1 + C_2)^3 \\ &+ (v + C_1)^3\Big]_x + 12\,(\partial_x - C_1)\,(v_{xx} + vv_x) = 0, \end{aligned} \tag{20}$$

where C_2 is an arbitrary constant, then u is a solution of the potential KP equation (1).

Proof For proving this theorem, we may just assume that v meets Eq. (4) in Theorem 2. Again u_{mx} and v_{nx} are simply abbreviated by ξ_m and η_n for positive integers m and n. First we substitute Eqs. (16) and (17) to potential KP equation (1) and consistent condition (18), together with our assumption. By the elimination of the coefficients ξ_4 and η_4, it leads to six alternatives. But if we get rid of the cases which eventually indicate again Eq. (4) or Eq. (5), only two different options remain. Namely,

$$\begin{aligned} &\text{(i)} \quad F = \frac{\xi_2}{\gamma} + F_1\left(\xi_0, \xi_1, \eta_0, \eta_1, \eta_2\right), \\ &\qquad G = -4\xi_3 + G_1\left(\xi_0, \xi_1, \xi_2, \eta_0, \eta_1, \eta_2\right), \\ &\text{(ii)} \quad F = \frac{\xi_2}{\gamma} + F_1\left(\xi_0, \xi_1, \eta_0, \eta_1, \eta_2\right), \\ &\qquad G = -4\xi_3 + 6\gamma F_{1_{\eta_2}}\eta_3 + G_1\left(\xi_0, \xi_1, \xi_2, \eta_0, \eta_1, \eta_2\right). \end{aligned} \tag{21}$$

Substituting the above two equations into potential KP equation (1) and consistent condition (18), respectively. After some cumbersome (but similar to what we did in the proof of Theorem 1), the results ensure straightforwardly that Eqs. (19) and (20) are the desired decompositions.

We can verify the integrability for these two variable BSTO systems in the sense of the existence of lax pairs. As being related to the Cole–Hopf transformation[28,29] $u = 2(\ln\psi)_x$, one can easily calculate to obtain the weak lax triple of Eq. (19) as follows:

$$\begin{aligned} &\psi_x = \frac{1}{2}\,(u - \lambda)\,\psi, \\ &\gamma\psi_y = \psi_{xx} + (C_1 - C_2 - v + \lambda)\,\psi_x + \frac{1}{2}\,(C_2 - \lambda)\,v\psi, \\ &\psi_t = -4\psi_{xxx} - 6\,(C_1 - C_2 - v + \lambda)\,\psi_{xx} \\ &\qquad - 3\,(C_1 - C_2 - v + \lambda)^2\,\psi_x \\ &\qquad - \frac{3}{2}\,(C_2 - \lambda)\,(2C_1 - C_2 - v + \lambda)\,v\psi, \end{aligned} \tag{22}$$

where λ is the spectral parameter. The weakness of lax triple here means that the variable coefficient BSTO system (19) can be produced by the consistency condition

$$\left[\psi_{xt} - \psi_{tx}\right]|_{(22)} = \left[\psi_{yt} - \psi_{ty}\right]|_{(22)} = \left[\psi_{xy} - \psi_{yx}\right]|_{(22)} = 0. \tag{23}$$

Note that spectral function ψ must satisfy the specific restriction (22) in this event. Furthermore, a strong lax triple of Eq. (19) also exists and possesses the form

$$\begin{aligned} &L\psi = \psi_x + \frac{1}{2}\,(u + \lambda)\,\psi, \\ &M\psi = \gamma\psi_y - \psi_{xx} - u\psi_x - \frac{1}{2}\left[(v - C_1 + C_2)\,u - C_2 v\right]\psi, \\ &N\psi = \psi_t + 4\psi_{xxx} + 6u\psi_{xx} + 3\left(u^2 + 2u_x\right)\psi_x \\ &\qquad + \frac{3}{2}\Big[(v - C_1 + C_2)\left(u^2 + 2u_x\right) + (v - C_1 + C_2)^2\,u \\ &\qquad + C_2\,(v - 2C_1 + C_2)\,v\Big]\psi. \end{aligned} \tag{24}$$

One can directly obtain the variable coefficient BSTO system (19) by the compatibility condition

$$[L, M] = [M, N] = [N, L] = 0, \tag{25}$$

in which the linear operators L, M, and N are defined by Eq. (24). Such a relation does not rely on the specific form of the spectral function ψ.[30]

Similarly, Eq. (20) possesses the weak lax triple

$$\begin{aligned} &\psi_x = \frac{1}{2}\,(u - \lambda)\,\psi, \\ &\gamma\psi_y = -\psi_{xx} + (C_1 - C_2 + v - \lambda)\,\psi_x \\ &\qquad + \frac{1}{2}\left(C_2 v - v^2 - 2v_x + \lambda v\right)\psi, \\ &\psi_t = -4\psi_{xxx} + 6\,(C_1 - C_2 + v - \lambda)\,\psi_{xx} \\ &\qquad - 3\,(C_1 - C_2 + v - \lambda)^2\,\psi_x \end{aligned}$$

$$-\left[\frac{3}{2}(C_2+\lambda)(2C_1-C_2+v-\lambda)v +3(\partial_x-C_1)\left(v^2+2v_x\right)\right]\psi, \tag{26}$$

where λ is the spectral parameter. And the strong lax triple has the form of

$$\begin{aligned}
L\psi &= \psi_x+\frac{1}{2}(u+\lambda)\psi,\\
M\psi &= \gamma\psi_y+\psi_{xx}+u\psi_x\\
&\quad -\frac{1}{2}\left[2v_x-(v+C_1-C_2)u+v^2-C_2v\right]\psi,\\
N\psi &= \psi_t+4\psi_{xxx}+6u\psi_{xx}+3\left(u^2+2u_x\right)\psi_x\\
&\quad -\frac{3}{2}\left[4v_{xx}+4(v-C_1)v_x-2(v+C_1-C_2)u_x\right.\\
&\quad -(u-v+C_2)(u-2C_1+C_2)v\\
&\quad \left.-(C_1-C_2)(u-C_1+C_2)u\right]\psi.
\end{aligned} \tag{27}$$

The variable coefficient BSTO system (20) can be obtained by the compatibility condition (25) together with Eq. (27) directly.

