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Quantum control based on three forms of Lyapunov functions

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This paper introduces the quantum control of Lyapunov functions based on the state distance, the mean of imaginary quantities and state errors. In this paper, the specific control laws under the three forms are given. Stability is analyzed by the LaSalle invariance principle and the numerical simulation is carried out in a 2D test system. The calculation process for the Lyapunov function is based on a combination of the average of virtual mechanical quantities, the particle swarm algorithm and a simulated annealing algorithm. Finally, a unified form of the control laws under the three forms is given.

Keywords: quantum system, Lyapunov function, particle swarm optimization, simulated annealing algorithms, quantum control

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1. Introduction

Quantum concepts were first proposed by the German physicist Planck in 1900 and they now play an important role in modern physics.[1,2] Planck hypothesized that the radiant energy in blackbody radiation is discontinuous and can only take integer multiples of the basic unit of energy, thus explaining the experimental phenomenon of blackbody radiation.[3] Later studies showed that this discontinuous separation property was not exclusive to energy, with other physical quantities, such as angular momentum,[4] spin,[5,6] and charge,[7] also showing this discontinuous quantization. This is fundamentally different from classical physics represented by Newtonian mechanics. The phenomenon of quantization is mainly manifested in the microscopic physical world. The physical theory that describes the microscopic physical world is quantum mechanics. After Planck proposed the quantum concept, it was perfected by Einstein, Bohr, de Broglie and others in the first half of the 20th century and a complete theory of quantum mechanics was established.

The development of laser technology[8,9] and a series of quantum information technologies has led to great interest in feedback control of quantum systems. Quantum systems can be divided into closed and open quantum systems. For closed quantum systems, the kinetic equations can be represented by the Schrödinger equation or the non-dissipative Liuville–von Neumann equation. This paper focuses on closed quantum systems, where open-loop control[10,11] currently dominates. In open-loop control, given the initial state and target state of a closed quantum system, we need to find a suitable control to make the initial state tend toward the target state.

Quantum control[12,13] problems usually translate as control over unitary transformation groups acting on state spaces. Special physical phenomena such as state superposition, state entanglement and continuous spectra in quantum systems lead to a series of unconventional control problems and their theoretical expressions. Quantum control research began at the end of the 1970s and is mainly limited to the modeling of quantum control systems, the analytical controllability and reversibility of quantum systems, and quantum filters.[14,15] Quantum control is the result of the intersection of quantum mechanics and control theory, with the goal of establishing universal theories and methods for controlling quantum systems. The research on quantum control comes from some important fields of modern quantum technology, such as quantum information processing,[16] bonding quantum chemistry, laser cooling, quantum optics[17,18] and new nanomaterials.[19,20] The Lyapunov function is named after the Russian mathematician Aleksandr Lyapunov and it plays an important role in stability theory of dynamic systems. Lyapunov stability was proposed in 1892. It employs a state vector function as a way to determine the stability of a system. It is used more widely than other criteria such as the algebraic criterion in classical control theory. It is suitable not only for univariate, linear, stationary systems, but also for multivariate, nonlinear, time-varying systems. The idea of quantum Lyapunov control is to construct a suitable Lyapunov function and design the corresponding control law by ensuring that the first-order partial derivative of the Lyapunov function with respect to time is not greater than zero. Various Lyapunov control methods[21–23] have been developed for quantum systems described by the Schrödinger equation or the Liuvile–von Neumann equation. This paper introduces the design of control laws under three classes of Lyapunov function in Ref. [24], including one based on state distance, one based on the mean of imaginary quantities and another based on state error. Based on the unknown parameters in the Lyapunov function of the mean of virtual mechanical quantities, we use machine learning algorithms for our

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calculations and finally give a unified form of the control law under the three forms.

Machine learning[25] is a multidisciplinary discipline that studies how computers simulate or implement human learning behaviors to acquire new knowledge or skills. Machine learning is a research hotspot in artificial intelligence and its theories and methods have been widely used to solve complex problems in engineering applications and scientific fields. Particle swarm optimization and simulated annealing algorithms are machine learning algorithms and, in this paper, we use both of these algorithms to help design unknown parameters for Lyapunov control.

