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Holevo bound independent of weight matrices for estimating two parameters of a qubit

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Holevo bound plays an important role in quantum metrology as it sets the ultimate limit for multi-parameter estimations, which can be asymptotically achieved. Except for some trivial cases, the Holevo bound is implicitly defined and formulated with the help of weight matrices. Here we report the first instance of an intrinsic Holevo bound, namely, without any reference to weight matrices, in a nontrivial case. Specifically, we prove that the Holevo bound for estimating two parameters of a qubit is equivalent to the joint constraint imposed by two quantum Cramér–Rao bounds corresponding to symmetric and right logarithmic derivatives. This weightless form of Holevo bound enables us to determine the precise range of independent entries of the mean-square error matrix, i.e., two variances and one covariance that quantify the precisions of the estimation, as illustrated by different estimation models. Our result sheds some new light on the relations between the Holevo bound and quantum Cramér–Rao bounds. Possible generalizations are discussed.

Keywords: quantum metrology, quantum Fisher information, Holevo bound, quantum multi-parameter estimation

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1. Introduction

Parameter estimation is ubiquitous in our quantitative description of the physical world around us. Metrology is the science and art of how to improve the precision of our estimation. Quantum metrology,[1–5] in comparison to its classical counterpart, employs non-classical resources to improve the precision of estimations beyond the classical limits and has found numerous applications in various different fields ranging from detection of gravitational fields[10–12] to imaging of biology samples.[13] Two main non-classical resources are at our disposal, namely, the quantum mechanical states, which can encode parameters in various ways and thus give rise to different statistical models, and the quantum mechanical measurements, which can be chosen to optimize the precision of estimations.

For single-parameter estimation, the precision as quantified by the mean-square error (MSE) has a fundamental limit placed by the inverse of the quantum Fisher information (QFI), known as quantum Cramér–Rao bound (QCRB).[14,15] QFI is in de facto a classical Fisher information (FI), which reflects the knowledge limitation of parameters in both theoretical and practical aspects, corresponding to optimal quantum measurement. Although the measurement for an optimal estimation depends on the exact value of the parameter to be estimated, there are adaptive strategies that can circumvent such a drawback. Thus for single-parameter estimation, the ultimate precision limit placed by QFI is attainable.

Simultaneously estimating multiple parameters might provide better precision than estimating them individually in the quantum scenario, which stimulated much interest in multi-parameter quantum metrology in recent years.[5–9] Unlike the case with single parameter estimation, the precision limit for classical multi-parameter estimation as placed by the classical FI matrix cannot be straightforwardly carried over to the quantum case. This is because, firstly, the optimal quantum measurements for each individual parameter might not be compatible. Secondly, the generalizations of the FI matrix to the quantum case are not unique, e.g., two of the most well-known QFI matrices corresponding to symmetric logarithmic derivative (SLD)[14,15] and right logarithmic derivative (RLD).[16,17] A family of general logarithmic derivatives introduced by Yamagata[21] also gives rise to QFI matrices.

For multi-parameter estimation, the figure of merit in quantifying the accuracy is the MSE matrix with diagonal entries being the variances and off-diagonal entries being the covariances of the parameters. Both QFI matrices arising from SLD and RLD provide lower bounds for the MSE matrix in terms of matrix inequalities, which are also referred to as QCRBs. Generally, these bounds, which share the advantages of being explicit and intrinsic, i.e., without reference to any weight matrix, are not always attainable. Holevo[1] proposed an asymptotically achievable bound[18–21] called Holevo bound (HB) as the ultimate limit of multi-parameter estimation. However, except for some trivial cases where the Holevo bound coincides with the QCRB arising from SLD or

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RLD, the Holevo bound is defined and formulated in an implicit manner, which restricts its wide applications. It is implicit in two aspects: first, it is defined by an optimization process and second, its formulation involves inevitably a weight matrix. One breakthrough is made by Suzuki[22] by deriving an analytical form of HB for a nontrivial case of estimating two parameters of a qubit.