Theorem 3 Let v be a solution of the potential KdV system (5) in Theorem 1. If u meets the variable coefficient BSTO system

$$\begin{aligned}
&\gamma u_y+u_{xx}+(u-C_4)u_x-[(u-C_4)(v+C_1)]_x\\
&+vv_x+v_{xx}+C_2=0,\\
&u_t+\left[4u_{xx}+3v_{xx}+6(u-v-C_1-C_4)u_x+(u-v-C_1-C_4)^3\right]_x\\
&+3C_1(C_1v_x-2v_{xx})+3v_x^2+C_3=0,
\end{aligned} \tag{28}$$

where C_4 is an arbitrary constant, then u is also a solution of the potential KP equation (1).

Proof Let v, as introduced in Eq. (17), be the solution of Eq. (5). Then a repeat of the procedure that was used to prove Theorem 2, would finally give Eq. (28), as desired.

System (28) is also an integrable system. The corresponding weak lax triple possesses the following form:

$$\begin{aligned}
\psi_x &= \frac{1}{2}(u-\lambda)\psi,\\
\gamma\psi_y &= -\psi_{xx}+(C_1+C_4+v-\lambda)\psi_x\\
&\quad -\frac{1}{2}\left(v_x+C_4v+\frac{1}{2}v^2-\lambda v-C_2x\right)\psi,\\
\psi_t &= -4\psi_{xxx}+6(C_1+C_4+v-\lambda)\psi_{xx}\\
&\quad -3(C_1+C_4+v-\lambda)^2\psi_x\\
&\quad +\frac{3}{2}\left[-v_{xx}+(C_1+C_4-\lambda)(C_1+C_4+v-\lambda)v\right.\\
&\quad \left.-C_1(C_1v-2v_x)-\int v_x^2\mathrm{d}x+\frac{v^3}{3}+\frac{C_3x}{3}\right]\psi,
\end{aligned} \tag{29}$$

where λ is the spectral parameter. And if an additional restriction $C_3=-3C_1C_2$ is accepted, we then have a strong lax triple for Eq. (28) as follows:

$$\begin{aligned}
L\psi &= \psi_x+\frac{1}{2}(u+\lambda)\psi,\\
M\psi &= \gamma\psi_y+\psi_{xx}+u\psi_x-\frac{1}{2}\left[v_x-(C_1+C_4+v)u\right.\\
&\quad \left.+\frac{1}{2}v^2+C_4v-C_2x\right]\psi,\\
N\psi &= \psi_t+4\psi_{xxx}+6u\psi_{xx}+3\left(u^2+2u_x\right)\psi_x\\
&\quad -\frac{3}{2}\left[v_{xx}-2(v+C_1+C_4)u_x\right.\\
&\quad -2C_1v_x(v-u+C_1+C_4)(v+C_1+C_4)u\\
&\quad -(C_1+C_4)v^2-(2C_1+C_4)C_4v\\
&\quad \left.-\frac{v^3}{3}+\int v_x^2\mathrm{d}x-\frac{C_3x}{3}\right]\psi.
\end{aligned} \tag{30}$$

Without loss of generality, one can validate the aforementioned three theorems directly by substituting them into the potential KP equation, respectively. These theorems all yield Bäcklund transformations for the potential KP equation (1). As a corollary of the above theorems, we can generally formulate the following theorem.

Theorem 4 (Bäcklund transformation theorem) Let u and v be functions satisfying the coupled equations system

$$\begin{aligned}
&\gamma(u-v)_y-(u+v)_{xx}-(u-v)(u-v)_x-C(u-v)_x=0,\\
&\left\{\partial_t+\partial_x^3+3(u+v)_{xx}+3\left[(u+v)_x+(u-v)^2\right]\partial_x\right\}(u-v)\\
&+3\gamma(u+v)_{xy}+3C\{[C+2(u-v)](u-v)_x+(u+v)_{xx}\}=0,
\end{aligned} \tag{31}$$

in which C is an arbitrary constant. Then u is a solution of the potential KP equation (1) iff v is a solution of the potential KP equation (1).

Proof The idea is to consider the combination of Eqs. (4) and (19). Suppose that both u and v meet Eq. (31). By substituting Eq. (31) into potential KP equation (1) directly, one can check that if u is a solution of Eq. (1), then so is v. The converse is immediate by symmetry.

This theorem is the same as the result obtained by Chen,[31] Levi,[32,33] *et al.* It indeed indicates a simple superposition principle for the potential KP equation. Precisely speaking, consider four distinct solutions u_0, u_1, u_2, u_{12}, and u_{21} for Eq. (31) with

$$\begin{aligned}
&\gamma(u_1-u_0)_y-(u_1+u_0)_{xx}-(u_1-u_0)(u_1-u_0)_x\\
&-C_1(u_1-u_0)_x=0,
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\gamma(u_{12}-u_1)_y-(u_3+u_1)_{xx}-(u_{12}-u_1)(u_{12}-u_1)_x\\
&-C_2(u_{12}-u_1)_x=0,
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\gamma(u_2-u_0)_y-(u_2+u_0)_{xx}-(u_2-u_0)(u_2-u_0)_x\\
&-C_2(u_2-u_0)_x=0,
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\gamma(u_{21}-u_2)_y-(u_{21}+u_2)_{xx}-(u_{21}-u_2)(u_{21}-u_2)_x\\
&-C_1(u_{21}-u_2)_x=0.
\end{aligned} \tag{35}$$