The order of this article is as follows: in Section 2, we describe the system model under study. In Section 3, the Lyapunov function based on state distance is considered and the control law under the system model is given. Combined with the principle of LaSalle invariance, the stability theorem is given and proved, and the numerical simulation is carried out under the 2D test system. In Section 4, the Lyapunov function based on the mean of imaginary quantities is considered and the control law under the system is given. The LaSalle invariance principle is used to give the stability theorem and prove it for the representation of the unknown parameter of the average value of virtual mechanical quantities. The particle swarm and simulated annealing algorithms are used to perform numerical simulations in a 2D test system. In Section 5, the Lyapunov function based on state error is studied, the control law and stability theorem of the system are given, numerical simulations are carried out in a 2D test system, and the unified form of control under the three forms is given. The conclusions are in Section 6.

2. System model

We consider a closed quantum system and assume that it is purely controllable. Its mathematical model is described by the following equation:

$$\mathrm{i}\hbar|\dot{\psi}(t)\rangle = (H_0 + u(t)H_1 + u^2(t)H_2)|\psi(t)\rangle, \tag{1}$$

where H_0 is the internal Hamiltonian of the system, H_1 and H_2 is the amount of control Hamiltonian in contact with the outside world. $\hbar$ is set to be 1 and $u(t)$ is the control field. H_0 can be expressed as $H_0 = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ and λ_i $(i = 1, 2, \ldots, n)$ is the eigenvalue corresponding to the eigenvector $|\lambda_i\rangle$ in H_0.

In this system where the Hamiltonian is $H(t) = H_0 + u(t)H_1 + u^2(t)H_2$, we consider a system with a Hamiltonian $H(t) = H_0 + u(t)H_1 + v(t)H_2$, where $v(t)$ is a second control system independent of $u(t)$. The system is controllable[26] and under the same conditions as $H(t) = H_0 + u(t)H_1 + u^2(t)H_2$, even with the Hamiltonian $H(t) = H_0 + u(t)H_1 + v(t)H_2$ all the target states that can be achieved can also be achieved by $H(t) = H_0 + u(t)H_1 + u^2(t)H_2$. $v(t) = u^2(t)$ is a special case. For the sake of theoretical controllability, we can think that u is independent of u^2 and then for u there is a functional relationship with u^2. It will play a role in the numerical algorithm of finding the control law.

For simplicity, we added a second control ω.[27] Therefore, we consider the following control system:

$$\mathrm{i}|\dot{\psi}(t)\rangle = (H_0 + u(t)H_1 + u^2(t)H_2 + \omega I)|\psi(t)\rangle, \tag{2}$$

where $\omega \in R$ is a new control that normalizes degrees of freedom and can be used to adjust the global phase without changing the system distribution. u and ω are independent of each other.

3. Lyapunov function based on state distance

3.1. Design of control laws

Here we choose the function based on the Hilbert–Schmidt distance between the controlled state $|\psi\rangle$ and the desired target state $|\psi_\mathrm{f}\rangle$ as the Lyapunov function, i.e.,

$$V_1 = \frac{1}{2}(1 - |\langle\psi_\mathrm{f}|\psi\rangle|^2), \tag{3}$$

where $|\langle\psi_\mathrm{f}|\psi\rangle|^2$ represents the transition probability from $|\psi\rangle$ to $|\psi_\mathrm{f}\rangle$.

Considering the control system of Eq. (2), in quantum control, the target state is usually the eigenstate of the Hamiltonian H_0, i.e., the target state $|\psi_\mathrm{f}\rangle$ satisfies the following conditions:

$$H_0|\psi_\mathrm{f}\rangle = \lambda_\mathrm{f}|\psi_\mathrm{f}\rangle. \tag{4}$$

The first-order time derivative of a V_1 can be calculated using

$$\begin{aligned}\dot{V}_1 &= -\langle\psi_\mathrm{f}|\psi\rangle\langle\psi_\mathrm{f}|\dot{\psi}\rangle \\ &= -u\mathrm{Im}\langle\psi|\psi_\mathrm{f}\rangle\langle\psi_\mathrm{f}|H_1|\psi\rangle - u^2\mathrm{Im}\langle\psi|\psi_\mathrm{f}\rangle\langle\psi_\mathrm{f}|H_2|\psi\rangle \\ &\quad -\omega\mathrm{Im}\langle\psi|\psi_\mathrm{f}\rangle\langle\psi_\mathrm{f}|\psi\rangle,\end{aligned} \tag{5}$$

where Im represents the imaginary part.

