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Citation: Chin. Phys. B, 2024, 33 (**3**): 030305. DOI: 10.1088/1674-1056/ad1748
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Complementary monogamy and polygamy properties among multipartite systems

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(Received 7 October 2023; revised manuscript received 1 December 2023; accepted manuscript online 20 December 2023)

Monogamy and polygamy relations are essential properties of quantum entanglement, which characterize the distributions of entanglement in multipartite systems. In this paper, we establish the general monogamy relations for γ-th $(0 \leq \gamma \leq \alpha, \alpha \geq 1)$ power of quantum entanglement based on unified-(q,s) entanglement and polygamy relations for δ-th $(\delta \geq \beta, 0 \leq \beta \leq 1)$ power of entanglement of assistance based on unified-(q,s) entanglement of assistance, which provides a complement to the previous research in terms of different parameter regions of γ and δ. These results are then applied to specific quantum correlations, *e.g.*, entanglement of formation, Renyi-q entanglement of assistance and Tsallis-q entanglement of assistance to get the corresponding monogamy and polygamy inequalities. Moreover, typical examples are presented for illustration.

Keywords: monogamy relation, polygramy relation

PACS: 03.67.Mn, 03.65.Ud **DOI:** 10.1088/1674-1056/ad1748

1. Introduction

In quantum information theory, entanglement is a vital resource due to its importance in quantum communication and quantum information processing. Although entanglement in bipartite quantum systems has been extensively studied with various applications, entanglement in multipartite quantum systems is much more complicated and the corresponding results are relatively few. One distinguishing feature of quantum entanglement, as compared to classical correlations, is that it cannot be freely distributed among the subsystems. For example, it is not possible to prepare three qubits in a way that any two qubits are maximally entangled. This property is called monogamy of entanglement,[1,2] quantitatively, $E(\rho_{A|BC}) \geq E(\rho_{AB}) + E(\rho_{AC})$, where E is a bipartite entanglement measure, ρ_{AB} and ρ_{AC} are the reduced density matrices of ρ_{ABC}. Futhermore, the monogamy property has emerged as the ingredient in some practical applications such as quantum cryptography, quantum teleportation, quantum computation, quantum key distribution.[3–5]

For the systems of three qubits, the first monogamy inequality was established by using concurrence[6] to quantify the shared entanglement among subsystems. However, it is failed in its generalization for higher-dimensional quantum systems. Later, an interesting observation was made showing that a entanglement measure may not satisfy the monogamy relations in itself, but satisfied after the α-th power of it. For example, it has been proved in Ref. [7] that the x-th power of the entanglement of formation and concurrence satisfied the monogamy inequalities for $x \geq \sqrt{2}$ and $x \geq 2$, respectively. Many finer monogamy inequalities of multi-qubit systems based on non-negative power are available in Refs. [8–11]. Recently, the authors[12] introduced the concept of monogamy relations without inequalities.

Monogamy inequality is about the restricted sharability of multipartite entanglement, providing a lower bound of entanglement, while its dual concept to the sharable entanglement, is known to have a polygamous property, which gives an upper bound in multipartite quantum systems. It is mathematically characterized as $E_{\mathrm{a}}(\rho_{ABC}) \leq E_{\mathrm{a}}(\rho_{AB}) + E_{\mathrm{a}}(\rho_{AC})$ for a three-party quantum state ρ_{ABC}, where the customary notion "$E_{\mathrm{a}}(\cdot)$" stands for the assisted entanglement. The polygamy inequality was first given in three-qubit systems using tangle of assistance[13,14] and generalized into multiqubit systems in terms of various assisted entanglements.[14–18] In recent years, monogamy and polygamy inequalities of multi-qubit entanglement have been further studied and extended in terms of non-negative power of entanglement measures and assisted entanglements. Moreover, the monogamy relations for the α-th $(0 \leq \alpha \leq 1)$ power and the polygamy relations for the β-th $(\beta \geq 1)$ power of unified-(q,s) entanglement and unified-(q,s) entanglement of assistance are obtained in Ref. [16].

In the present paper, complementary monogamy and polygamy inequalities among multipartite systems are investigated. We provide general monogamy inequalities for γ-th $(0 \leq \gamma \leq \alpha, \alpha \geq 1)$ power of quantum entanglement based

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on unified-(q,s) entanglement and δ-th $(\delta \geq \beta, 0 \leq \beta \leq 1)$ power of entanglement of assistance based on unified-(q,s) entanglement of assistance, which turn out to be tighter than the existing ones in literature. As a by-product, we derive the corresponding monogamy relations of specific quantum correlations such as entanglement of formation, Renyi-q entanglement and Tsallis-q entanglement by setting the parameters q and s. We take Tsallis-2 entanglement as an example to illustrate in detail. Applying the unified-(q,s) entanglement of assistance to specific quantum correlations, *e.g.*, Renyi-q entanglement of assistance and Tsallis-q entanglement of assistance, the corresponding new class of polygamy relations is obtained, which are complementary to the existing ones[16] with different regions of parameter δ.

2. Monogamy of multiqubit relations for unified entanglement

Due to the flexibility in parameter selection, the unified-(q,s) entropy is interesting both in theory and in applications and has been studied in various fields. For any quantum state $\rho \in \mathcal{H}$, the unified-(q,s) entropy is defined as[19,20]

$$S_{q,s}(\rho) = \frac{1}{(1-q)s}\left((\mathrm{Tr}\rho^q)^s - 1\right), \tag{1}$$

with $q,s \geq 0$, $q \neq 1$, and $s \neq 0$. For a bipartite pure state $|\psi\rangle_{AB} \in \mathcal{H}_{AB}$, its unified-$(q,s)$ entanglement (UE) can be defined based on unified-(q,s) entropy as[17]

$$E_{q,s}(|\psi\rangle_{AB}) = S_{q,s}(\rho_A), \tag{2}$$

where $\rho_A = \mathrm{Tr}_B|\psi\rangle_{AB}\langle\psi|$ is the reduced density matrix onto subsystem A and $q,s \geq 0$. Furthermore, considering mixed state $\rho_{AB} \in \mathcal{H}_{AB}$, one can define its UE via convex-roof extension

$$E_{q,s}(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle_{AB}\}} \sum_i p_i E_{q,s}(|\psi_i\rangle_{AB}), \tag{3}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$ and $\sum_i p_i = 1$. Since the UE in Eq. (3) is continuous with respect to the parameters q and s, UE reduces to Tsallis-q entanglement (TE)[18,21] as s tends to 1, and converges to Renyi-q entanglement (RE)[22,23] as s goes to 0. With any nonnegative s and q tends to 1, UE converges to entanglement of formation (EoF)

$$\lim_{q\to 1} E_{q,s}(\rho_{AB}) = E_{\mathrm{f}}(\rho_{AB}). \tag{4}$$

Thus, UE is one of the most general classes of bipartite entanglement measures, with RE, TE, and EoF as special classes of UE.[17]