In this work, we show that Suzuki's analytical HB for estimating two parameters of a qubit can be further made intrinsic by removing its dependence on the weight matrices. In detail, we show that HB in this specific case is equivalent to the joint constraint imposed by two QCRBs arising from SLD and RLD. This intrinsic form in terms of pure matrix inequalities enjoys the advantage that we can determine explicitly the precise range of two variances and one covariance in the MSE matrix, which is illustrated by some examples. The rest of the paper is structured as follows. In Section 2, we briefly review the necessary background for the formulation of HB. In Section 3 we recapitulate Suzuki's explicit HB with given weight for a 2-parameter estimation in a qubit model. Next in Section 4, we derive an explicit and intrinsic HB in this specific case. In Section 5, we analyze four different situations and illustrate the precise range of the variances and their covariance of two parameters in different statistical models. In Section 6, we prove that two QCR bounds corresponding to SLD and RLD cover all the QCR bounds corresponding to the whole one-parameter family of logarithmic derivatives introduced by Yamagata in the two-parameter qubit models. Finally, we conclude with some discussions.

2. Preliminary

To begin, let us recall some pertinent backgrounds for the formulation of Holevo bound for multi-parameter estimation. When simultaneously estimating n parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ encoded in a probability distribution $\{p_\mu\}$, we employ MSE matrix $\boldsymbol{C}$ with entries

$$C_{jk} = \sum_\mu (\hat{\theta}_{j,\mu} - \theta_j)(\hat{\theta}_{k,\mu} - \theta_k) p_\mu \tag{1}$$

to characterize the precision of our estimation. Here $\hat{\theta}_{j,\mu}$ denotes the estimator of the j-th parameter corresponding in the event of obtaining outcome μ. We suppose that our estimation is locally unbiased, i.e., the mean of the estimator of each parameter and its derivative satisfy

$$\sum_\mu \hat{\theta}_{j,\mu} p_\mu = \theta_j, \ \ \frac{\partial}{\partial \theta_k} \sum_\mu \hat{\theta}_{j,\mu} p_\mu = \delta_{jk}, \tag{2}$$

near the true value $\boldsymbol{\theta}$ with $j = 1, 2, \ldots, n$. Then for the classical estimation n parameters, it holds the following classical CR bound:

$$\boldsymbol{C} \geq \boldsymbol{\mathcal{F}}^{-1}, \ \ \mathcal{F}_{jk} = \sum_\mu \frac{\partial_j p_\mu \partial_k p_\mu}{p_\mu}, \tag{3}$$

where FI matrix $\boldsymbol{\mathcal{F}}$ quantifies the information of parameters encoded in data.

In quantum metrology, we have at our disposal two nonclassical resources, namely, quantum mechanical states and measurements. The resulting probability distribution encodes also the information of the parameter and presents a classical estimation problem. Different choices of measurements $\{M_\mu\}$ which are described by a positive operator-valued measure (POVM)

$$M_\mu \geq 0 \ \ (\forall \mu), \ \ \sum_\mu M_\mu = I, \tag{4}$$

lead to different statistical models and the optimal one is determined by the so-called symmetric logarithmic derivative (SLD) in the case of single-parameter estimation. For estimating n parameters the SLD can be defined for each parameter

$$\partial_j \rho = \frac{\rho \boldsymbol{L}_j + \boldsymbol{L}_j \rho}{2}, \ \ j = 1, 2, \ldots, n, \tag{5}$$

and the corresponding QCRB reads

$$\boldsymbol{C} \geq \boldsymbol{J}^{-1}, \ \ J_{jk} = \frac{1}{2} \mathrm{Tr}(\rho \{\boldsymbol{L}_j, \boldsymbol{L}_k\}). \tag{6}$$

However, if there are two or more parameters, QCRB can not always be simultaneously attained so the simultaneous optimization for all parameter's precisions is unknown even though individual optimizations for every single parameter are clear. This is because the optimal measurements determined by SLDs corresponding to different parameters are in general not compatible.

