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Higher-dimensional Chen–Lee–Liu equation and asymmetric peakon soliton

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Integrable systems play a crucial role in physics and mathematics. In particular, the traditional $(1+1)$-dimensional and $(2+1)$-dimensional integrable systems have received significant attention due to the rarity of integrable systems in higher dimensions. Recent studies have shown that abundant higher-dimensional integrable systems can be constructed from $(1+1)$-dimensional integrable systems by using a deformation algorithm. Here we establish a new $(2+1)$-dimensional Chen–Lee–Liu (C–L–L) equation using the deformation algorithm from the $(1+1)$-dimensional C–L–L equation. The new system is integrable with its Lax pair obtained by applying the deformation algorithm to that of the $(1+1)$-dimension. It is challenging to obtain the exact solutions for the new integrable system because the new system combines both the original C–L–L equation and its reciprocal transformation. The traveling wave solutions are derived in implicit function expression, and some asymmetry peakon solutions are found.

Keywords: higher dimensional Chen–Lee–Liu equation, Lax integrable system, deformation algorithm, implicit traveling wave solutions

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1. Introduction

Since the establishment of the inverse scattering transform (IST) method,[1] research of integrable systems that can be solved by using the IST method has attracted great attention. By now, a vast number of integrable systems have been established and the various remarkable and harmonious properties of those systems have also been explored. It is generally believed that IST integrable systems possess infinite symmetries and conservation laws,[2,3] analytic behaviors related to Painlevé property,[4,5] Hirota bilinear form and τ function,[6] Darboux transformation and Bäcklund transformation, as well as the nonlinear superposition principle,[7] Hamiltonian and bi-Hamiltonian structure, and recursion operator.[8] Due to the remarkable properties, soliton theories and the related integrable systems have been popularly used to describe the remarkable nonlinear phenomena in different branches of physics, for example, the particle physics and nuclear physics,[9,10] condensed matter physics,[11–16] fluid dynamics,[17,18] field theory,[19–22] cosmology,[23] and nonlinear optics.[24–26]

However, most research to date has mainly concentrated on lower-dimensional integrable systems, specifically $(1+1)$-dimensional or $(2+1)$-dimensional systems due to the scarcity of higher-dimensional integrable systems. The prototypical examples of integrable systems are the famous Korteweg–de Vries (KdV) equation and the nonlinear Schrödinger (NLS) equation in $(1+1)$-dimensions, as well as the Kadomtsev–Petviashvili (KP)[27] and Davey–Stewartson (DS)[28] equations in $(2+1)$-dimensions, respectively. Usually, the KP equation is regarded as an extension of the KdV equation, while the DS equation is the corresponding extension of the NLS equation. For $(n+1)$-dimensional $(n \geq 3)$ integrable KdV and NLS equations, there are a few developments available except for the KP and DS equations in $(3+1)$-dimensions[29] and $(4+1)$-dimensions.[30]

Consequently, a significant and challenging open problem in nonlinear integrable systems pertains to the existence of higher-dimensional integrable evolution equations. To deal with this problem, great efforts have been taken to exploration of integrable systems in higher dimensions. So far, there have been two types of effective approaches for searching for higher-dimensional integrable systems. One approach is to construct higher-dimensional integrable systems from lower-dimensional ones by using Painlevé analysis.[29] The other approach involves obtaining integrable equations in higher dimensions is complexifying time.[30,31] Although higher-dimensional integrable systems can be established by using different approaches, the newly obtained systems are considered traditional integrable systems because they possess the Painlevé property and can be solved by using the IST method.

Recent studies[32–36] have demonstrated that arbitrary low-dimensional integrable systems can be extended to higher-dimensional integrable systems by using a deformation algorithm and a modified deformation algorithm. It is shown the novel higher-dimensional integrable systems deformed from the lower dimensional systems possess completely unique properties and structures. Similar to the traditional integrable

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systems, the newly obtained nonlinear integrable systems have been proved to be integrable with Lax pairs and possess the infinitely many symmetries. However, the traditional research methods and techniques of integrable systems, such as the IST and the bilinear transformation, are no longer effective to search for the exact solutions. Because the higher-dimensional systems include the original lower dimensional systems and their several reciprocal systems, this leads to the difficulty to solve the systems.