Unified-(q,s) entanglement of assistance (UEoA), as a dual quantity to UE, is defined as[24]

$$E^{\mathrm{a}}_{q,s}(\rho_{AB}) = \max_{\{p_i,|\psi_i\rangle_{AB}\}} \sum_i p_i E_{q,s}(|\psi_i\rangle_{AB}) \tag{5}$$

for each $q,s \geq 0$, where the maximum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i|\psi\rangle_{AB}\langle\psi|$ and $\sum_i p_i = 1$.

Similarly, as UEoA in Eq. (5) is continuous for the parameters q and s assures that UEoA reduces to Renyi-q entanglement of assistance (REoA)[23] and Tsallis-q entanglement of assistance (TEoA)[18] when s tends to 0 or 1, respectively. For any nonnegative s, with q tends to 1, UEoA reduces to entanglement of assistance (EoA)[24]

$$\lim_{q\to 1} E^{\mathrm{a}}_{q,s}(\rho_{AB}) = E^{\mathrm{a}}(\rho_{AB}). \tag{6}$$

Using UE in Eq. (3) to quantify bipartite quantum entanglement, the monogamy inequality was established in multiqubit systems; for any N-qubit state $\rho_{A_1A_2\cdots A_N}$ and its two-qubit reduced density matrices $\rho_{A_1A_i}$ with $i = 2,\ldots,N$, we have

$$E_{q,s}(\rho_{A_1|A_2\ldots A_N}) \geq \sum_{i=2}^{N} E_{q,s}(\rho_{A_1A_i}), \tag{7}$$

for $q \geq 2$, $0 \leq s \leq 1$, and $qs \leq 3$.[17] Furthermore, in Ref. [16] the authors presented monogamy inequality based on UE in multiqubit systems as

$$E^{\alpha}_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}}) \geq \sum_{i=1}^{N-1} (2^{\alpha}-1)^{i-1} E^{\alpha}_{q,s}(\rho_{AB_i}) \tag{8}$$

for $0 \leq \alpha \leq 1$, $q \geq 2$ and $0 \leq s \leq 1$, $qs \leq 3$.

It was also shown that UEoA can be used to characterize the polygamy of multiqubit entanglement as[24]

$$E^{\mathrm{a}}_{q,s}(\rho_{A_1|A_2\cdots A_N}) \leq \sum_{j=2}^{N} E^{\mathrm{a}}_{q,s}(\rho_{A_1A_j}) \tag{9}$$

for $1 \leq q \leq 2$ and $-q^2+4q-3 \leq s \leq 1$. It is further improved that, when $\beta \geq 1$, $-q^2+4q-3 \leq s \leq 1$, and $1 \leq q \leq 2$,[16]

$$(E^{\mathrm{a}}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^{\beta} \leq \sum_{j=1}^{N-1} (2^{\beta}-1)^{j-1} (E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta} \tag{10}$$

for any multiqubit state $\rho_{A_1A_2\cdots A_N}$.

3. Tighter monogamy relations for multiqubit for unified entropy

In this section, a corresponding new class of polygamy relations is present which are complementary to the existing ones. We start with the following lemma which gives a useful inequality in the proof of the main theorems.

Lemma 1 For any real numbers x and t, if $0 \leq x \leq 1$ and $t \geq k \geq 1$, then we have

$$(1+t)^x \geq 1 + \frac{(1+k)^x - 1}{k^x} t^x, \tag{11}$$

$$0 < \frac{(1+k)^x - 1}{k^x} \leq 1. \tag{12}$$

Proof Let $k \geq 1$. First, we can construct a binary function $f(x,y) = (1+y)^x - y^x$ with $0 \leq x \leq 1, 0 < y \leq 1/k$. Then, based on the fact that

$$\frac{\partial f}{\partial y} = x[(1+y)^{x-1} - y^{x-1}] \leq 0,$$

the binary function $f(x,y)$ is a decreasing function of y, i.e.,

$$f(x,y) \geq f\left(x, \frac{1}{k}\right) = \frac{(1+k)^x - 1}{k^x}.$$

Set $y = 1/t$ with $t \geq k \geq 1$, we obtain

$$(1+t)^x \geq 1 + \frac{(1+k)^x - 1}{k^x} t^x. \tag{13}$$

Hence, for all $t \geq k \geq 1$, it holds that

$$0 < \frac{(1+k)^x - 1}{k^x} \leq \frac{(1+t)^x - 1}{t^x}. \tag{14}$$

Fixed $k \geq 1$ and letting t in expression (14) go to $+\infty$, we get expression (12).

Making use of Lemma 1 in triqubit system $H_A \otimes H_B \otimes H_C$, we get the following.