According to the Bianchi's permutability theorem,[34] we have here that $u_{12}=u_{21}\equiv u_3$. By some suitable linear combinations for these four equations, it follows that

$$2\left(u_2-u_1\right)_{xx}=\partial_x\left[\left(u_2-u_1-C_1+C_2\right)\left(u_1+u_2-u_0-u_3\right)\right]. \tag{36}$$

Consequently, we calculate by integrating Eq. (36) with respect to x once to obtain a desired explicit superposition formula for the potential KP equation as follows:

$$\begin{aligned} u_3 \;=\; & -u_0+u_1+u_2+2\left[\ln\left(u_1-u_2+C_1-C_2\right)\right]_x \\ & +\frac{F\left(y,t\right)}{u_1-u_2+C_1-C_2}, \end{aligned} \tag{37}$$

where $F\equiv F\left(y,t\right)$ is a function preserving the latter equation in Eq. (31). Indeed, we can similarly consider the corresponding t-parts for Eqs. (32)–(35), i.e.,

$$\begin{aligned} &\left\{\partial_t+\partial_x^3+3\left(u_1+u_0\right)_{xx}+3\left[\left(u_1+u_0\right)_x+\left(u_1-u_0\right)^2\right]\partial_x\right\} \\ &\times\left(u_1-u_0\right)+3\gamma\left(u_1+u_0\right)_{xy} \\ &+3C_1\left\{\left[C_1+2\left(u_1-u_0\right)\right]\left(u_1-u_0\right)_x+\left(u_1+u_0\right)_{xx}\right\}=0, \end{aligned} \tag{38}$$

$$\begin{aligned} &\left\{\partial_t+\partial_x^3+3\left(u_3+u_1\right)_{xx}+3\left[\left(u_3+u_1\right)_x+\left(u_3-u_1\right)^2\right]\partial_x\right\} \\ &\times\left(u_3-u_1\right)+3\gamma\left(u_3+u_1\right)_{xy} \\ &+3C_2\left\{\left[C_2+2\left(u_3-u_1\right)\right]\left(u_3-u_1\right)_x+\left(u_3+u_1\right)_{xx}\right\}=0, \end{aligned} \tag{39}$$

$$\begin{aligned} &\left\{\partial_t+\partial_x^3+3\left(u_2+u_0\right)_{xx}+3\left[\left(u_2+u_0\right)_x+\left(u_2-u_0\right)^2\right]\partial_x\right\} \\ &\times\left(u_2-u_0\right)+3\gamma\left(u_2+u_0\right)_{xy} \\ &+3C_2\left\{\left[C_2+2\left(u_2-u_0\right)\right]\left(u_2-u_0\right)_x+\left(u_2+u_0\right)_{xx}\right\}=0, \end{aligned} \tag{40}$$

$$\begin{aligned} &\left\{\partial_t+\partial_x^3+3\left(u_3+u_2\right)_{xx}+3\left[\left(u_3+u_2\right)_x+\left(u_3-u_2\right)^2\right]\partial_x\right\} \\ &\times\left(u_3-u_2\right)+3\gamma\left(u_3+u_2\right)_{xy} \\ &+3C_1\left\{\left[C_1+2\left(u_3-u_2\right)\right]\left(u_3-u_2\right)_x+\left(u_3+u_2\right)_{xx}\right\}=0. \end{aligned} \tag{41}$$

We combine these equations in the same way as the y-parts and the result shows that F must meet the constraint

$$\begin{aligned} F\cdot\Big[&\left(u_2-u_1\right)_x+\frac{1}{2}\left(u_1-u_2+C_1-C_2\right) \\ &\times\left(2u_0-u_1-u_2-C_1-C_2\right)+\frac{1}{2}F\Big]=0. \end{aligned} \tag{42}$$

With the superposition principle (37), we can now construct a series of novel solutions for the potential KP equation in a remarkably straightforward manner. More precisely, if we have three solutions u_0, u_1, and u_2 for Eq. (1) such that u_0 and u_1 are solutions for the coupled equation system (31) with parameter $C=C_1$, and that u_0 and u_2 are solutions for the coupled equation system (31) with parameter $C=C_2$, then Eq. (37) provides a new solution for Eq. (1) distinct from the three initial solutions.

Next, we will apply this superposition principle to derive some novel solutions for the potential KP equation. Observe that with the trivial solution $v=0$, Eq. (31) reduces to Eq. (4). Therefore, it is convenient to consider, for example, any two solutions u_1 and u_2 for Eq. (4) with $u_0=0$. From this, in order to find some solutions for the potential KP equation (1), it suffices to apply Eq. (37) to solutions of Eq. (4).

For instance, the standard truncated Painlevé method[22,23] reveals that Eq. (4) has the following multiple soliton solution:

$$v=2\ln\left(\phi\right)_x, \tag{43}$$

where ϕ takes the form of

$$\phi=1+\sum_{i=1}^{N}\exp\left(k_ix+r_iy-\omega_it+\xi_i\right), \tag{44}$$

with the wave numbers k_i, r_i, the frequencies ω_i, and the original positions ξ_i. We can also confirm that the dispersion relations here satisfy that

$$r_i=-\frac{k_i^2}{\gamma},\quad \omega_i=4k_i^3,\quad i=1,\ldots,N. \tag{45}$$

To utilize Eq. (37), we initially set $u_0=0$. Let u_1 be a single-soliton solution (taking $N=1$ in Eq. (44)) with wave number k_1 and let u_2 be a two-soliton solution (taking $N=2$ in Eq. (44)) with wave numbers k_2 and k_3, respectively. Then we must have that both C_1 and C_2 in Eq. (37) are zero because of the fact that these two solutions should satisfy Eq. (4) since $u_0=0$. In this situation, it follows that the superposition u_3 obtained from Eq. (37) remains a solution for the potential KP equation (we here take the arbitrary function F to be zero). Figure 1 shows a specific instance with the parameters setting: $k_1=-0.5$, $k_2=0.2$, $k_3=0.4$, $\xi_1=0$, $\xi_2=5$, $\xi_3=10$, and $\gamma=1$, at different times.