Therefore, in order to further investigate this new class of integrable systems, this manuscript aims to extend the significant $(1+1)$-dimensional Chen–Lee–Liu (C–L–L) equation to higher dimensions. The C–L–L equation,[37] also referred to as the derivative nonlinear Schrödinger (DNLS) Ⅱ equation, generally can be written as

$$u_t = \mathrm{i}u_{xx} + 2uvu_x, \quad v_t = -\mathrm{i}v_{xx} + 2uvv_x, \tag{1}$$

where $v = u^*$ is the complex conjugate of u, and $\mathrm{i} \equiv \sqrt{-1}$.

First introduced by Chen *et al.*,[37] the C–L–L equation (1) is known to be a completely integrable system and can be solved exactly by the IST. Great developments have been made by various authors. For example, the N-soliton solution of the C–L–L equation was obtained by using the dependent variable transformation and the Hirota method in Ref. [38]. Furthermore, in Ref. [39], the Wronskian technique was utilized to derive the double Wronskian solutions of the C–L–L equation. The researchers of Ref. [40] investigated the Darboux transformation and constructed the rational soliton solutions for the C–L–L equation. Additionally, the classical Lie symmetries were employed to recover bright, dark, and singular solitons for this model.[41] By using the PINN deep learning method, the rogue wave solution on a periodic background for the C–L–L equation was successfully obtained.[42] In Ref. [43], Fan showed that the C–L–L equation can be reduced from the generalized Kaup–Newell (KN) hierarchy exactly, known as the DNLS Ⅱ equation.

Due to the completely integrability, the C–L–L equation is of significant importance in various fields of physics. It is commonly used to describe the dynamics of soliton propagation through different forms of optical waveguides, which are known as optic solitons.[44–46] In optical fiber setting, the C–L–L equation is usually associated with the self-steepening phenomena.[47] The C–L–L equation can also be used as a model to describe the propagation of finite-amplitude Alfvén waves in the presence of sound waves.[48–50] It is found the C–L–L equation governs the wave propagation on a discrete nonlinear transmission line.[51] Additionally, the C–L–L equation is crucial in the study of ferromagnetic spin chains. It helps to understand the generation of solitons of different forms through the interplay of various magnetic interactions.[52]

In this article, we aim to establish the higher-dimensional C–L–L equation (1) from the $(1+1)$-dimensional case by using the deformation algorithm. The organization of this paper is as follows: Section 2 reproduces the detailed description of the deformation algorithm used in Ref. [32]. In Section 3, we construct a C–L–L equation in $(2+1)$-dimensions by presenting the conservation laws of the known C–L–L equation. The Lax pair of the newly obtained $(2+1)$-dimensional C–L–L equation is also found from the Lax pair of the equation in $(1+1)$-dimensions. That demonstrates the integrability of the obtained system. The relations of the C–L–L equation and its reciprocal system are shown in this section. Though it is very difficult to find the exact solutions by using the traditional methods, Section 4 successfully finds the exact traveling wave structures of the higher-dimensional C–L–L equation.

2. Deformation algorithm

The deformation algorithm proposed in Ref. [32] provides an effective way to construct higher-dimensional integrable systems from $(1+1)$-dimensional ones. Here we just reproduce the deformation algorithm and related theories in detail.

Deformation algorithm If a $(1+1)$-dimensional integrable local nonlinear system

$$u_t = F(u,\ u_x,\ \ldots,\ u_{xn}), \tag{2}$$

with $u_{xn} = \partial_x^n u$ and $u = (u_1,\ u_2,\ \ldots,\ u_m)$ keeps the conservation laws

$$\rho_{it} = J_{ix},\ i = 1,\ 2,\ \ldots, D-1, \tag{3}$$

where the conserved densities $\rho_i = \rho_i(u)$ are dependent only on the field u, while the conserved flows $J_i = J_i(u,\ u_x,\ \ldots,\ u_{xN})$ can be field derivative dependent, then a deformed $D+1$-dimensional system can be obtained as follows:

$$\hat{T}u = F(u,\ \hat{L}u,\ \ldots,\ \hat{L}^n u), \tag{4}$$

with the deformation operators given by

$$\hat{L} \equiv \partial_x + \sum_{i=1}^{D-1} \rho_i \partial_{x_i},\ \ \hat{T} \equiv \partial_t + \sum_{i=1}^{D-1} \bar{J}_i \partial_{x_i}, \tag{5}$$