Theorem 1 Let $\alpha \geq 1$, $k \geq 1$, $0 \leq \gamma \leq \alpha$, $\mathcal{K}_{(\alpha,\gamma)} = [(1+k)^{\gamma/\alpha} - 1]/k^{\gamma/\alpha}$ and let ρ_{ABC} be any mixed state of $H_A \otimes H_B \otimes H_C = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Then

(i) When $E_{q,s}(\rho_{AB}) \geq kE_{q,s}(\rho_{AC})$, it holds that

$$\left(E_{q,s}(\rho_{A|BC})\right)^\gamma \geq \left(E_{q,s}(\rho_{AC})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)} \left(E_{q,s}(\rho_{AB})\right)^\gamma. \tag{15}$$

(ii) When $kE_{q,s}(\rho_{AB}) \leq E_{q,s}(\rho_{AC})$, it holds that

$$\left(E_{q,s}(\rho_{A|BC})\right)^\gamma \geq \left(E_{q,s}(\rho_{AB})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)} \left(E_{q,s}(\rho_{AC})\right)^\gamma. \tag{16}$$

Proof First note that by expression (12) in Lemma 1, $0 < \mathcal{K}_{(\alpha,\gamma)} \leq 1$. It was proved in Ref. [17] that

$$E_{q,s}(\rho_{A|BC}) \geq E_{q,s}(\rho_{AB}) + E_{q,s}(\rho_{AC}). \tag{17}$$

(I) Suppose that $E_{q,s}(\rho_{AB}) \geq kE_{q,s}(\rho_{AC})$. When $E_{q,s}(\rho_{AC}) = 0$, since $0 < \mathcal{K}_{(\alpha,\gamma)} \leq 1$, the inequality expression (17) yields that

$$\begin{aligned} &\left(E_{q,s}(\rho_{A|BC})\right)^\gamma \geq \left(E_{q,s}(\rho_{AB})\right)^\gamma \geq \mathcal{K}_{(\alpha,\gamma)} \left(E_{q,s}(\rho_{AB})\right)^\gamma \\ =\ &\left(E_{q,s}(\rho_{AC})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)} \left(E_{q,s}(\rho_{AB})\right)^\gamma, \end{aligned}$$

leading to expression (15). Next we assume that $E_{q,s}(\rho_{AC}) > 0$ and take

$$t = \frac{\left(E_{q,s}(\rho_{AB})\right)^\alpha}{\left(E_{q,s}(\rho_{AC})\right)^\alpha},$$

$x = \gamma/\alpha$. Noticing that $(X+Y)^\alpha \geq X^\alpha + Y^\alpha$ for all $X, Y \geq 0$, the monotonicity of the exponential function together with expression (17) and Lemma 1 readily gives

$$\begin{aligned} \left(E_{q,s}(\rho_{A|BC})\right)^\gamma &= \left(E_{q,s}(\rho_{A|BC})\right)^{\alpha x} \\ &\geq \left(E_{q,s}(\rho_{AB}) + E_{q,s}(\rho_{AC})\right)^{\alpha x} \\ &\geq \left(\left(E_{q,s}(\rho_{AB})\right)^\alpha + \left(E_{q,s}(\rho_{AC})\right)^\alpha\right)^x \\ &= \left(E_{q,s}(\rho_{AC})\right)^{\alpha x} (1+t)^x \\ &\geq \left(E_{q,s}(\rho_{AC})\right)^{\alpha x} \left[1 + \frac{(1+k)^x - 1}{k^x} t^x\right] \\ &= \left(E_{q,s}(\rho_{AC})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)} \left(E_{q,s}(\rho_{AB})\right)^\gamma. \end{aligned}$$

(II) Suppose that $kE_{q,s}(\rho_{AB}) \leq E_{q,s}(\rho_{AC})$. When $E_{q,s}(\rho_{AB}) = 0$, since $0 < \mathcal{K}_{(\alpha,\gamma)} \leq 1$, the inequality (16) follows from expression (17). When $E_{q,s}(\rho_{AB}) > 0$, using Lemma 1 for $t = \left(E_{q,s}(\rho_{AC})\right)^\alpha / \left(E_{q,s}(\rho_{AB})\right)^\alpha$ and $x = \gamma/\alpha$ yields expression (16).

We point out that the monogamy inequalities given by Theorem 1 can be regarded as a complement to the previous work[16] in a sense that the value region of the parameter γ for UE is different. Stated briefly, it is well known that some quantum entanglement measures E such as UE[17] satisfies the original monogamy relations, say $E(\rho_{A|BC}) \geq E(\rho_{AB}) + E(\rho_{AC})$, while others, such as concurrence, does not satisfy the monogamy relations itself, but satisfies after some certain powers of it. For instance, as can be seen in Ref. [9], there exists a real number α such that for any quantum correlation measure Q, Q^x is monogamous if x belongs to the interval $[\alpha, +\infty)$. Different from that results, the monogamy relations (15) holds for the interval $[0, \alpha]$. So, in this sense, Theorem 1 gives monogamy inequalities that are complementary to the existing ones with different regions of the parameter γ for UE.

More specifically, by using Theorem 1 repeatedly, we have the following theorem for multipartite quantum systems.

Theorem 2 Let $0 \leq \gamma \leq \alpha$, $q \geq 2$, $0 \leq s \leq 1$, $qs \leq 3$, $\mathcal{K}_{(\alpha,\gamma)} = [(1+k)^{\gamma/\alpha} - 1]/k^{\gamma/\alpha}$, and let $\rho_{AB_1\cdots B_{N-1}}$ be any N-qubit state with $kE_{q,s}(\rho_{AB_i}) \leq E_{q,s}(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for every $i = 1, \ldots, N-2$. Then it holds that

$$\begin{aligned} &\left(E_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}})\right)^\gamma \\ \geq\ &\left(E_{q,s}(\rho_{AB_1})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)} \left(E_{q,s}(\rho_{AB_2})\right)^\gamma \\ &+ \cdots + \left(\mathcal{K}_{(\alpha,\gamma)}\right)^{N-2} \left(E_{q,s}(\rho_{AB_{N-1}})\right)^\gamma. \end{aligned} \tag{18}$$

Proof Using Theorem 1 continuously, we have

$$\begin{aligned} &\left(E_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}})\right)^\gamma \\ &\geq \left(E_{q,s}(\rho_{AB_1})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)} \left(E_{q,s}(\rho_{A|B_2\cdots B_{N-1}})\right)^\gamma \\ &\geq \left(E_{q,s}(\rho_{AB_1})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)} \left(E_{q,s}(\rho_{AB_2})\right)^\gamma \\ &\quad + \left(\mathcal{K}_{(\alpha,\gamma)}\right)^2 \left(E_{q,s}(\rho_{A|B_3\cdots B_{N-1}})\right)^\gamma \\ &\geq \cdots \\ &\geq \left(E_{q,s}(\rho_{AB_1})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)} \left(E_{q,s}(\rho_{AB_2})\right)^\gamma \\ &\quad + \cdots + \left(\mathcal{K}_{(\alpha,\gamma)}\right)^{N-2} \left(E_{q,s}(\rho_{AB_{N-1}})\right)^\gamma. \end{aligned}$$