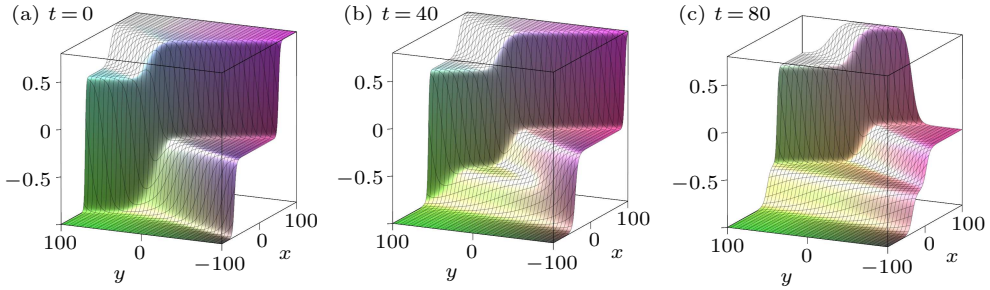


Fig. 1. These three pictures are all obtained by the parameters setting $k_1=-0.5$, $k_2=0.2$, $k_3=0.4$, $\xi_1=0$, $\xi_2=5$, $\xi_3=10$, and $\gamma=1$, but occur at distinct time.

3. Symmetry reduction

In the preceding section, we obtain three specific novel decompositions for the potential KP equation (1) (i.e., those three variable coefficients systems in Theorems 2 and 3). They are all integrable systems because of the existence of lax triples. In this section, we study the symmetries of these three systems, and give the corresponding symmetry constraints, respectively.

Recall that a symmetry $\boldsymbol{\sigma}$ of a partial differential equation system

$$P(x_i, u_{x_i}, u_{x_i x_j}, \ldots) \equiv P(u) = 0,$$

can be defined as a solution of its linearized equation

$$P'\boldsymbol{\sigma} \equiv \frac{\mathrm{d}}{\mathrm{d}\varepsilon} P(u+\varepsilon\boldsymbol{\sigma})|_{\varepsilon=0} = 0,$$

which means that the system is invariant under the transformation

$$u \to u + \varepsilon\boldsymbol{\sigma},$$

with an infinitesimal ε.

We start with the variable coefficient system (19) in which v agrees with Eq. (4). The corresponding symmetry of this system is

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix},$$

which should satisfy the following equations:

$$\begin{cases} \sigma^v_{xx} + \gamma\sigma^v_y + (v-C_1)\,\sigma^v_x + v_x\sigma^v = 0, \\ 4\sigma^v_{xxx} + \sigma^v_t + 6(v-C_1)\,\sigma^v_{xx} + 3\left[(v-C_1)^2 + 4v_x\right]\sigma^v_x \\ +6\left[(v-C_1)\,v_x + v_{xx}\right]\sigma^v = 0, \\ -\sigma^u_{xx} + \gamma\sigma^u_y - \left[(u-v+C_1-C_2)\,\sigma^u\right]_x + \left[(u-C_2)\,\sigma^v\right]_x = 0, \\ 4\sigma^u_{xxx} + 6(u-v+C_1-C_2)\,\sigma^u_{xx} + \sigma^u_t \\ +3\left[(u-v+C_1-C_2)^2 - 2v_x + 4u_x\right]\sigma^u_x \\ +3\left[(u-C_2)(2v-u-2C_1+C_2) - 2u_x\right]\sigma^v_x \\ +6\left[(u-v+C_1-C_2)(u_x-v_x) + u_{xx}\right]\sigma^u \\ -6\left[(u-v+C_1-C_2)\,u_x - (u-C_2)\,v_x + u_{xx}\right]\sigma^v = 0. \end{cases} \tag{46}$$

According to the standard Lie point symmetry method,[35–37] we may assume that the general Lie symmetries possess the form

$$\begin{cases} \sigma^u = \xi(x,y,t,u,v)u_x + \eta(x,y,t,u,v)u_y \\ \qquad + \tau(x,y,t,u,v)u_t - r(x,y,t,u,v), \\ \sigma^v = \xi(x,y,t,u,v)v_x + \eta(x,y,t,u,v)v_y \\ \qquad + \tau(x,y,t,u,v)v_t - r(x,y,t,u,v), \end{cases} \tag{47}$$

where ξ, η, τ, r and s are functions to be determined with respect to $\{x,y,t,u,v\}$. We next substitute them into the original symmetry Eq. (46). With some complex but straightforward calculations, one can obtain that

$$\begin{cases} \sigma^u = (-3c_1t + c_5)\,u_t + \left(-c_1x + \dfrac{c_2y}{\gamma} + c_4\right)u_x \\ \qquad + (-6\gamma c_2t - 2c_1y + c_3)\,u_y - c_1(u - C_2), \\ \sigma^v = (-3c_1t + c_5)\,v_t + \left(-c_1x + \dfrac{c_2y}{\gamma} + c_4\right)v_x \\ \qquad + (-6\gamma c_2t - 2c_1y + c_3)\,v_y - \left[c_1(v - C_1) + c_2\right], \end{cases} \tag{48}$$

where c_i $(i = 1,2,3,4,5)$ are all arbitrary constants.