and the deformed flows presented by

$$\bar{J}_i = J_i|_{u_{xj} \to \hat{L}^j u,\ j=1,\ 2,\ \ldots,\ N}. \tag{6}$$

If the $(1+1)$-dimensional system (2) is integrable with the Lax pair

$$\Phi_x = M\Phi,\ \ \Phi_t = N\Phi, \tag{7}$$

then it follows that the $(D+1)$-dimensional system can also be integrable with a Lax pair

$$\begin{aligned} &\hat{L}\Phi = \hat{M}\Phi,\ \ \hat{T}\Phi = \hat{N}\Phi, \\ &\{\hat{M},\ \hat{N}\} = \{M,\ N\}|_{\partial_x \to \hat{L},\ u_{xj} \to (\hat{L}^j u)}, \end{aligned} \tag{8}$$

which can be obtained directly by employing the deformation operators to the Lax pair for the $(1+1)$-dimensional system.

Lou *et al.*[32] have verified the correctness of the deformation algorithm for nearly all the known local integrable systems. Subsequently, the deformation algorithm has been further developed to a modified deformation algorithm. The result of Ref. [33] shows that the nonlinear systems with weak Lax pair can also be deformed to higher-dimensional integrable systems. The famous $(1+1)$-dimensional Cammasa–Holm (CH) equations are extended to some new higher-dimensional CH equations and some novel kinds of asymmetric peakon solutions are derived.

3. An integrable (2+1)-dimensional C–L–L equation

In this section, we deform the known $(1+1)$-dimensional C–L–L equation to a $(2+1)$-dimensional C–L–L equation by using the deformation algorithm.

The conservation law satisfying the deformation algorithm for the C–L–L equation (1) possesses the form

$$(uv)_t = [u^2v^2 - \mathrm{i}(uv_x - vu_x)]_x, \tag{9}$$

where the conserved density is $\rho = uv$, and the conserved flow is $J = u^2v^2 - \mathrm{i}(uv_x - vu_x)$. Thus, the deformation operators can be obtained directly by applying the deformation algorithm to the C–L–L equation (1), say,

$$\hat{L} = \partial_x + (uv)\partial_y, \quad \hat{T} = \partial_t + \bar{J}_1\partial_y, \tag{10}$$

and the corresponding deformed flow is

$$\bar{J}_1 = u^2v^2 - \mathrm{i}[(u\hat{L}v - v\hat{L}u)]. \tag{11}$$

Acting the deformation operators (10) to the C–L–L equation (1), we derive a specific $(2+1)$-dimensional C–L–L equation

$$\hat{T}u = \mathrm{i}\hat{L}^2u + 2uv\hat{L}u, \quad \hat{T}v = -\mathrm{i}\hat{L}^2v + 2uv\hat{L}v, \tag{12}$$

i.e.,

$$\begin{aligned} u_t &= \mathrm{i}u_{xx} + u^2v^2(\mathrm{i}u_y + u)_y + 2\mathrm{i}u(u_yv)_x + 2u(u_xv + \mathrm{i}uvu_yv_y), \\ v_t &= -\mathrm{i}v_{xx} - u^2v^2(\mathrm{i}v_y - v)_y - 2\mathrm{i}v(uv_y)_x + 2v(uv_x - \mathrm{i}uvu_yv_y). \end{aligned} \tag{13}$$

The integrability of the deformed C–L–L equation (13) can be ensured by finding its Lax pair. We start from the Lax pair of the $(1+1)$-dimensional C–L–L equation (1) expressed as

$$\psi_x = U\psi, \quad \psi_t = V\psi, \tag{14}$$

where

$$U = \mathrm{i}\begin{pmatrix} r & \lambda v \\ \lambda u & -r \end{pmatrix},$$

$$V = 2rU + \begin{pmatrix} \frac{1}{2}(uv_x - u_xv) & \lambda v_x \\ -\lambda u_x & \frac{1}{2}(u_xv - uv_x) \end{pmatrix}, \tag{15}$$

with

$$2r = uv - \lambda^2. \tag{16}$$

By examining the compatibility condition $\psi_{xt} = \psi_{tx}$ of Eqs. (14)–(16), the C–L–L equation (1) can be successfully obtained.