Let $I_1 = \mathrm{Im}\langle\psi|\psi_\mathrm{f}\rangle\langle\psi_\mathrm{f}|H_1|\psi\rangle$, $I_2 = \mathrm{Im}\langle\psi|\psi_\mathrm{f}\rangle\langle\psi_\mathrm{f}|H_2|\psi\rangle$, $I_3 = \mathrm{Im}\langle\psi|\psi_\mathrm{f}\rangle\langle\psi_\mathrm{f}|\psi\rangle$, then Eq. (5) can be abbreviated as

$$\dot{V}_1 = -uI_1 - u^2I_2 - \omega I_3.$$

For simplicity and efficiency, here we design the control law guaranteeing $\dot{V}_1 \leq 0$ in the following special form, so that uI_1, u^2I_2, ωI_3 are all non-negative.[24]

Here we take

$$\begin{cases} u = \frac{I_1}{(1+\frac{1}{2}I_2)^2}, \\ \omega = I_3. \end{cases} \tag{6}$$

3.2. Stability analysis

For stability analysis, we consider the asymptotic stability of the system, i.e., the gradual convergence to the target state. We will analyze the asymptotic convergence to the goal state via LaSalle's invariance principle. In this part, we first provide the stability theorem, then give the proof.

Theorem 1 Considering control (6) and system (2), if the spectrum of H_0 is not degenerate, then the largest invariant set of the closed-loop system is $S^{2N-1}\cap E_1$, $E_1=\{|\psi\rangle:\langle\psi_f|H_k|\psi\rangle=0$ and $\mathrm{Im}\langle\psi_f|\psi\rangle=0,\ k=1,2\}$; if all the solutions of $\langle\psi_f|H_k|\psi\rangle=0$ and $\mathrm{Im}\langle\psi_f|\psi\rangle=0$ are within the equivalence class of $|\psi_f\rangle$, the closed loop system will be asymptotically stable.

Proof From the point of view of control (6), $\dot{V}_1=0$ is equivalent to $u=0$, $\omega=0$, then system (2) can be written as

$$\mathrm{i}|\dot{\psi}(t)\rangle=H_0|\psi(t)\rangle. \tag{7}$$

Suppose that the state of the t_0 at the moment $|\psi(t_0)\rangle$ satisfies $\dot{V}_1(t_0)=0$, that is,

$$\mathrm{Im}\langle\psi_f|H_k|\psi(t_0)\rangle=0,$$
$$\mathrm{Im}\langle\psi_f|\psi(t_0)\rangle=0.$$

If the system state stops evolving to the target state, $u(t_0+\Delta t)=0$ and $\omega(t_0+\Delta t)=0$, i.e.,

$$\mathrm{Im}\langle\psi_f|H_k|\psi(t_0+\Delta t)\rangle=0, \tag{8}$$
$$\mathrm{Im}\langle\psi_f|\psi(t_0+\Delta t)\rangle=0, \tag{9}$$

where $\Delta t\in(0,\infty)$, combined with the solution of Eq. (7)

$$|\psi(t_0+\Delta t)\rangle=\mathrm{e}^{-\mathrm{i}H_0\Delta t}|\psi(t_0)\rangle.$$

Substituting into Eqs. (8) and (9), we get

$$\mathrm{Im}\langle\psi_f|\mathrm{e}^{\mathrm{i}H_0\Delta t}H_k|\psi(t_0)\rangle=0, \tag{10}$$
$$\mathrm{Im}\langle\psi_f|\mathrm{e}^{\mathrm{i}H_0\Delta t}|\psi(t_0)\rangle=0. \tag{11}$$

Consider $\mathrm{e}^{\mathrm{i}H_0\Delta t}=1+\frac{\mathrm{i}H_0\Delta t}{1!}+\frac{(\mathrm{i}H_0\Delta t)^2}{2!}+\cdots$. Equations (10) and (11) can be written as

$$\langle\psi_f|H_k|\psi(t_0)\rangle=0,\quad \mathrm{Im}\langle\psi_f|\psi(t_0)\rangle=0.$$

The maximum set of invariants for closed closed-loop systems can be fully characterized.

Lassalle's principle guarantees that a closed-loop system with Eq. (6) control laws (2) converges on the maximum invariant set E_1 contained in $\dot{V}_1=0$.

Remark 1 In the proof of stability, we use Lassalle's principle of invariance. Lassalle's invariant principle is a generalization of Lyapunov's second method. Here, the stability of the system at $\dot{V}_1=0$ in the complex sphere plane is mainly discussed.