Theorem 2 gives a new class of monogamy relations for multiqubit states, which includes inequality (8) as a special case since inequality (18) reduces to inequality (8) when

$\alpha = 1$ and $k = 1$. For $k > 1$, the inequality (18) is tighter than the inequality (8), as $[(1+k)^{\gamma/\alpha} - 1]/k^{\gamma/\alpha} \geq 2^{\gamma/\alpha} - 1$, where the equality holds only for $\alpha = \gamma$. Particularly, for $E_{q,s}(\rho_{AB}) \geq E_{q,s}(\rho_{AC})$ and $\alpha \geq 1$, taking $\gamma = 1/2$, $k = 1$, one has

$$\sqrt{E_{q,s}(\rho_{A|BC})} \geq \sqrt{E_{q,s}(\rho_{AC})} + (2^{1/2\alpha} - 1)\sqrt{E_{q,s}(\rho_{AB})}.$$

And the tighter monogamy inequality is

$$\sqrt{E_{q,s}(\rho_{A|BC})} \geq \sqrt{E_{q,s}(\rho_{AC})} + (\sqrt{2} - 1)\sqrt{E_{q,s}(\rho_{AB})}.$$

Example 1 Let us consider the three-qubit state $|\psi\rangle_{ABC}$ in the generalized Schmidt decomposition form

$$\begin{aligned} |\psi\rangle_{ABC} \;=\; & \lambda_0|000\rangle + \lambda_1 \mathrm{e}^{\mathrm{i}\varphi}|100\rangle + \lambda_2|101\rangle \\ & + \lambda_3|110\rangle + \lambda_4|111\rangle, \end{aligned} \tag{19}$$

where $\lambda_i \geq 0$, $i = 0,1,2,3,4$ and $\sum_{i=0}^{4} \lambda_i^2 = 1$. From the definition of UE in Eq. (3), when $q = 2$ and s tends to 1, we can compute $E_{2,1}$ (i.e., Tsallis-2 entanglment T_2) that $E_{2,1}(|\psi\rangle_{A|BC}) = 1 - \mathrm{Tr}\rho_A^2$, $E_{2,1}(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle_{AB}\}} \sum_i p_i E_{2,1}(|\psi_i\rangle_{AB})$, where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi\rangle_{AB}\langle\psi|$ with $\sum_i p_i = 1$ and $\rho_A = \mathrm{Tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$, $\rho_{AB} = \mathrm{Tr}_C|\psi\rangle_{ABC}\langle\psi|$. Then we get $E_{2,1}(|\psi\rangle_{A|BC}) = 2\lambda_0^2(\lambda_2^2 + \lambda_3^2 + \lambda_4^2)$, $E_{2,1}(\rho_{AB}) = 2\lambda_0^2\lambda_2^2$, and $E_{2,1}(\rho_{AC}) = 2\lambda_0^2\lambda_3^2$. Taking $\lambda_2 = \sqrt{2}/2$, $\lambda_0 = \lambda_1 = \lambda_3 = \lambda_4 = 1/2\sqrt{2}$, one can get $\left(E_{2,1}(|\psi\rangle_{A|BC})\right)^\gamma = (3/16)^\gamma$, $(E_{2,1}(\rho_{AC}))^\gamma + (2^{\gamma/\alpha} - 1)\left(E_{2,1}(\rho_{AB})\right)^\gamma = (1/32)^\gamma + (2^{\gamma/\alpha} - 1)(1/8)^\gamma$. Let $k = 2$, we have $(E_{2,1}(\rho_{AC}))^\gamma + \mathcal{K}_{(\alpha,\gamma)}\left(E_{2,1}(\rho_{AB})\right)^\gamma = (1/32)^\gamma + ((3/2)^{\gamma/\alpha} - (1/2)^{\gamma/\alpha})(1/8)^\gamma$. One can see that our result is better than the results in inequality (8) for $0 \leq \gamma \leq 1$ and $\alpha \geq 1$, see Fig. 1. To be more clear, set $z = \mathcal{K}_{(\alpha,\gamma)}\left(E_{2,1}(\rho_{AB})\right)^\gamma - (2^{\gamma/\alpha} - 1)\left(E_{2,1}(\rho_{AB})\right)^\gamma = ((3/2)^{\gamma/\alpha} - (1/2)^{\gamma/\alpha})(1/8)^\gamma - (2^{\gamma/\alpha} - 1)(1/8)^\gamma$, i.e., z represents the difference of the UE between inequalities (18) and (8) on the right side, see Fig. 2.

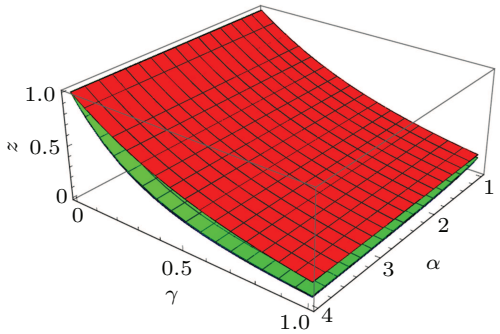


Fig. 1. The axis z represents the lower bounds of UE of $|\psi\rangle_{ABC}$, which are functions of α, γ. The red surface represents the UE of the state $|\psi\rangle$, blue surface represents the lower bound from our result, green surface (just below the green one) represents the lower bound from the result in Ref. [16].

Although Theorem 2 gives a new class of monogamy relations for multiqubit states, however, its condition is not always satisfied. To get ride of the strict condition for inequality (18), we give out a universal monogamy inequality as follow.