To obtain the corresponding symmetry reductions, we consider the symmetry constraint equation

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix} = 0.$$

Or equivalently, to solve the characteristic equations

$$\begin{aligned} \frac{\mathrm{d}t}{-3c_1t + c_5} &= \frac{\mathrm{d}x}{-c_1x + \gamma^{-1}c_2y + c_4} \\ &= \frac{\mathrm{d}y}{-6\gamma c_2t - 2c_1y + c_3} \\ &= \frac{\mathrm{d}u}{c_1(u - C_2)} = \frac{\mathrm{d}v}{c_1(v - C_1) + c_2}. \end{aligned}$$

The result shows that there are four special cases

(i) $c_1 \neq 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, and $c_5 = 0$;
(ii) $c_1 = 0$, $c_2 \neq 0$, $c_3 = 0$, $c_4 = 0$, and $c_5 \neq 0$;
(iii) $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 \neq 0$, and $c_5 = 0$;
(iv) $c_1 = 0$, $c_2 = 0$, $c_3 \neq 0$, $c_4 = 0$, and $c_5 = 0$.

The first case gives a so-called scaling symmetry, which leads to four group invariants as

$$\begin{aligned} &a = yx^{-2}, \quad b = tx^{-3}, \\ &U(a,b) = (u - C_2)\,x, \quad V(a,b) = (v - C_1)\,x, \end{aligned} \tag{49}$$

and hence we conclude that the group invariant solutions for the system of Eqs. (4) and (19) possess the form

$$u = C_2 + \frac{F(a,b)}{x}, \quad v = C_1 + \frac{G(a,b)}{x}, \tag{50}$$

where F and G are undetermined functions with respect to the group invariants a and b as shown in Eq. (49). Note that it doesn't matter what the specific value we select for c_1, the invariants a and b will eventually be of such forms. The decided procedure for the specific forms of F and G can be completed by substituting Eq. (50) into the original system Eqs. (4) and (19). The result shows that F and G must satisfy the following equations:

$$\begin{aligned} &8\gamma G_{aa}a + 12\gamma G_{ab}b + 8G_a^2a^2 + 18G_b^2a^2 + 24G_aG_bab \\ &+ \left[-2G^2a + (8a - 2\gamma)\,G + 12\gamma\right]G_a \\ &+ \left(-3G^2b + 12Gb + 1\right)G_b - G^3 + 2G^2 = 0, \\ &18\gamma F_{bb}b^2 + 12\gamma F_{ab}ab + 12F_aG_ba^2b + 24F_aF_ba^2b \\ &+ 8F_aG_ba^3 + 18F_bG_bab^2 + 12F_bG_aa^2b + \left[-2\left(F^2 - G^2\right)a^2\right. \\ &\left. + 2(2Ga + 4a - \gamma)\,Fa + 2(2a + \gamma)\,Ga + 2\gamma(4a - \gamma)\right]F_a \end{aligned}$$

$$
\begin{aligned}
&+\left[-3F^2ab+6\left(Ga+2a-\gamma\right)Fb-3G^2ab+6\left(a+\gamma\right)Gb\right.\\
&\left.+36\gamma b+a\right]F_b+8F_a^2a^3+2F\left(F+2\right)G_aa^2\\
&+\left[F^2a+\left(-3Ga-2a+2\gamma\right)F+G^2a-2\left(a+2\gamma\right)G-4\gamma\right]F\\
&+3F\left(Fa+2a+2\gamma\right)G_bb+18F_b^2ab^2=0.
\end{aligned}
$$

For the second case, one can easily see that it is related to a rotation group. And some similar arguments show that the corresponding group invariant solution is of the form

$$u=F(a,b),\quad v=t+G(a,b),\tag{51}$$

where a and b read

$$a=-3\gamma t^2-y,\quad b=-2t^3-\gamma^{-1}yt+x,$$

if c_2 and c_5 have both been set to 1 for convenience. And the constraint equations, obtained by substituting Eq. (51) into the system of Eqs. (4) and (19), read as

$$
\begin{aligned}
&4\gamma G_{ab}+2G_b^2+2\gamma\left(G-C_1\right)G_a+\left[\left(G-C_1\right)^2+\frac{a}{\gamma}\right]G_b+1=0,\\
&-4\gamma F_{ab}+2F_bG_b+2F_b^2-2\gamma\left(F-G+C_1-C_2\right)F_a\\
&+4\gamma\left(F-C_2\right)G_a-\left(F-C_2\right)^2G_b\\
&+\left[\left(F-G+C_1-C_2\right)^2+\frac{a}{\gamma}\right]F_b=0.
\end{aligned}
$$

Case (iii) demonstrates the translation group on x, and trivially indicates an uninteresting case that both u and v are constants.

The remaining case (i.e., case (iv)) is related to the translation invariance with y. The result shows that u and v are constrained to $(1+1)$-dimensional situation. In such a case, we have that

$$u=F(x,t),\quad v=G(x,t),\tag{52}$$

and

$$
\begin{aligned}
&G_t+2G_x^2+\left(C_1-G\right)^2G_x=0,\\
&F_t+2F_x^2+\left[2G_x+\left(F-G+C_1-C_2\right)^2\right]F_x\\
&-\left(F-C_2\right)^2G_x=0.
\end{aligned}
$$