The quantum version of the classical FI matrix is not unique. As an alternative, the right logarithm derivative (RLD) $\tilde{\boldsymbol{L}}_j$ is defined by

$$\partial_j \rho = \rho \tilde{\boldsymbol{L}}_j, \tag{7}$$

which also places a lower bound to the MSE matrix in the form of the following QCRB:

$$\boldsymbol{C} \geq \tilde{\boldsymbol{J}}^{-1}, \ \ \tilde{J}_{jk} = \mathrm{Tr}(\rho \tilde{\boldsymbol{L}}_j \tilde{\boldsymbol{L}}_k^\dagger). \tag{8}$$

Both these two QCRBs enjoy the advantages of being explicit and intrinsic, i.e., no optimization and ad hoc weight matrix are involved, and the disadvantage of being not attainable in general. The main result of this paper is to show that these two bounds when consider simultaneously, are equivalent to the HB for estimating two parameters of a qubit, and thus can be asymptotically achieved.

By considering compatible observables Holevo proposed the so-called HB. Inevitably the formulation of HB involves a kind of weight matrix $\boldsymbol{W} \geq 0$, which is positive semidefinite. In terms of the lower bounds for the weighted trace of the MSE matrix, the HB of Nagaoka's form reads

$$\mathrm{Tr}(\boldsymbol{W}\boldsymbol{C}) \geq C_{\mathrm{H}}[\boldsymbol{W}] := \min_{X \in \mathcal{X}} \mathrm{Tr}\,(\boldsymbol{W}\boldsymbol{E}_X)$$

$$+\mathrm{Tr}\left|\sqrt{\boldsymbol{W}}\boldsymbol{F}_X\sqrt{\boldsymbol{W}}\right|, \tag{9}$$

where $\mathrm{Tr}|\boldsymbol{K}| = \mathrm{Tr}\sqrt{\boldsymbol{K}^\dagger\boldsymbol{K}}$ is the trace norm for an arbitrary $\boldsymbol{K}$. Here we have denoted $\boldsymbol{E}_X = \mathrm{Re}\,\boldsymbol{\Delta}_X$ and $\boldsymbol{F}_X = \mathrm{Im}\,\boldsymbol{\Delta}_X$, the real and imaginary parts of

$$[[\Delta_X]]_{jk} = \langle X_j X_k\rangle_\rho - \langle X_j\rangle_\rho\langle X_k\rangle_\rho, \tag{10}$$

with notation $\langle\boldsymbol{A}\rangle_\rho = \mathrm{Tr}(\rho\boldsymbol{A})$ and $X = (X_1, X_2, \ldots, X_n) \in \mathcal{X}$ being a set of n observables constructed from an arbitrary POVM $\{M_\mu\}$, satisfying the unbiased conditions

$$X_j = \sum_\mu (\hat{\theta}_{j,\mu} - \theta_j) M_\mu, \tag{11}$$

with $j,k = 1,2,\ldots,n$. We note that the HB can be viewed as some kind of uncertainty relation of some sets of jointly measurable unbiased observables $X \in \mathcal{X}$. In fact we have[23]

$$\boldsymbol{C} \geq \boldsymbol{\Delta}_X,$$

which holds for all measurements M_μ and corresponding estimators $\{\hat{\theta}_{j,\mu}\}$, from which the HB of Nagaoka's form Eq. (9) can be derived by noting that while $\boldsymbol{C}$ is real symmetric and $\boldsymbol{\Delta}_X$ is Hermitian.

3. Suzuki's analytical Holevo bound

As mentioned above, the HB $C_\mathrm{H}[\boldsymbol{W}]$ is in general not explicitly given, i.e., optimization has to be carried out over all possible observables of form (Eq. (11)) for arbitrary measurement M_μ and estimator $\hat{\theta}_{j,\mu}$ and depends on weight matrix $\boldsymbol{W}$. In 2016, Suzuki[22] worked out the first instance of analytical HB for the nontrivial case of estimating two parameters in a qubit model.