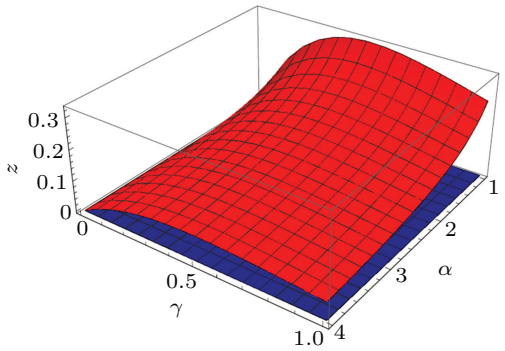


Fig. 2. The red surface represents the difference of the UE between inequalities (18) and (8) on the right side. The blue surface is zero plane of z.

Theorem 3 Let $0 \leq \gamma \leq \alpha$, $q \geq 2$, $0 \leq s \leq 1$, $qs \leq 3$, $\mathcal{K}_{(\alpha,\gamma)} = [(1+k)^{\gamma/\alpha} - 1]/k^{\gamma/\alpha}$, and let $\rho_{AB_1B_2\cdots B_{N-1}}$ be any N-qubit quantum state with $kE_{q,s}(\rho_{AB_i}) \leq E_{q,s}(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i = 1,2,\ldots,m$ and $E_{q,s}(\rho_{AB_j}) \geq kE_{q,s}(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m+1,\ldots,N-2$, $\forall\ 1 \leq m \leq N-3$, $N \geq 4$. Then it holds that

$$\begin{aligned} & \left(E_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}})\right)^\gamma \\ \geq\; & \left(E_{q,s}(\rho_{AB_1})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)}\left(E_{q,s}(\rho_{AB_2})\right)^\gamma \\ & + \cdots + \left(\mathcal{K}_{(\alpha,\gamma)}\right)^{m-1}\left(E_{q,s}(\rho_{AB_m})\right)^\gamma \\ & + \left(\mathcal{K}_{(\alpha,\gamma)}\right)^{m+1}\left(\left(E_{q,s}(\rho_{AB_{m+1}})\right)^\gamma + \cdots + \left(E_{q,s}(\rho_{AB_{N-2}})\right)^\gamma\right) \\ & + \left(\mathcal{K}_{(\alpha,\gamma)}\right)^m\left(E_{q,s}(\rho_{AB_{N-1}})\right)^\gamma. \end{aligned} \tag{20}$$

Proof For N-qubit state $\rho_{AB_1B_2\cdots B_{N-1}}$, from the inequality (15), we have

$$\begin{aligned} & \left(E_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}})\right)^\gamma \\ \geq\; & \left(E_{q,s}(\rho_{AB_1})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)}\left(E_{q,s}(\rho_{A|B_2\cdots B_{N-1}})\right)^\gamma \\ \geq\; & \left(E_{q,s}(\rho_{AB_1})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)}\left(E_{q,s}(\rho_{AB_2})\right)^\gamma \\ & + \left(\mathcal{K}_{(\alpha,\gamma)}\right)^2\left(E_{q,s}(\rho_{A|B_3\cdots B_{N-1}})\right)^\gamma \\ \geq\; & \cdots \\ \geq\; & \left(E_{q,s}(\rho_{AB_1})\right)^\gamma + \mathcal{K}_{(\alpha,\gamma)}\left(E_{q,s}(\rho_{AB_2})\right)^\gamma \\ & + \cdots + \left(\mathcal{K}_{(\alpha,\gamma)}\right)^{m-1}\left(E_{q,s}(\rho_{AB_m})\right)^\gamma \\ & + \left(\mathcal{K}_{(\alpha,\gamma)}\right)^m\left(E_{q,s}(\rho_{A|B_{m+1}\cdots B_{N-1}})\right)^\gamma. \end{aligned} \tag{21}$$

Similarly, as $\left(E_{q,s}(\rho_{AB_j})\right)^\gamma \geq k\left(E_{q,s}(\rho_{A|B_{j+1}\cdots B_{N-1}})\right)^\gamma$ for $j = m+1,\ldots,N-2$, we get

$$\begin{aligned} & \left(E_{q,s}(\rho_{A|B_{m+1}\cdots B_{N-1}})\right)^\gamma \\ \geq\; & \mathcal{K}_{(\alpha,\gamma)}\left(E_{q,s}(\rho_{AB_{m+1}})\right)^\gamma + \left(E_{q,s}(\rho_{A|B_{m+2}\cdots B_{N-1}})\right)^\gamma \\ \geq\; & \mathcal{K}_{(\alpha,\gamma)}\left(\left(E_{q,s}(\rho_{AB_{m+1}})\right)^\gamma + \cdots + \left(E_{q,s}(\rho_{AB_{N-2}})\right)^\gamma\right) \\ & + \left(E_{q,s}(\rho_{AB_{N-1}})\right)^\gamma. \end{aligned} \tag{22}$$

Combining inequalities (21) and (22), we get Theorem 3.

Theorem 3 gives a general monogamy inequality satisfied by the γ-th power of UE for the case of $0<\gamma<\alpha$ and $\alpha\geq 1$ with less constriction. Specifically, if $\gamma=1/2$ and $k=\alpha=1$, we obtain the monogamy inequality established by the UE

$$\sqrt{E_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}})}\geq\sum_{i=1}^{m}(\sqrt{2}-1)^{i-1}\sqrt{E_{q,s}(\rho_{AB_i})}$$
$$+(\sqrt{2}-1)^{m+1}\sum_{j=m+1}^{N-2}\sqrt{E_{q,s}(\rho_{AB_j})}$$
$$+(\sqrt{2}-1)^{m}\sqrt{E_{q,s}(\rho_{AB_{N-1}})},$$

which was absent in Ref. [16].

4. Polygamy relations for multiqubit systems

As a dual relation to monogamy inequality, polygamy inequality is one of the hot issues in the study of quantum information theory in recent years. Being an intriguing feature of quantum entanglement, it is also closely related to many quantum information and communication processing tasks. In this section, we will provide a class of polygamy inequalities in multiqubit systems based on UEoA, which are tighter than the existing ones. To this end, we shall first give the following Lemma.