3.3. Numerical simulation

We numerically simulate a 2D test system with H_0, H_1, H_2, where H_0, H_1, H_2 are

$$H_0=\begin{bmatrix}1&0\\0&2\end{bmatrix},\ H_1=\begin{bmatrix}1&1\\1&1/2\end{bmatrix},\ H_2=\begin{bmatrix}1/3&0\\-1&1\end{bmatrix}. \tag{12}$$

In this case, in order to achieve the target state $|\psi_f\rangle=|1\rangle$, the Lyapunov control of Eq. (6) is used and the total length of time is T.

In Fig. 1, we give the feedback of equation u corresponding to Eq. (6) and the evolution of the example in Eq. (12).

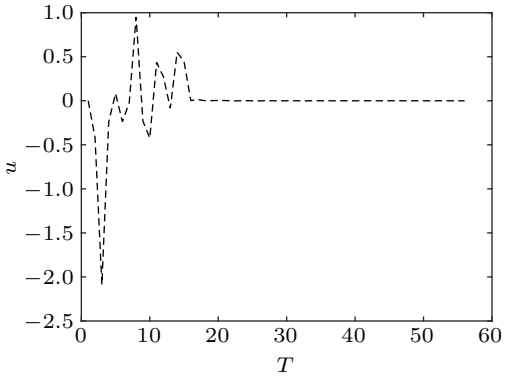


Fig. 1. Controlling the evolution of u. Initial conditions $|\psi(t_0)\rangle=|0\rangle$. The system is defined by Eq. (12), using feedback (6).

4. Lyapunov function based on the mean of imaginary quantities

4.1. Design of control laws

Suppose that the Hermitian operator P is a mechanical quantity of a quantum system. According to quantum theory, if the system is in the eigenstate of P, then the mean value of P is the eigenvalue corresponding to the eigenstate of P. From here, we consider the average of P as a Lyapunov function, i.e.,

$$V_2=\langle\psi|P|\psi\rangle. \tag{13}$$

Considering the control system of Eq. (2), the target state satisfies Eq. (4) and the V_2 first derivative with respect to time is

$$\begin{aligned}\dot{V}_2&=\langle\dot{\psi}|P|\psi\rangle+\langle\psi|P|\dot{\psi}\rangle\\&=\mathrm{i}\langle\psi|[H_0+\omega I,P]|\psi\rangle+u\mathrm{i}\langle\psi|[H_1,P]|\psi\rangle\\&\quad+u^2\mathrm{i}\langle\psi|[H_2,P]|\psi\rangle,\end{aligned} \tag{14}$$

where $[.,.]$ denotes the switcher of two matrices, such as $[A,B]=AB-BA$. A and B are Hermitian operators. Because $\mathrm{tr}([A,B])=\mathrm{tr}(AB)-\mathrm{tr}(BA)=0$, transposers are not determined by signs. For the convenience of P construction , one can make $[H_0,P]=0$.

Order $I_1 = \langle\psi|[H_1,P]|\psi\rangle$, $I_2 = \langle\psi|[H_2,P]|\psi\rangle$, then Eq. (14) can be abbreviated as

$$\dot{V}_2 = uI_1 + u^2 I_2.$$

For simplicity and effectiveness, here we design the control law of guaranteeing $\dot{V}_2 \leq 0$ in the following special form:

$$\begin{cases} u = -\dfrac{I_1}{(1+\frac{1}{2}I_2)^2}, \\ \omega, \ \text{any value can be taken.} \end{cases} \tag{15}$$

4.2. Stability analysis

A general theorem for the maximum invariant set of closed-loop systems is given as follows.

Theorem 2 Considering the control (15) and the system (2). Given the following conditions:

(i) $[H_0,P]=0$,

(ii) $\lambda_{ij} \neq \lambda_{lm}, (i,j) \neq (l,m)$,

(iii) $P_i \neq P_j, i \neq j$,

where $\lambda_{ij} = \lambda_i - \lambda_j$, $(i,j = 1,2,\ldots,n)$, λ_i is the eigenvalue of H_0 corresponding to the eigenvector $|\lambda_i\rangle$, and P_i $(1,2,\ldots,n)$ is the eigenvalue of P corresponding to the eigenvector $|\lambda_i\rangle$. Then the largest invariant set of the closed-loop system is $S^{2N-1}\cap E_2$, $E_2 = \{|\psi\rangle : \langle\lambda_j|H_k|\lambda_i\rangle\langle\lambda_i|\rho|\lambda_j\rangle = 0,\ k = 1,2,\ i,j = 1,2,\ldots,n\}$, where $\rho = |\psi\rangle\langle\psi|$.