We next discuss the variable coefficient system (20) in Theorem 2. Again, we can find that the symmetry of such a system must follow the four equations:

$$
\begin{cases}
\sigma^v_{xx}+\gamma\sigma^v_y+\left(v-C_1\right)\sigma^v_x+v_x\sigma^v=0,\\
4\sigma^v_{xxx}+\sigma^v_t+6\left(v-C_1\right)\sigma^v_{xx}+3\left[\left(v-C_1\right)^2+4v_x\right]\sigma^v_x\\
+6\left[\left(v-C_1\right)v_x+v_{xx}\right]\sigma^v=0,\\
\sigma^u_{xx}+2\sigma^v_{xx}+\gamma\sigma^u_y+\left[\left(u-v-C_1+C_2\right)\sigma^u\right]_x\\
-\left[\left(u-2v+C_2\right)\sigma^v\right]_x=0,\\
4\sigma^u_{xxx}+12\sigma^v_{xxx}+6\left(u-v-C_1+C_2\right)\sigma^u_{xx}+12\left(v-C_1\right)\sigma^v_{xx}\\
+\sigma^u_t+3\left[4u_x-2v_x+\left(u-v-C_1+C_2\right)^2\right]\sigma^u_x\\
-3\left[2u_x-8v_x+\left(u-2v+C_2\right)\left(u-2C_1+C_2\right)\right]\sigma^v_x\\
+6\left[\left(u-v-C_1+C_2\right)\left(u_x-v_x\right)+u_{xx}\right]\sigma^u-6\left[\left(u-v\right.\right.\\
\left.\left.-C_1+C_2\right)u_x-\left(u-2C_1+C_2\right)v_x+u_{xx}-2v_{xx}\right]\sigma^v=0.
\end{cases}\tag{53}
$$

And after some complicated computation, the result shows that

$$
\begin{cases}
\sigma^u=\left(c_1t+c_2\right)u_t+\left(\dfrac{c_1x}{3}-\dfrac{c_3y}{6\gamma^2}+c_5\right)u_x\\
\quad+\left(\dfrac{2c_1y}{3}+c_3t+c_4\right)u_y+\dfrac{c_1\left(u-2C_1+C_2\right)\gamma+c_3}{3\gamma},\\
\sigma^v=\left(c_1t+c_2\right)v_t+\left(\dfrac{c_1x}{3}-\dfrac{c_3y}{6\gamma^2}+c_5\right)v_x\\
\quad+\left(\dfrac{2c_1y}{3}+c_3t+c_4\right)v_y+\dfrac{2c_1\left(v-C_1\right)\gamma+c_3}{6\gamma},
\end{cases}\tag{54}
$$

where c_i $(i=1,2,3,4,5)$ are all arbitrary constants. Furthermore, it is clear that the above formulas, when set to zero, are equivalent to the characteristic equations

$$
\begin{aligned}
\frac{\mathrm{d}t}{c_1t+c_2}&=\frac{6\gamma^2\mathrm{d}x}{2\gamma^2c_1x-c_3y+6\gamma^2c_5}=\frac{3\mathrm{d}y}{2c_1y+3c_3t+3c_4}\\
&=\frac{3\gamma\mathrm{d}u}{-c_1\left(u-2C_1+C_2\right)\gamma-c_3}\\
&=\frac{6\gamma\mathrm{d}v}{-2c_1\left(v-C_1\right)\gamma-c_3}.
\end{aligned}
$$

The corresponding symmetry reductions need to be discussed in four different alternatives listed as follows:

(i) $c_1\neq0$, $c_2=0$, $c_3=0$, $c_4=0$, and $c_5=0$;

(ii) $c_1=0$, $c_2\neq0$, $c_3\neq0$, $c_4=0$, and $c_5=0$;

(iii) $c_1=0$, $c_2=0$, $c_3=0$, $c_4=0$, and $c_5\neq0$;

(iv) $c_1=0$, $c_2=0$, $c_3=0$, $c_4\neq0$, and $c_5=0$.

For the first case, we solve the characteristic equations to obtain that

$$u=2C_1-C_2+\frac{F(a,b)}{x},\quad v=C_1+\frac{G(a,b)}{x},\tag{55}$$

in which $a=yx^{-2}$ and $b=tx^{-3}$, are similar to the first case we discussed in Eq. (4). The arbitrary functions F and G must satisfy the following equations:

$$
\begin{aligned}
&8\gamma G_{aa}a+12\gamma G_{ab}b+8G_a^2a^2+18G_b^2b^2\\
&+\left[24G_bab-2G^2a+2\left(4a-\gamma\right)G+12\gamma\right]G_a\\
&+\left(-3G^2b+12Gb+1\right)G_b+2G^2=0,\\
&8\gamma\left(F_{aa}+G_{aa}\right)a+12\gamma\left(F_{ab}+G_{ab}\right)b+8F_a^2a^2+18G_b^2b^2\\
&+\left[24F_bab+8G_aa^2+12G_bab-2F^2a+2\left(2Ga+4a-\gamma\right)F\right.\\
&\left.-2G^2a+2\left(2a+\gamma\right)G+12\gamma\right]F_a+\left[12G_aab+18G_bb^2\right.\\
&\left.-3F^2b+6\left(G+2\right)bF-3G^2b+6Gb+1\right]F_b\\
&+\left[2F^2a+4Fa-4\gamma\left(G-3\right)\right]G_a+3b\left(F+2\right)FG_b\\
&-\left[F^2-\left(3G-2\right)F+G^2-2G\right]F=0.
\end{aligned}
$$

For v in Eq. (55) should be certainly the solution of the BSTO system (4) and u in Eq. (55) should be the solution of the variable coefficient BSTO system (20).

The above discussion is based on the scaling transformation as well as we see in the case of Eq. (4), and the consequence seems to be similar except for the specific forms of F and G. It is reasonable since the different parts between these two theorems make certainly no formal effects on the scaling symmetry.