Lemma 2 For any real numbers x, t, and k, if $x\geq 1$ and $t\geq k$, then we have

$$(1+t)^x\leq 1+\frac{(1+k)^x-1}{k^x}t^x, \tag{23}$$
$$\frac{(1+k)^x-1}{k^x}\geq 1. \tag{24}$$

Proof Analogously to Lemma 1, construct a binary function $f(x,y)=(1+y)^x-y^x$ with $x\geq 1$, $0<y\leq 1/k$. Obviously, $f(x,y)$ is an increasing function of y due to the fact that $\partial f/\partial y=x[(1+y)^{x-1}-y^{x-1}]\geq 0$. Therefore, $f(x,y)\leq f(x,1/k)=[(1+k)^x-1]/k^x$. Set $y=1/t$ with $t\geq k$, we obtain

$$(1+t)^x\leq 1+\frac{(1+k)^x-1}{k^x}t^x. \tag{25}$$

Furthermore,

$$\frac{(1+k)^x-1}{k^x}\geq\frac{(1+t)^x-1}{t^x} \tag{26}$$

is valid for all $t\geq k$. Fixed k and letting t in inequality (26) go to $+\infty$, we get inequality (24).

Using the similar method to the proof of Theorem 1 and Lemma 2,we have

Theorem 4 For any mixed state ρ_{ABC} of $H_A\otimes H_B\otimes H_C=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$, let $0\leq\beta\leq 1$, $\delta\geq\beta$, $-q^2+4q-3\leq s\leq 1$, $1\leq q\leq 2$, $k\leq 1$ and $\mathcal{K}_{(\beta,\delta)}=[(1+k)^{\delta/\beta}-1]/k^{\delta/\beta}$. Then

(i) When $E^{\mathrm{a}}_{q,s}(\rho_{AC})\geq kE^{\mathrm{a}}_{q,s}(\rho_{AB})$, it holds that

$$(E^{\mathrm{a}}_{q,s}(\rho_{A|BC}))^{\delta}\leq(E^{\mathrm{a}}_{q,s}(\rho_{AB}))^{\delta}+\mathcal{K}_{(\beta,\delta)}(E^{\mathrm{a}}_{q,s}(\rho_{AC}))^{\delta}. \tag{27}$$

(ii) When $E^{\mathrm{a}}_{q,s}(\rho_{AB})\geq kE^{\mathrm{a}}_{q,s}(\rho_{AC})$, it holds that

$$(E^{\mathrm{a}}_{q,s}(\rho_{A|BC}))^{\delta}\leq(E^{\mathrm{a}}_{q,s}(\rho_{AC}))^{\delta}+\mathcal{K}_{(\beta,\delta)}(E^{\mathrm{a}}_{q,s}(\rho_{AB}))^{\delta}. \tag{28}$$

Proof For $k\leq 1$ and $\delta\geq\beta\geq 0$, we see from the inequality (24) in Lemma 2 that $\mathcal{K}_{(\beta,\delta)}\geq 1$. According to Ref. [16], for $0\leq\beta\leq 1$,

$$(E^{\mathrm{a}}_{q,s}(\rho_{A|BC}))^{\beta}\leq(E^{\mathrm{a}}_{q,s}(\rho_{AB}))^{\beta}+(E^{\mathrm{a}}_{q,s}(\rho_{AC}))^{\beta}. \tag{29}$$

(I) Suppose that $E^{\mathrm{a}}_{q,s}(\rho_{AC})\geq kE^{\mathrm{a}}_{q,s}(\rho_{AB})$. When $E^{\mathrm{a}}_{q,s}(\rho_{AB})=0$, since $\mathcal{K}_{(\beta,\delta)}\geq 1$, the inequality (29) yields that for $\delta\geq\beta\geq 0$,

$$\begin{aligned}&(E^{\mathrm{a}}_{q,s}(\rho_{A|BC}))^{\delta}\leq(E^{\mathrm{a}}_{q,s}(\rho_{AC}))^{\delta}\leq\mathcal{K}_{(\beta,\delta)}(E^{\mathrm{a}}_{q,s}(\rho_{AC}))^{\delta}\\&=(E^{\mathrm{a}}_{q,s}(\rho_{AB}))^{\delta}+\mathcal{K}_{(\beta,\delta)}(E^{\mathrm{a}}_{q,s}(\rho_{AC}))^{\delta},\end{aligned} \tag{30}$$

leading to inequality (27). Next we assume that $E^{\mathrm{a}}_{q,s}(\rho_{AB})>0$ and take $t=[(E^{\mathrm{a}}_{q,s}(\rho_{AC}))^{\beta}]/[(E^{\mathrm{a}}_{q,s}(\rho_{AB}))^{\beta}]$, $x=\delta/\beta$. Making use of inequality (29) and Lemma 2, similarly as the proof of Theorem 1, we have

$$\begin{aligned}(E^{\mathrm{a}}_{q,s}(\rho_{A|BC}))^{\delta}&=(E^{\mathrm{a}}_{q,s}(\rho_{A|BC}))^{\beta x}\\&\leq((E^{\mathrm{a}}_{q,s}(\rho_{AB}))^{\beta}+(E^{\mathrm{a}}_{q,s}(\rho_{AC}))^{\beta})^{x}\\&=(E^{\mathrm{a}}_{q,s}(\rho_{AB}))^{\beta x}(1+t)^{x}\\&\leq(E^{\mathrm{a}}_{q,s}(\rho_{AB}))^{\beta x}\left[1+\frac{(1+k)^x-1}{k^x}t^x\right]\\&=(E^{\mathrm{a}}_{q,s}(\rho_{AB}))^{\beta x}+\mathcal{K}_{(\beta,\delta)}(E^{\mathrm{a}}_{q,s}(\rho_{AC}))^{\beta x}.\end{aligned}$$

(II) Suppose that $E^{\mathrm{a}}_{q,s}(\rho_{AB})\geq kE^{\mathrm{a}}_{q,s}(\rho_{AC})$ holds. When $E^{\mathrm{a}}_{q,s}(\rho_{AC})=0$, since $\mathcal{K}_{(\beta,\delta)}\geq 1$, the inequality (28) follows from inequality (29). When $E^{\mathrm{a}}_{q,s}(\rho_{AC})>0$, using Lemma 2 for $t=(E^{\mathrm{a}}_{q,s}(\rho_{AB}))^{\beta}/(E^{\mathrm{a}}_{q,s}(\rho_{AC}))^{\beta}$ and $x=\delta/\beta$ yields inequality (28).