The Lax pair of the $(2+1)$-dimensional can be directly derived by using the deformation algorithm to the Lax pair of the $(1+1)$-dimensional C–L–L equations (14)–(16) according to the deformation algorithm. Acting the deformation operators to the Lax pair of the $(1+1)$-dimensional C–L–L equation, we can see that the Lax pair of the $(2+1)$-dimensional C–L–L equation (13) is finally given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = \begin{pmatrix} \mathrm{i}r - uv\partial_y & \mathrm{i}\lambda v \\ \mathrm{i}\lambda u & -\mathrm{i}r - uv\partial_y \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{17}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{18}$$

with

$$\begin{aligned} M_{11} &= 2\mathrm{i}r^2 + \frac{1}{2}u(v_x + uvu_y) - \frac{1}{2}v(u_x + uvu_y) - \bar{J}_1\partial_y, \\ M_{12} &= 2\mathrm{i}\lambda rv + \lambda(v_x + uvv_y), \\ M_{21} &= 2\mathrm{i}\lambda ru - \lambda(u_x + uvu_y), \\ M_{22} &= -2\mathrm{i}r^2 - \frac{1}{2}u(v_x + uvu_y) + \frac{1}{2}v(u_x + uvu_y) - \bar{J}_1\partial_y, \end{aligned} \tag{19}$$

or, equivalently,

$$\hat{M}\phi = 0, \quad \hat{N}\phi = 0, \tag{20}$$

where

$$\begin{aligned} \hat{M} &= \begin{pmatrix} \mathrm{i}r - \hat{L} & \mathrm{i}\lambda v \\ \mathrm{i}\lambda u & -\mathrm{i}r - \hat{L} \end{pmatrix}, \\ \hat{N} &= \begin{pmatrix} 2\mathrm{i}r^2 - \frac{1}{2}(u\hat{L}v - v\hat{L}u) - \hat{T} & 2\mathrm{i}\lambda rv + \lambda\hat{L}v \\ 2\mathrm{i}\lambda ru - \lambda\hat{L}u & -2\mathrm{i}r^2 + \frac{1}{2}(v\hat{L}u - u\hat{L}v) - \hat{T} \end{pmatrix}, \end{aligned} \tag{21}$$

and

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{22}$$

One can check the compatibility conditions (20)–(22) for

$$[\hat{M}, \hat{N}] = \hat{M}\hat{N} - \hat{N}\hat{M} = 0$$

to yield the $(2+1)$-dimensional C–L–L equation (13).

Clearly, the C–L–L equation (13) will be simplified to the $(1+1)$-dimensional C–L–L equation (1) when $\{u, v\}$ is independent of y. However, when $\{u, v\}$ is independent of x, the $(2+1)$-dimensional C–L–L equation (13) will transform into a different $(1+1)$-dimensional equation reading

$$u_t = u^2v^2(\mathrm{i}u_y + u)_y + 2\mathrm{i}u^2vu_yv_y,$$

$$v_t = -u^2v^2(\mathrm{i}v_y - v)_y - 2\mathrm{i}uv^2u_yv_y. \tag{23}$$

One can easily find that the $(1+1)$-dimensional equation (23) has a special conservation law where the conserved density is described by $\rho = (uv)^{-1}$, and the conserved flow is given by $J = -\mathrm{i}(uv_y - u_yv) + uv$, i.e.,

$$(u^{-1}v^{-1})_t = [-\mathrm{i}(uv_y - u_yv) + uv]_y. \tag{24}$$

For this nonlinear equation (23) in $(1+1)$-dimensions and its corresponding conservation law (24), by applying the deformation algorithm, we can find that the $(2+1)$-dimensional C–L–L equation (13) is reobtained. Equation (23) can be considered as a reciprocal C–L–L equation associated with the conservation law (24).

It should be emphasized that though the Lax pair (17)–(19), or (20)–(22) guarantees the integrability of the new $(2+1)$-dimensional C–L–L equation (13), finding exact solutions for the higher-dimensional equation (13) still remains challenging. It implies that the deformation algorithm involves a reciprocal transformation of the conservation law (9) and the original $(1+1)$-dimensional C–L–L equation by the deformation relations $\partial_x \to \rho\partial_y$ and $\partial_t \to \partial_t + J\partial_y$. Thus, we can conclude that the $(2+1)$-dimensional C–L–L equation (13) is combined with both the original C–L–L equation (1) and its reciprocal equation (23). Therefore, the traditional approaches and methods, for example, the IST, the Hirota bilinear direct method, and the Darboux and Bäcklund transformations can no longer be directly used to search for the exact solutions for the new obtained higher-dimensional equation (13).