Proof Here, we outline the basic idea of the proof through the LaSalle principle of invariance and explain the corresponding stability conditions. Since $[H_0,P]=0$, then $[H_0+\omega I,P] = 0$, where ω can take any value, for the convenience of proof, take $\omega = -\lambda_{\rm f}$. From the point of view of control (15), $\dot{V}_2 = 0$ is equivalent to $u=0$, then system (2) can be written as

$$\mathrm{i}|\dot{\psi}(t)\rangle = (H_0 - \lambda_{\rm f} I)|\psi(t)\rangle. \tag{16}$$

Suppose the state of the t_0 at the moment $|\psi(t_0)\rangle$ satisfies $\dot{V}_2(t_0) = 0$, that is,

$$\langle\psi(t_0)|[H_k,P]|\psi(t_0)\rangle = 0.$$

If the system state stops evolving to the target state, $u(t_0+\Delta t) = 0$, i.e.,

$$\langle\psi(t_0+\Delta t)|[H_k,P]|\psi(t_0+\Delta t)\rangle = 0, \tag{17}$$

where $\Delta t \in (0,\infty)$, equation (16) has the solution

$$|\psi(t_0+\Delta t)\rangle = \mathrm{e}^{-\mathrm{i}(H_0-\lambda_{\rm f} I)\Delta t}|\psi(t_0)\rangle.$$

The status at the t_0 of the system is expressed as

$$|\psi(t_0)\rangle = \sum_{i=1}^{n} c_i(t_0)|\lambda_i\rangle. \tag{18}$$

Then the system at $t_0+\Delta t$ is denoted as

$$\begin{aligned} |\psi(t_0+\Delta t)\rangle &= \mathrm{e}^{-\mathrm{i}(H_0-\lambda_{\rm f} I)\Delta t}|\psi(t_0)\rangle \\ &= \mathrm{e}^{-\mathrm{i}(H_0-\lambda_{\rm f} I)\Delta t}\sum_{i=1}^{n} c_i(t_0)|\lambda_i\rangle \\ &= \mathrm{e}^{\mathrm{i}\lambda_{\rm f} I\Delta t}\sum_{i=1}^{n} c_i(t_0)\,\mathrm{e}^{-\mathrm{i}\lambda_i\Delta t}|\lambda_i\rangle. \end{aligned} \tag{19}$$

Substituting Eq. (19) into Eq. (17) yields

$$\sum_{k=1}^{2}\sum_{i,j=1}^{n}(P_i - P_j)c_i(t_0)c_j^*(t_0)\,\mathrm{e}^{\mathrm{i}\lambda_{ji}\Delta t}\langle\lambda_j|H_k|\lambda_i\rangle = 0, \tag{20}$$

where $\lambda_{ij} = \lambda_i - \lambda_j$, equation (20) can be further written as

$$\sum_{k=1}^{2}\sum_{i,j=1}^{n}(P_i - P_j)\langle\lambda_j|H_k|\lambda_i\rangle\langle\lambda_i|\rho(t_0)|\lambda_j\rangle\,\mathrm{e}^{\mathrm{i}\lambda_{ij}\Delta t} = 0, \tag{21}$$

where $\rho(t_0) = |\psi(t_0)\rangle\langle\psi(t_0)|$, because $P_i \neq P_j$, so $\langle\lambda_j|H_k|\lambda_i\rangle \neq 0$, equation (21) is the formula

$$\langle\lambda_j|H_k|\lambda_i\rangle\langle\lambda_i|\rho(t_0)|\lambda_j\rangle = 0.$$

The LaSalle principle of invariance guarantees that a closed-loop system with Eq. (15) control laws (2) converges on the maximum invariant set E_2 contained in $\dot{V}_2 = 0$.

4.3. The construction of P

4.3.1. Particle swarm optimization algorithm

Particle swarm optimization is a kind of machine learning algorithm, which has the advantages of fast convergence speed, few parameters as well as simple and easy implementation. The fitness function is used as the optimization objective function to evaluate the quality of the particle position, decide whether to update the historical optimal position of the particle individual and the historical optimal position of the group, and ensure that the particle searches in the direction of the optimal solution, so that the global optimal solution can be found.