By using Theorem 4 repeatedly, with the similar method in the proof of Theorem 2, we have the following theorem for multipartite quantum systems.

Theorem 5 Let $0\leq\beta\leq 1$, $\delta\geq\beta$, $-q^2+4q-3\leq s\leq 1$, $1\leq q\leq 2$, $k\leq 1$ and $\mathcal{K}_{(\beta,\delta)}=[(1+k)^{\delta/\beta}-1]/k^{\delta/\beta}$. For any N-qubit quantum state $\rho_{AB_1\cdots B_{N-1}}$ with $k(E^{\mathrm{a}}_{q,s}(\rho_{AB_i}))^{\beta}\leq\sum_{j=i+1}^{N-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta}$ for $i=1,\ldots,N-2$, it holds that

$$(E^{\mathrm{a}}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^{\delta}\leq\sum_{j=1}^{N-1}(\mathcal{K}_{(\beta,\delta)})^{j-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\delta}. \tag{31}$$

Proof By the inequality (23) in Lemma 2,

$$\begin{aligned}&(E^{\mathrm{a}}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^{\beta x}\\&\leq(E^{\mathrm{a}}_{q,s}(\rho_{AB_1}))^{\beta x}+\mathcal{K}_{(\beta,\delta)}\left(\sum_{j=2}^{N-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta}\right)^{x}\\&\leq(E^{\mathrm{a}}_{q,s}(\rho_{AB_1}))^{\beta x}+\mathcal{K}_{(\beta,\delta)}(E^{\mathrm{a}}_{q,s}(\rho_{AB_2}))^{\beta x}\\&\quad+(\mathcal{K}_{(\beta,\delta)})^{2}\left(\sum_{j=2}^{N-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta}\right)^{x}\\&\leq\cdots\end{aligned}$$

$$\leq (E^{\mathrm{a}}_{q,s}(\rho_{AB_1}))^{\beta x}+\mathcal{K}_{(\beta,\delta)}(E^{\mathrm{a}}_{q,s}(\rho_{AB_2}))^{\beta x} +\cdots+(\mathcal{K}_{(\beta,\delta)})^{N-2}(E^{\mathrm{a}}_{q,s}(\rho_{AB_{N-1}}))^{\beta x}.$$

Setting $\beta x=\delta$ in the above formula gives the claim in the theorem.

Recently, for any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$, another class of multiqubit polygamy inequalities in terms of the β-th power was found in Ref. [16].

$$(E^{\mathrm{a}}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^{\beta}\leq\sum_{j=1}^{N-1}(2^{\beta}-1)^{j-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta} \quad (32)$$

for $\beta\geq 1$, $-q^2+4q-3\leq s\leq 1$, $1\leq q\leq 2$. Since $\mathcal{K}_{(\beta,\delta)}\leq 2^{\beta}-1$ with $0\leq\beta\leq 1$, formula (31) in Theorem 5 obviously yields a tighter polygamy relation with stringent upper bounds than expression (32) in Ref. [16].

Example 2 Let us consider again the three-qubit state $|\psi\rangle_{ABC}$ of Eq. (19). From the definition of UEoA in Eq. (5), when $q=2$ and s tends to 1, we have $T_2(|\psi\rangle_{A|BC})=1-\mathrm{Tr}\rho_A^2$, $T_2(\rho_{AB})=\max_{\{p_i,|\psi_i\rangle_{AB}\}}\sum_i p_i T_2(|\psi_i\rangle_{AB})$, where the maximum is taken over all possible pure state decompositions of $\rho_{AB}=\sum_i p_i|\psi\rangle_{AB}\langle\psi|$ and $\sum_i p_i=1$ and $\rho_A=\mathrm{Tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$, $\rho_{AB}=\mathrm{Tr}_C|\psi\rangle_{ABC}\langle\psi|$. Then we have $T^{\mathrm{a}}_2(|\psi\rangle_{A|BC})=2\lambda_0^2(\lambda_2^2+\lambda_3^2+\lambda_4^2)$, $T^{\mathrm{a}}_2(\rho_{AB})=2\lambda_0^2(\lambda_2^2+\lambda_4^2)$, and $T^{\mathrm{a}}_2(\rho_{AC})=2\lambda_0^2(\lambda_3^2+\lambda_4^2)$. Let $\lambda_2=\sqrt{2}/2$, $\lambda_0=\lambda_1=\lambda_3=\lambda_4=1/2\sqrt{2}$, one can get $(T^{\mathrm{a}}_2(|\psi\rangle_{A|BC}))^{\beta}=(3/16)^{\beta}$, $(T^{\mathrm{a}}_2(\rho_{AB}))^{\beta}=(5/32)^{\beta}$, and $(T^{\mathrm{a}}_2(\rho_{AC}))^{\beta}=(1/16)^{\beta}$. Thus, from the result in Ref. [16], one has $(T^{\mathrm{a}}_2(\rho_{AB}))^{\beta}+(2^{\beta}-1)(T^{\mathrm{a}}_2(\rho_{AC}))^{\beta}=(5/32)^{\beta}+(2^{\beta}-1)(1/16)^{\beta}$. Specially, let $k=2$ and $\beta=1$, then we have $(T^{\mathrm{a}}_2(\rho_{AB}))^{\delta}+\mathcal{K}_{(1,\delta)}(T^{\mathrm{a}}_2(\rho_{AC}))^{\delta}=(5/32)^{\delta}+((3/2)^{\delta}-(1/2)^{\delta})(1/16)^{\delta}$. It is obvious that the result in this work is better than inequality (32) of Ref. [12] for $\delta\geq 1$, see Fig. 3.