4. Asymmetric soliton solutions for the higher-dimensional C–L–L equation

Though the traditional methods for finding the exact solutions of the higher integrable systems are not effective, the traveling wave solutions for the C–L–L equation (13) can be successfully constructed. In this section, we show the details in searching for the traveling wave solutions of the C–L–L equation (13).

Generally, the traveling wave solution of the system (13) possesses the following form:

$$\begin{aligned} &u = U(X)\exp\{\mathrm{i}[k_0x + p_0y + \omega_0t + V(X)]\},\\ &v = U(X)\exp\{-\mathrm{i}[k_0x + p_0y + \omega_0t + V(X)]\},\\ &X \equiv kx + py + \omega t, \end{aligned} \tag{25}$$

where k_0, p_0, ω_0, k, p, and ω are constants to be further determined. Substituting the solution (25) into Eq. (13), then separating the real and imaginary parts, we can find that the functions $U(X)$ and $V(X)$ are constrained by the following equations:

$$V_X = \frac{c_1}{U} - \frac{2p_0 - 1}{2p} + \frac{2pkk_0 - 2k^2p_0 + p\omega + k^2}{2p^2U^2(pU^2 + k)},$$

$$\begin{aligned} &4U^3(pU^2+k)^2U_{XX} + 8pU^4(pU^2+k)U_X^2 + 3U^8\\ &+ 4C_1U^6 + C_2U^4 - \omega_1^2 = 0, \end{aligned} \tag{26}$$

where c_1 is the integral constant, and C_1, C_2, and ω_1 are defined as

$$\begin{aligned} C_1 =& 2pc_1 + 2k_0 - 2kp^{-1}p_0 + kp^{-1},\\ \omega_1 =& 2c_1k + 2p^{-1}kk_0 - 2k^2p^{-2}p_0 + p^{-1}\omega + k^2p^{-2},\\ C_2 =& 4p^2c_1^2 + 8kc_1 - 4k_0^2 - 4\omega_0 + 2p^{-1}(2k+\omega)(2p_0+1)\\ &- k^2p^{-2}(2p_0+3)(2p_0-1). \end{aligned} \tag{27}$$

The solution for $U(X)$ determined by Eq. (26) can be directly given by the special elliptic integral

$$\begin{aligned} &\int^U \frac{f(pf^2+k)}{\sqrt{-f^8 - 2C_1f^6 - C_2f^4 + C_0f^2 - \omega_1^2}}\mathrm{d}f\\ &= -\frac{X - X_0}{2}, \end{aligned} \tag{28}$$

with C_0 and X_0 being two integral constants.

In order to find the results for the special elliptic integral, it is natural to assume the arbitrary constants C_1, C_2 and ω_1 to satisfy the following conditions:

$$\begin{aligned} &C_1 = -a^2 - b,\ \ C_2 = a^4 + 4a^2b + c^2,\\ &C_0 = \frac{a^2}{2}(a^2b + c^2),\ \ \omega_1^2 = c^2a^4. \end{aligned} \tag{29}$$

With these particular parameter relations (29), the solution for the elliptic integral (28) simplifies to the single soliton solution implicitly, say

$$\begin{aligned} &\frac{A(-U^4 + 2bU^2 - c^2)}{[(a^2-b)U^2 - a^2b + c^2]^2}\\ &= \tanh^2\left[\frac{\sqrt{A}}{a^2p+k}\left(p\arctan\left(\frac{U^2-b}{\sqrt{-U^4+2bU^2-c^2}}\right)\right.\right.\\ &\left.\left. + X - X_0\right)\right], \end{aligned} \tag{30}$$

with $A = -a^4 + 2a^2b - c^2$.

Figure 1 illustrates different soliton structures with various parameters selected. Figure 1(a) depicts the dark (gray) soliton of the $(2+1)$-dimensional C–L–L equation (13) with a relatively small deformation parameter ($p = 1/1000$). It can be observed that the dark soliton solution (30) is almost symmetric with respect to the center of the soliton. In Fig. 1(b), for a relatively large deformation parameter ($p = 1$), the dark (gray) soliton solution (30) of the (2+1)-dimensional C–L–L equation (13) exhibits significant asymmetry relative to the soliton center. Figure 1(c) is the dark peak soliton structure described by Eq. (30) when $c = 0$.