For the remaining cases, we eliminate the scaling symmetry. Case (ii) leads to the group invariant solutions related to the rotation symmetry. For a special instance, let $c_2 = c_3 = 1$, then the group invariant solutions possess the following forms:

$$u = -\frac{t}{3\gamma} + F(a,b), \quad v = -\frac{t}{6\gamma} + G(a,b), \tag{56}$$

in which

$$a = t^2 - 2y, \quad b = -\frac{t^3}{18\gamma^2} + \frac{ty}{6\gamma^2} + x. \tag{57}$$

To make it possible that u and v are certainly the solutions of the original Eqs. (4) and (20), respectively, we substitute Eq. (56) into these equations to obtain the implicit equations for u and v, namely,

$$\begin{aligned}
&8\gamma G_{ab} + 2G_b^2 + 4\gamma(G - C_1)G_a \\
&+ \left[(G - C_1)^2 - \frac{a}{12\gamma^2}\right] G_b - \frac{1}{6\gamma} = 0, \\
&8\gamma(F_{ab} + G_{ab}) + 4\gamma(F - G - C_1 + C_2)F_a + 2G_bF_b \\
&+ \left[(F - G - C_1 + C_2)^2 - \frac{a}{12\gamma^2}\right] F_b + 2F_b^2 - 8\gamma(C_1 - G)G_a \\
&- (F - 2C_1 + C_2)^2 G_b - \frac{1}{3\gamma} = 0.
\end{aligned}$$

As for case (iii), we again obtain an x-translation-invariant solution as trivial as the similar situation we discussed in Eq. (19). The last case (case (iv)) is related to the translation invariance on y-axis of the PDE systems (4) and (20). It follows that

$$u = F(x,t), \quad v = G(x,t), \tag{58}$$

where F and G are arbitrary functions satisfying that

$$\begin{aligned}
&G_t + 2G_x^2 + (C_1 - G)^2 G_x = 0, \\
&F_t + 2F_x^2 + \left[2G_x + (F - G - C_1 + C_2)^2\right] F_x \\
&- (F - 2C_1 + C_2)^2 G_x = 0.
\end{aligned}$$

Finally we consider Theorem 3. Assume similarly that the symmetry of our system, Eqs. (5) and (28), is of the form

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix},$$

where σ^u and σ^v are the symmetries, related to u and v, being of the form (47), respectively. We first list the corresponding symmetry equations for this system as follows:

$$\begin{cases}
C_1\sigma_x^v - \gamma\sigma_y^v = 0, \\
\sigma_t^v + 3\left(2v_x + C_1^2\right)\sigma_x^v + \sigma_{xxx}^v = 0, \\
\sigma_{xx}^u + \sigma_{xx}^v + \gamma\sigma_y^u + \left[(u - v - C_1 - C_4)\,\sigma^u\right]_x \\
- \left[(u - v - C_4)\,\sigma^v\right]_x = 0, \\
4\sigma_{xxx}^u + 3\sigma_{xxx}^v + 6(u - v - C_1 - C_4)\,\sigma_{xx}^u - 6C_1\sigma_{xx}^v + \sigma_t^u \\
+3\left[4u_x - 2v_x + (u - v - C_1 - C_4)^2\right]\sigma_x^u \\
-3\left[2u_x - 2v_x + (u - v - 2C_1 - C_4)(u - v - C_4)\right]\sigma_x^v \\
+6\left[u_{xx} + (u - v - C_1 - C_4)(u_x - v_x)\right](\sigma^u - \sigma^v) = 0.
\end{cases} \tag{59}$$

Again, by substituting Eq. (47) into the above equations, we can obtain that

$$\begin{cases}
\sigma^u = (c_1t + c_2)\,u_t + \left(\dfrac{c_1x}{3} - \dfrac{c_1C_1y}{3\gamma} + c_3t + c_4\right)u_x + \left(\dfrac{2c_1(3C_1\gamma t + y)}{3} + c_5\right)u_y \\
\qquad - \dfrac{\left[2(6C_1C_2t + 4C_3t + C_4 - u)c_1 + \left(x - 3C_1^2t\right)c_3 + 6c_6\right]\gamma + (6C_2c_1 + C_1c_3)y}{6\gamma}, \\
\sigma^v = (c_1t + c_2)\,v_t + \left(\dfrac{c_1x}{3} - \dfrac{c_1C_1y}{3\gamma} + c_3t + c_4\right)v_x + \left(\dfrac{2c_1(3C_1\gamma t + y)}{3} + c_5\right)v_y \\
\qquad - \dfrac{\left[2(6C_1C_2t + 4C_3t - v)c_1 + \left(x - 3C_1^2t\right)c_3 + 6c_6\right]\gamma + (6C_2c_1 + C_1c_3)y}{6\gamma},
\end{cases} \tag{60}$$

where c_i $(i = 1,2,3,4,5,6)$ are all arbitrary constants. To give the corresponding symmetry reduction, it is equivalent to solving the characteristic equations

$$\begin{aligned}
\frac{\mathrm{d}t}{c_1t + c_2} &= \frac{3\gamma\mathrm{d}x}{c_1\gamma x - c_1C_1y + 3\gamma c_3t + 3\gamma c_4} = \frac{3\mathrm{d}y}{2c_1(3C_1\gamma t + y) + 3c_5}, \\
\frac{\mathrm{d}t}{c_1t + c_2} &= \frac{6\gamma\mathrm{d}u}{\left[2(6C_1C_2t + 4C_3t + C_4 - u)c_1 + \left(x - 3C_1^2t\right)c_3 + 6c_6\right]\gamma + (6C_2c_1 + C_1c_3)y} \\
&= \frac{6\gamma\mathrm{d}v}{\left[2(6C_1C_2t + 4C_3t - v)c_1 + \left(x - 3C_1^2t\right)c_3 + 6c_6\right]\gamma + (6C_2c_1 + C_1c_3)y}.
\end{aligned}$$

It follows that four alternatives:

(i) $c_1 \neq 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_5 = 0$, and $c_6 = 0$;

(ii) $c_1 = 0$, $c_2 \neq 0$, $c_3 \neq 0$, $c_4 = 0$, $c_5 = 0$, and $c_6 = 0$;

(iii) $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 \neq 0$, $c_5 = 0$, and $c_6 = 0$;

(iv) $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_5 \neq 0$, and $c_6 = 0$.