Consider the scenario of simultaneously estimating two parameters, e.g., θ_1 and θ_2 for a general mixed qubit state

$$\rho = \frac{1}{2}(I + \boldsymbol{s}\cdot\boldsymbol{\sigma}), \tag{12}$$

with Bloch vector $\boldsymbol{s}$ depending on $\theta_{1,2}$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ being the set of three Pauli operators. Two QFI matrices corresponding to SLD and RLD read,[22] respectively,

$$[[J]]_{jk} = \boldsymbol{s}_j\cdot\boldsymbol{s}_k + \frac{(\boldsymbol{s}_j\cdot\boldsymbol{s})(\boldsymbol{s}_k\cdot\boldsymbol{s})}{1-s^2}, \tag{13a}$$

$$[[\tilde{J}]]_{jk} = \frac{\boldsymbol{s}_j\cdot\boldsymbol{s}_k - \mathrm{i}(\boldsymbol{s}_j\times\boldsymbol{s}_k)\cdot\boldsymbol{s}}{1-s^2}, \tag{13b}$$

where $\boldsymbol{s}_j = \partial_j\boldsymbol{s}$ denotes the j-th partial derivative of Bloch vector and $s = |\boldsymbol{s}|$. Their inverses are given by

$$\boldsymbol{J}^{-1} = \frac{\boldsymbol{\epsilon}\boldsymbol{J}\boldsymbol{\epsilon}^\mathrm{T}}{\det\boldsymbol{J}},\quad \tilde{\boldsymbol{J}}^{-1} = \frac{\boldsymbol{\epsilon}\tilde{\boldsymbol{J}}\boldsymbol{\epsilon}^\mathrm{T}}{\det\tilde{\boldsymbol{J}}},\quad \boldsymbol{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{14}$$

with determinants $\det\tilde{\boldsymbol{J}} = (\det\boldsymbol{J})/(1-s^2)$ and

$$\det\boldsymbol{J} = \frac{(\boldsymbol{s}_1\times\boldsymbol{s}_2)^2 - [(\boldsymbol{s}_1\times\boldsymbol{s}_2)\cdot\boldsymbol{s}]^2}{1-s^2}. \tag{15}$$

Two QFI matrices are related via

$$\boldsymbol{J}^{-1} - \tilde{\boldsymbol{J}}^{-1} = \tilde{\boldsymbol{S}} + \mathrm{i}c\boldsymbol{\epsilon}, \tag{16}$$

where $\tilde{\boldsymbol{S}} = \boldsymbol{\epsilon}\boldsymbol{S}\boldsymbol{\epsilon}^\mathrm{T}$ is of rank-1 with

$$S_{jk} = \frac{(\boldsymbol{s}_j\cdot\boldsymbol{s})(\boldsymbol{s}_k\cdot\boldsymbol{s})}{(1-s^2)\det\boldsymbol{J}},\quad c = \frac{(\boldsymbol{s}_1\times\boldsymbol{s}_2)\cdot\boldsymbol{s}}{\det\boldsymbol{J}}. \tag{17}$$

We see that two diagonal entries of $\tilde{\boldsymbol{J}}^{-1}$ are no larger than those of $\boldsymbol{J}^{-1}$.

Let $\boldsymbol{W} \geq 0$ be an arbitrary weight matrix with normalization $\det\boldsymbol{W} = 1$ and Suzukis' explicit HB reads (in those terms defined above)[22]

$$C_\mathrm{H}[\boldsymbol{W}] = \mathrm{Tr}(\boldsymbol{W}\boldsymbol{J}^{-1}) + \begin{cases} 2|c| - s_W, & |c| \geq s_W, \\ \dfrac{|c|^2}{s_W}, & |c| < s_W, \end{cases} \tag{18}$$

where $s_W = \mathrm{Tr}(\boldsymbol{W}\tilde{\boldsymbol{S}})$. From this explicit form, we can immediately identify the QCRB $\boldsymbol{C} \geq \boldsymbol{J}^{-1}$ corresponding to SLD since the second term is always non-negative. When the second term becomes positive this bound cannot be attained. We note that the explicit HB $\mathrm{Tr}(\boldsymbol{C}\boldsymbol{W}) \geq C_\mathrm{H}[\boldsymbol{W}]$ with $C_\mathrm{H}[\boldsymbol{W}]$ given in Eq. (18) allows itself to be recast into a single inequality