We provide a method that uses particle swarm optimization algorithms to find P to achieve our control goals

$$P = \sum_{i=1}^{n} P_i|\lambda_i\rangle\langle\lambda_i|.$$

In particle swarm calculation, all possible solutions to an optimization problem are called particles. Assuming that in the D-dimensional search space, there are n particles, each representing a solution, then the position vector of the i-th particle is

$$X_{id} = (x_{i1}, x_{i2}, \ldots, x_{iD}).$$

The velocity vector of the i-th particle is

$$V_{id} = (v_{i1}, v_{i2}, \ldots, v_{iD}).$$

The optimal position searched for by the i-th particle is

$$P_{id,\rm pbest} = (p_{i1}, p_{i2}, \ldots, p_{iD}).$$

The optimal location for the group search is

$$P_{d,\text{gbest}} = (p_{1,\text{gbest}}, p_{2,\text{gbest}}, \ldots, p_{D,\text{gbest}}).$$

The following two equations are commonly used to guide how to update a particle's velocity vector and position vector:

$$v_{id}^{k+1} = \eta v_{id}^{k} + c_1 r_1 (p_{id,\text{pbest}}^{k} - x_{id}^{k}) + c_2 r_2 (p_{d,\text{gbest}}^{k} - x_{id}^{k}), \quad (22)$$

$$x_{id}^{k+1} = x_{id}^{k} + v_{id}^{k+1}, \quad (23)$$

where n is the particle swarm size, i is the particle number, $i = 1, 2, \ldots, n$, D is the particle dimension, d is the particle dimension number, $d = 1, 2, \ldots, D$, k represents the number of iterations, η is the inertia weight, c_1, c_2 is a learning factor, and r_1, r_2 is a random number on the interval $[0, 1]$. v_{id}^{k} is the velocity vector of particle i in the d-dimension in the k-th iteration and x_{id}^{k} is the position vector of the d-dimensional particle i in the k-th iteration. $p_{id,\text{pbest}}^{k}$ is the optimal position of particle i after the k-th iteration. $p_{d,\text{gbest}}^{k}$ is the optimal solution in the entire particle population after the k-th iteration.

For system (2), we want to find an optimal set of parameter P_1 in the control law (15) to $P_2, \ldots, P_n$ the system to reach the target state. In particle swarms, a $P_1, P_2, \ldots, P_n$ is defined as a particle to search for optimal P. The search space is constructed by having each component of these particles take a value in $[a, b]$ and the probability of transition from the initial state of the system to the desired target state is selected as the fitness function for evaluating the mass of the particles. The particle swarm algorithm should constantly update the optimal position of the individual and the global optimal position of the entire particle swarm and update the position and velocity of each particle until the optimal P is found.

4.3.2. Simulated annealing algorithm

Here, we use another optimization algorithm, the simulated annealing algorithm. It is an optimization algorithm that simulates the cooling process of solids. The simulation process is to first heat the solid to a certain temperature. As the temperature increases, the particles inside the solid slowly become disordered, resulting in the internal energy increasing. Subsequently, the solid is allowed to cool slowly. As the temperature drops, the internal particles slowly tend to order, eventually reaching an equilibrium state. After reaching normal temperature, the system finally reaches the ground state and the internal energy is reduced to the minimum. The algorithm simulates this process to achieve the purpose of optimization. The algorithm starts from the initial solution, performs a random search in the solution space, and finally searches for the optimal value.

The basic steps of the algorithm are to determine the initial temperature value of T_0 of the annealing process, randomly generate the particle state at this temperature and a solution x_0, and calculate the objective function value f, which is the internal energy E_{x0} in this state. At the next cooling temperature, we add disturbances to the current solution x_i and generate a new solution x_j, calculate the corresponding internal energy E_{xi} and E_{xj}, judge whether to accept the new solution according to the Metropolis criterion, and update the local optimal solution and the global optimal solution. The above process is repeated. Here, the probability of transition from the initial state of the system to the desired target state is selected as the objective function.

The two different P's obtained by the particle swarm optimization algorithm and the simulated annealing algorithm are interesting and worthy of further study.