(Note that with all c_i $(i = 1,2,3,4,5)$ being zero, the only possible value for c_6 is also zero.)

The first case leads to some scaling invariance, which gives that

$$\begin{aligned} u =& C_3 t + \frac{C_2 y}{\gamma} + C_4 \\ &+ t^{-\frac{1}{3}} F\left(-\frac{6C_1\gamma t - y}{t^{\frac{2}{3}}}, -\frac{3C_1^2\gamma t - C_1 y - \gamma x}{t^{\frac{1}{3}}\gamma} \right), \end{aligned} \tag{61}$$

$$\begin{aligned} v =& C_3 t + \frac{C_2 y}{\gamma} \\ &+ t^{-\frac{1}{3}} G\left(-\frac{6C_1\gamma t - y}{t^{\frac{2}{3}}}, -\frac{3C_1^2\gamma t - C_1 y - \gamma x}{t^{\frac{1}{3}}\gamma} \right). \end{aligned} \tag{62}$$

With a simple reword

$$a = -\frac{6C_1\gamma t - y}{t^{\frac{2}{3}}}, \quad b = -\frac{3C_1^2\gamma t - C_1 y - \gamma x}{t^{\frac{1}{3}}\gamma}, \tag{63}$$

we now can give the constraint conditions for F and G as follows:

$$\begin{aligned} &3G_{bbb} + 9G_b^2 - G_b b - G = 0, \\ &-3G_{bbb} - 12\gamma F_{ab} + 6(F-G)G_{bb} + 6F_b^2 - 3G_b^2 \\ &-2(3\gamma F + 3\gamma G + a)F_a + \left[3(F-G)^2 + 6G_b - b\right]F_b \\ &-3(F-G)^2 G_b - F = 0. \end{aligned}$$

For case (ii), similarly we obtain the reduction of u and v (with $c_2 = c_3 = 1$) as

$$u = H(x,y,t) + F(a,b), \tag{64}$$

$$v = H(x,y,t) + G(a,b), \tag{65}$$

in which

$$\begin{aligned} &H(x,y,t) = -\frac{\left(3C_1^2 - t\right)x}{6} + \frac{C_1 t y}{6\gamma} - \frac{t^3}{18}, \\ &a = y, \quad b = t^2 - 2x. \end{aligned} \tag{66}$$

By substituting Eqs. (64) and (65) into the system described in Theorem 3 directly, we obtain the constraints for F and G

$$\begin{aligned} &\gamma^3 G_{aaa} + 3C_1\gamma^2 G_a^2 + 3C_1\gamma\left(C_1^3 - 2C_2\right)G_a \\ &+C_1^3\left(-3C_1C_2 + \frac{C_1 a}{6\gamma} - C_3 - \frac{b}{12} + \frac{3C_2^2}{C_1^2}\right) = 0, \\ &-\gamma^3 G_{aaa} + 8C_1^3\gamma F_{ab} - C_1^2\Big[4\gamma G_a + C_1(F-G-C_1-C_4)^2 \\ &+2\left(C_1^3 + C_2\right)\Big]F_b + 2C_1\gamma^2(F-G-2C_1-C_4)G_{aa} \\ &-C_1\left[C_1(F-G-2C_1-C_4)^2 - C_1^3 - 2C_2\right]\gamma G_a \\ &-2C_1^3\gamma(F-G-C_1-C_4)F_a + 8C_1^3 F_b^2 - C_1\gamma^2 G_a^2 \\ &+\frac{C_1}{2}\left(C_1^3 - 2C_2\right)(F-G-C_1-C_4)^2 \\ &+\frac{1}{6}\left[3C_1^6 - 12C_1^3 C_2 - \left(\frac{b}{2} + 6C_3\right)C_1^2 - 6C_2^2\right] + \frac{C_1^3 a}{6\gamma} = 0. \end{aligned}$$

The remaining two alternatives lead to some translation invariant solutions. For $c_4 \neq 0$ (case (iii)), a trivial constant solution would be obtained. For $c_5 \neq 0$ (case (iv)), u and v are restricted to be functions with respect to x and t. And if we take $c_5 = 1$, the result shows that

$$u = F(x,t), \quad v = G(x,t), \tag{67}$$

with

$$G = -\frac{C_2 x}{C_1} + \frac{\left(3C_1^3 + C_1^2 C_3 - 3C_2^2\right)t}{C_1^2} + C_5,$$

where C_5 is an arbitrary constant, and

$$\begin{aligned} &F_t + 2F_x^2 + \left[\frac{-2C_2}{C_1} + (F-G-C_1-C_4)^2\right]F_x \\ &+2C_2(F-G-C_1-C_4) \\ &+(F-G-4C_1-C_4)(F-G-C_4)\frac{C_2}{C_1} \\ &-\left(\frac{C_2}{C_1}\right)^2 - C_3 = 0. \end{aligned}$$

4. Summary and discussion

This study offers a fresh perspective on the decompositions of the potential KP equation. The findings elucidate the relationship between the potential KP equation, the BSTO system, and the BSTO–KdV system. Our research establishes a superposition principle for the solutions of the potential KP equation, subject to a restriction associated with the BSTO decomposition. With these decompositions, one can easily apply the superposition principle to construct, for instance, a $(1+2)$-soliton solution for the potential KP equation. Not only is such a superposition feasible, but we can also consider an m-soliton solution and an n-soliton solution, the sum of which remains valid. Generally, it indicates that any solutions for the BSTO system (4) can be amalgamated to derive a solution for the potential KP equation.

The symmetry of the BSTO decompositions and the BSTO-KdV decomposition is also examined. The corresponding symmetry constraints are presented in Section 3. For most nontrivial cases, the scaling symmetry plays a significant role. These decompositions are all linked to some novel integrable systems. Future research could further investigate the potential applications of this solution in various domains of mathematical physics.

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