$$\sqrt{\mathrm{Tr}\,[\boldsymbol{W}(\boldsymbol{C}-\boldsymbol{J}^{-1})] + s_W - 2|c|} \geq \sqrt{s_W} - \frac{|c|}{\sqrt{s_W}}, \tag{19}$$

together with an implicit condition $\mathrm{Tr}\left[\boldsymbol{W}(\boldsymbol{C}-\boldsymbol{J}^{-1})\right] + s_W \geq 2|c|$ in order for the square root to be well-defined. We note that both HB and Suzuki's explicit form Eq. (18) are homogenous with respect to transformation $\boldsymbol{W} \to \lambda\boldsymbol{W}$ for some positive λ. That is, for a weight matrix $\boldsymbol{W}'$ with $\det\boldsymbol{W}' = \lambda^2 \neq 0, 1$, the corresponding HB $C_\mathrm{H}[\boldsymbol{W}']$ is obtained by HB $\lambda C_\mathrm{H}[\boldsymbol{W}'/\lambda]$, which assumes the same form as Eq. (18) with $|c|$ replaced by $\sqrt{\det\boldsymbol{W}'}|c|$.

4. Holevo bound independent of weight matrices

As shown above, the formulation of HBs, analytical or not, is implicitly formulated in terms of weight matrices. Our main result is to remove this dependence on the weight matrix to get the first instant of an explicit and intrinsic HB in a nontrivial case.

Theorem 1 The Holevo bound for estimating two parameters of a qubit is equivalent to the joint constraint imposed by two QCRBs arising from SLD and RLD

$$\boldsymbol{C} \geq \boldsymbol{J}^{-1},\quad \boldsymbol{C} \geq \tilde{\boldsymbol{J}}^{-1}. \tag{20}$$

Proof First, we shall prove that the explicit HB (Eq. (18)), which holds true for all weight matrices $\boldsymbol{W} \geq 0$ with $\det\boldsymbol{W} = 1$, leads to QCRBs (Eq. (20)). As the second term in the explicit HB (Eq. (18)) is non-negative in both cases, we have

$\mathrm{Tr}\,\boldsymbol{W}\boldsymbol{C} \geq \mathrm{Tr}\,\boldsymbol{W}\boldsymbol{J}^{-1}$ for all positive symmetric weight matrix $\boldsymbol{W}$ with $\det\boldsymbol{W}=1$ we then get (see Lemma 1 in section 1 of the supplementary materials) $\boldsymbol{C} \geq \boldsymbol{J}^{-1}$ as both $\boldsymbol{C}$ and $\boldsymbol{J}^{-1}$ are real symmetric. Furthermore, an arbitrary Hermitian 2×2 matrix $\tilde{\boldsymbol{W}} \geq 0$ can be decomposed into $\boldsymbol{W}_{\mathrm{S}} + \mathrm{i}y\boldsymbol{\epsilon}$ with $\boldsymbol{W}_{\mathrm{S}}$ being real symmetric and $y^2 \leq \det\boldsymbol{W}_{\mathrm{S}}$. Suppose $|y|>0$ we have $\det\boldsymbol{W}_{\mathrm{S}}>0$ so that we can assume $\det\boldsymbol{W}_{\mathrm{S}}=1$ so that $|y|\leq 1$. Then by using relation Eq. (16) we have