4.4. Numerical simulation

4.4.1. Numerical simulation of the particle swarm optimization algorithm

The 2D test system (12) was used for numerical simulation and the particle swarm algorithm was used to find the optimal P for control (15). We take $[a, b]$ as $[0, 1]$. Other parameter choices in the particle swarm algorithm are $\eta = 0.3$, $c_1 = c_2 = 1.5$. The number of particles is 10, the number of dimensions is 2 and the termination condition is set to 100 iterations. Through particle swarm calculations, the optimal P is found as

$$P = \text{diag}[0.8147, 0.1576].$$

In this case, in order to achieve the target state $|\psi_f\rangle = |1\rangle$, the Lyapunov control of Eq. (15) is used and the total length of time is T.

In Fig. 2, we give the feedback that u corresponds to Eq. (15) and the evolution of the Eq. (12) example.

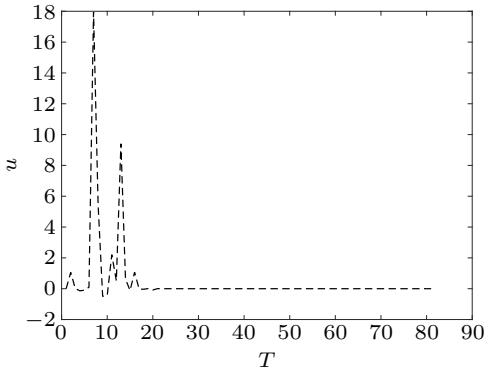


Fig. 2. Controlling the evolution of u. Initial conditions $|\psi(t_0)\rangle = |0\rangle$. The system is defined by Eq. (12), using feedback (15).

4.4.2. Numerical simulation of the simulated annealing algorithm

The 2D test system (12) was used for numerical simulation and the simulated annealing algorithm was used to find the optimal P of the control (15). The initial temperature value

T_0 is 1000, the cooling parameter is 0.9 and the length of the Markov chain is 20.

$$P = \mathrm{diag}[0.9094, 0.3343].$$

In this case, in order to achieve the target state $|\psi_f\rangle = |1\rangle$, the Lyapunov control of Eq. (15) is used and the total length of time is T. In Fig. 3, we give the feedback of the equation u corresponding to Eq. (15) and the evolution of the example in Eq. (12).

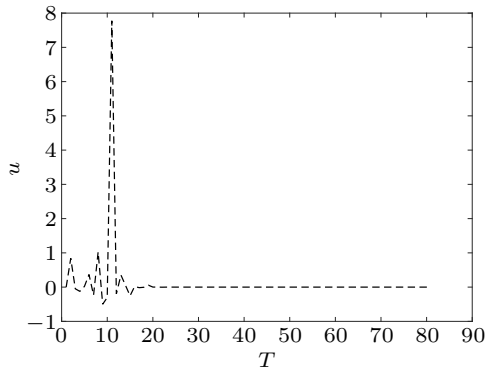


Fig. 3. Controlling the evolution of u. Initial conditions $|\psi(t_0)\rangle = |0\rangle$. The system is defined by Eq. (12), using feedback (15).

5. Lyapunov function based on state error

5.1. Design of control laws

The basis of the state error is to achieve control by continuously reducing the error between the controlled state and the target state. The Lyapunov function is then

$$V_3 = \frac{1}{2}\langle \psi - \psi_f | \psi - \psi_f \rangle. \tag{24}$$

Considering a control system of Eq. (2), the target state satisfies Eq. (4) and the V_3 first derivative of time is

$$\begin{aligned} \dot{V}_3 &= \frac{1}{2}\langle \dot{\psi} | \psi - \psi_f \rangle + \frac{1}{2}\langle \psi - \psi_f | \dot{\psi} \rangle \\ &= (\lambda_f + \omega)\mathrm{Im}\langle \psi | \psi_f \rangle + u\mathrm{Im}\langle \psi | H_1 | \psi_f \rangle \\ &\quad + u^2 \mathrm{Im}\langle \psi | H_2 | \psi_f \rangle. \end{aligned} \tag{25}$$

Let $u_0 = \lambda_f + \omega$ (λ_f are the eigenvalues of the target state), $I_1 = \mathrm{Im}\langle \psi | H_1 | \psi_f \rangle$, $I_2 = \mathrm{Im}\langle \psi | H_2 | \psi_f \rangle$, $I_3 = \mathrm{Im}\langle \psi | \psi_f \rangle$. Then, equation (25) can be abbreviated as

$$\dot{V}_3 = u_0 I_3 + u I_1 + u^2 I_2.$$

For simplicity and effectiveness, here we design the control law of guaranteeing $\dot{V}_3 \le 0$ in the following special form:

$$\begin{cases} u = -\dfrac{I_1}{(1 + \frac{1}{2} I_2)^2}, \\ \omega = u_0 - \lambda_f = -I_3 - \lambda_f. \end{cases} \tag{26}$$

5.2. Stability analysis

Combined with the principle of LaSalle invariance, the stability theorem is given.