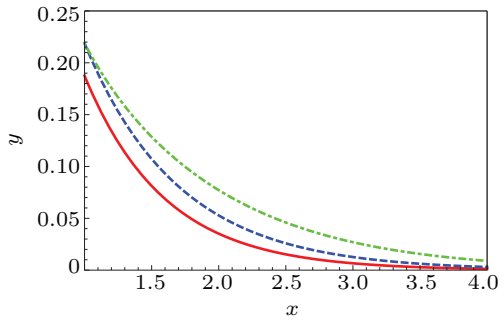


Fig. 3. The axis y denotes the upper bound of the UEoA of $|\psi\rangle_{ABC}$, which are functions of x-th power of quantum relations. The red solid line represents the UEoA of $|\psi\rangle_{ABC}$ in Eq. (19), blue dashed line represents the upper bound of our result, green dot-dashed line represents the upper bound given in Ref. [16].

With a similar consideration of Theorem 3, we have the following widespread result with less constriction compared with Theorem 5.

Theorem 6 Let $0\leq\beta\leq 1$, $\delta\geq\beta$, $-q^2+4q-3\leq s\leq 1$, $1\leq q\leq 2$, $k\leq 1$ and $\mathcal{K}_{(\beta,\delta)}=[(1+k)^{\delta/\beta}-1]/k^{\delta/\beta}$. For any N-qubit quantum state $\rho_{AB_1\cdots B_{N-1}}$ with $k(E^{\mathrm{a}}_{q,s}(\rho_{AB_i}))^{\beta}\leq\sum_{j=i+1}^{N-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta}$ for $i=1,\ldots,m$ and $(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta}\geq k\sum_{l=j+1}^{N-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_l}))^{\beta}$ for $j=m+1,\ldots,N-2$, $\forall\ 1\leq m\leq N-3$, $N\geq 4$, it holds that

$$\begin{aligned}(E^{\mathrm{a}}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^{\beta}\leq&\sum_{i=1}^{m}(\mathcal{K}_{(\beta,\delta)})^{i-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_m}))^{\beta}\\&+(\mathcal{K}_{(\beta,\delta)})^{m+1}((E^{\mathrm{a}}_{q,s}(\rho_{AB_{m+1}}))^{\beta}\\&+\cdots+(E^{\mathrm{a}}_{q,s}(\rho_{AB_{N-2}}))^{\beta})\\&+(\mathcal{K}_{(\beta,\delta)})^{m}(E^{\mathrm{a}}_{q,s}(\rho_{AB_{N-1}}))^{\beta}. \quad (33)\end{aligned}$$

Proof From inequality (9) and Lemma 2, we have

$$\begin{aligned}&(E^{\mathrm{a}}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^{\beta x}\\&\leq(E^{\mathrm{a}}_{q,s}(\rho_{AB_1}))^{\beta}+\mathcal{K}_{(\beta,\delta)}\left(\sum_{j=2}^{N-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta}\right)^{x}\\&\leq(E^{\mathrm{a}}_{q,s}(\rho_{AB_1}))^{\beta x}+\mathcal{K}_{(\beta,\delta)}(E^{\mathrm{a}}_{q,s}(\rho_{AB_2}))^{\beta x}\\&\quad+(\mathcal{K}_{(\beta,\delta)})^{2}\left(\sum_{j=3}^{N-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta}\right)^{x}\\&\leq\cdots\\&\leq(E^{\mathrm{a}}_{q,s}(\rho_{AB_1}))^{\beta x}+\cdots+(\mathcal{K}_{(\beta,\delta)})^{m-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_m}))^{\beta x}\\&\quad+(\mathcal{K}_{(\beta,\delta)})^{m}\left(\sum_{j=m+1}^{N-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta}\right)^{x}. \quad (34)\end{aligned}$$

Similarly, as $(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta}\geq k\sum_{l=j+1}^{N-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_l}))^{\beta}$ for $j=m+1,\ldots,N-2$, we can get

$$\begin{aligned}&\left(\sum_{j=m+1}^{N-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta}\right)^{x}\\&\leq\mathcal{K}_{(\beta,\delta)}(E^{\mathrm{a}}_{q,s}(\rho_{AB_{m+1}}))^{\beta x}+\left(\sum_{j=m+2}^{N-1}(E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta}\right)^{x}\\&\leq\mathcal{K}_{(\beta,\delta)}((E^{\mathrm{a}}_{q,s}(\rho_{AB_{m+1}}))^{\beta x}+\cdots+(E^{\mathrm{a}}_{q,s}(\rho_{AB_{N-2}}))^{\beta x})\\&\quad+(E^{\mathrm{a}}_{q,s}(\rho_{AB_{N-1}}))^{\beta x}. \quad (35)\end{aligned}$$

Combining inequalities (34) and (35), we then obtain Theorem 6.

5. Conclusion

Entanglement monogamy and polygamy are two fundamental properties of multipartite entanglement. Based on unified-(q,s) entropy, we provide a characterization of complementary relations for multiqubit states. On the one hand, we derive a class of complementary monogamy inequalities for multiqubit entanglement based on the γ-th ($0\leq\gamma\leq\alpha,\alpha\geq 1$) power of unified-$(q,s)$ entanglement, which turn out to be tighter than the previous results in Ref. [12]. On the other hand, we established polygamy relations as a dual property of monogamy in terms of the δ-th ($\delta\geq\beta,0\leq\beta\leq 1$) power of unified-$(q,s)$ entanglement of assistance.

We mention that since unified-(q,s) entropy entanglement is a general bipartite entanglement measure, our work indeed gives a general class of the complementary monogamy

and polygamy inequalities. By application of the results above, the corresponding monogamy and polygamy relations for special cases of unified-(q,s) entropy entanglement such as RE and TE can be obtained immediately. Furthermore, monogamy and polygamy relations can be interpreted as one class of distributions of entanglement in multipartite systems, from this point of view, tighter monogamy relations mean better characterizations of the entanglement distribution. Therefore, our results complement and unify the previous results for monogamy relations in literature, which also shed light on the study of the monogamy and polygamy inequalities about quantum correlations.

Acknowledgements

Project supported by the National Natural Science Foundation of China (Grant No. 12175147), the Disciplinary Funding of Beijing Technology and Business University, the Fundamental Research Funds for the Central Universities (Grant No. 2022JKF02015), and the Research and Development Program of Beijing Municipal Education Commission (Grant No. KM202310011012).

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