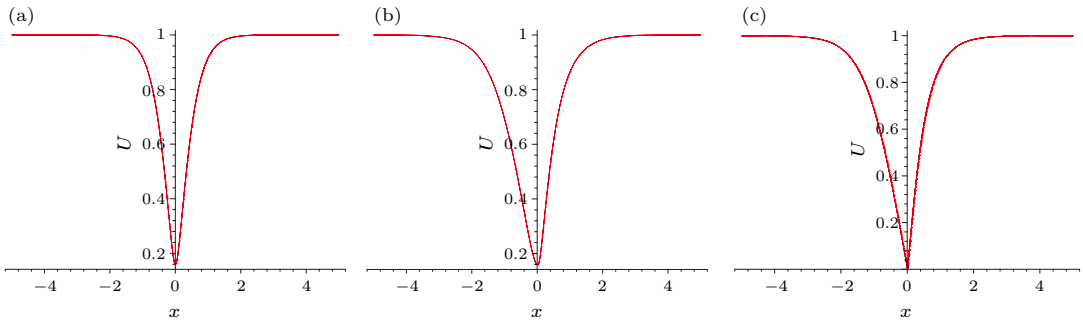


Fig. 1. Soliton solution given by Eq. (30) of the higher-dimensional C–L–L equation (13) with the parameters $a = 1$, $b = 20$, $k = 2$, and $p = 1/1000$, $X_0 = 0$, and $c = 1$ in (a), $p = 1$, $X_0 = -1.55$, and $c = 1$ in (b), and the dark peakon soliton solution for $p = 1$, $X_0 = -1.56$, and $c = 0$ in (c).

5. Summary and discussion

In summary, a $(2+1)$-dimensional C–L–L equation is successfully constructed by using the deformation algorithm applied to the $(1+1)$-dimensional traditional C–L–L equation. The obtained $(2+1)$-dimensional C–L–L equation is found to possess a Lax pair which is presented by applying the same deformation algorithm to the Lax pair of the lower dimensional C–L–L equation. However, it is shown that the traditional methods for solving integrable systems cannot be directly applied to the $(2+1)$-dimensional C–L–L equation since it contains both the original C–L–L equation and its reciprocal equation. Further studies show that the $(2+1)$-dimensional C–L–L equation possesses a soliton solution given by a complex implicit elliptic integral. The dark (gray) soliton maintains an approximate central symmetric form under small deformations but exhibits clear central symmetry breaking under large deformations. The dark soliton is a novel type of dark peak soliton.

It should be mentioned that the $(1+1)$-dimensional NLS equation has been extended to a $(2+1)$-dimensional one in Ref. [34]. Similar to the $(2+1)$-dimensional C–L–L equation, the $(2+1)$-dimensional NLS equation has some special implicit asymmetric dark soliton solutions for large deformations. However, the peakon solution for the $(2+1)$-dimensional NLS equation has not yet been found. This imply that the $(2+1)$-dimensional C–L–L equation and the $(2+1)$-dimensional NLS equation obtained from the deformation algorithm may have different properties. Further studies about the higher-dimensional systems deformed from $(1+1)$-dimensional integrable systems should be done.

From the results presented in this article, it is evident that high-dimensional integrable systems not only exist but also exhibit great complexity. By exploiting the diverse conservation laws associated with low-dimensional integrable systems, a plethora of high-dimensional integrable systems can be derived. The Lax integrability of these systems can be remarkably preserved, with the exception of Painlevé integrability, which proves to be challenging to maintain. The deformation techniques employed in this study enable the direct derivation of the Lax pairs of the deformed systems from the Lax pairs of their original models.

Although there have been many effective methods for constructing exact solutions for traditional integrable systems, the study of new systems poses a significant challenge as these traditional solution methods are no longer applicable. Therefore, all research methods need to be re-examined and developed. In the past, the study of the reciprocal equation of the original equation required the use of reciprocal transformations to transform back to the original equation and then to transform back to the reciprocal equation again. However, for the new models we obtain, this effective method is no longer applicable because the original model and the reciprocal model are combined into the same model. This significantly increases the difficulty of studying these models, but it also provides us with new research opportunities. Although the deformation algorithm applies to all integrable systems, each obtained new integrable system has its own characteristics. Therefore, it still requires further studies so as to make breakthroughs in research on these special models.

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