Theorem 3 Considering the control (26), system (2) and eigenstates of H_0, $|\psi_f\rangle$, if the H_0 does not degenerate, then the maximum invariant set of a closed-loop system is $S^{2N-1} \cap E_3$, $E_3 = \{|\psi\rangle : \langle \psi_f | H_k | \psi \rangle = 0 \text{ and } \mathrm{Im}\langle \psi_f | \psi \rangle = 0, k = 1, 2\}$. If $\langle \psi_f | H_k | \psi \rangle = 0$ and $\mathrm{Im}\langle \psi_f | \psi \rangle = 0$, all solutions are in the equivalent class of the target state $|\psi_f\rangle$ and the closed loop system will be progressively stable, gradually converging to the target state $|\psi_f\rangle$.

The proof of Theorem 3 can be combined with the proof of Theorem 1, which is not provided here.

5.3. Uniform form

The unified forms of the V_1, V_2, and V_3, Lyapunov functions are given in Ref. [24], i.e.,

$$V = \langle \psi - \alpha \psi_f | Q | \psi - \alpha \psi_f \rangle,$$

(i) $\alpha = 0$, $Q = (1/2)(I - |\psi_f\rangle\langle\psi_f|)$,
(ii) $\alpha = 0$,
(iii) $\alpha = 1$, $Q = (1/2)I$,
where $\alpha \in [0, 1]$, Q is a Hermitian operator.

Combining Eqs. (6), (15), and (26), we give the unified form of control under the three forms of Lyapunov function

$$\begin{cases} u = k\dfrac{I_1}{(1 + \frac{1}{2} I_2)^2}, \\ \omega, \ \text{discussion in a specific form.} \end{cases}$$

Among them, I_1, the I_2 should be determined according to the specific form of the formula

(i) $k = 1$, $\omega = \mathrm{Im}\langle \psi | \psi_f \rangle \langle \psi_f | \psi \rangle$,
(ii) $k = -1$, ω can take any value,
(iii) $k = -1$, $\omega = -\mathrm{Im}\langle \psi | \psi_f \rangle - \lambda_f$.

5.4. Numerical simulation

Consider a 2D test system of type (12) and a control system of type (26). In this case, in order to reach the target state $|\psi_f\rangle = |1\rangle$, $\lambda_f = 1$, the Lyapunov control of formula (26) is used and the total length of time is T.

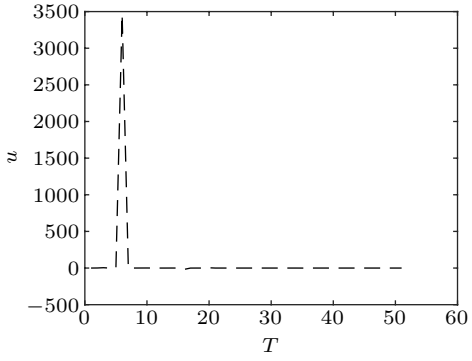


Fig. 4. Controlling the evolution of u. Initial conditions $|\psi(t_0)\rangle = |0\rangle$. The system is defined by Eq. (12), using feedback (26).

In Fig. 4, we give the feedback that u corresponds to Eq. (26) and the evolution of Eq. (12).

6. Conclusion

This paper summarizes control under the three Lyapunov functions for pure closed quantum systems, gives a unified form of the control law, points out and proves the key problem of progressive stability of closed-loop systems in combination with the Lasalle invariance principle, and for a 2D test system, the Lyapunov function in three forms is numerically simulated. When studying the Lyapunov function based on the average value of virtual mechanical quantities, we combined the particle swarm algorithm and simulated annealing algorithm to give unknown parameters. Numerical results show that the parameters given by particle swarm algorithm and simulated annealing algorithm can converge the state to the target state. Therefore, particle swarm optimization and simulated annealing algorithms can be used to assist the design of Lyapunov control in complex quantum systems.

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