$$\begin{aligned}\mathrm{Tr}\,\tilde{\boldsymbol{W}}(\boldsymbol{C}-\tilde{\boldsymbol{J}}^{-1}) &= \mathrm{Tr}\,\tilde{\boldsymbol{W}}(\boldsymbol{C}-\boldsymbol{J}^{-1}) + \mathrm{Tr}\,\tilde{\boldsymbol{S}}\tilde{\boldsymbol{W}} + \mathrm{i}c\,\mathrm{Tr}\,\tilde{\boldsymbol{W}}\boldsymbol{\epsilon}\\ &= \mathrm{Tr}\,\boldsymbol{W}_{\mathrm{S}}(\boldsymbol{C}-\boldsymbol{J}^{-1}) + s_W + 2yc\\ &\geq 2|c| + 2yc \geq 0\end{aligned}$$

for an arbitrary $\tilde{\boldsymbol{W}} \geq 0$, from which it follows $\boldsymbol{C} \geq \tilde{\boldsymbol{J}}^{-1}$. This proves the necessity.

Second, to prove the sufficiency of constraints (Eq. (20)), we consider first the general case $\tilde{\boldsymbol{S}} \neq 0$, i.e., $\boldsymbol{s}_1\cdot\boldsymbol{s}, \boldsymbol{s}_2\cdot\boldsymbol{s}$ do not equal to zero at the same time. Consider an arbitrary 2×2 weight matrix $\boldsymbol{W}$ satisfying $\det\boldsymbol{W}=1$. In the 2-dimensional space in which the weight matrix lives we introduce a basis $\{|e\rangle, |\tilde{e}\rangle = \boldsymbol{\epsilon}\,|e\rangle\}$ where

$$|e\rangle = \frac{(\boldsymbol{s}_1\cdot\boldsymbol{s}, \boldsymbol{s}_2\cdot\boldsymbol{s})^{\mathrm{T}}}{\sqrt{(1-s^2)(\det\boldsymbol{J})\mathrm{Tr}\,\boldsymbol{S}}}. \tag{21}$$

For convenience, we have adopted Dirac bracket notations for a 2-dimensional real vector. In this basis the weight matrix reads

$$\boldsymbol{W} = \gamma|\tilde{e}\rangle\langle\tilde{e}| + \frac{1+u^2}{\gamma}|e\rangle\langle e| + u\left(|e\rangle\langle\tilde{e}| + |\tilde{e}\rangle\langle e|\right), \tag{22}$$

where $u = \langle e|W|\tilde{e}\rangle$ and $\gamma = s_W/b^2$ with $b=\sqrt{\mathrm{Tr}\,\boldsymbol{S}}$ since $\tilde{\boldsymbol{S}} = b^2|\tilde{e}\rangle\langle\tilde{e}|$. We note that there are only two free parameters u and $\gamma>0$. For matrix $\boldsymbol{A} = \boldsymbol{C}-\boldsymbol{J}^{-1} \geq 0$ we denote its elements in the same basis as

$$A_e = \langle e|\boldsymbol{A}|e\rangle,\ \ A_{\tilde{e}} = \langle\tilde{e}|\boldsymbol{A}|\tilde{e}\rangle,\ \ A_{e\tilde{e}} = \langle e|\boldsymbol{A}|\tilde{e}\rangle, \tag{23}$$

and optimize

$$\begin{aligned}&\min_W\left\{\frac{\sqrt{s_W}}{b}\sqrt{\mathrm{Tr}\,\boldsymbol{W}\boldsymbol{A}+s_W-2|c|}-\frac{s_W}{b}\right\}\\ =&\min_{\gamma,u}\left\{-b\gamma+\sqrt{\gamma^2A_{\tilde{e}}+(1+u^2)A_e+2u\gamma A_{e\tilde{e}}+\gamma^2b^2-2|c|\gamma}\right\}\\ =&\min_{\gamma,u}\left\{-b\gamma+\sqrt{\gamma^2k+A_e+A_e\left(u+\frac{\gamma A_{e\tilde{e}}}{A_e}\right)^2-2|c|\gamma}\right\}\\ =&\min_{\gamma}\left\{-b\gamma+\sqrt{k\left(\gamma-\frac{|c|}{k}\right)^2+A_e-\frac{|c|^2}{k}}\right\}\\ =&\frac{1}{k}\left(\sqrt{(A_ek-|c|^2)(k-b^2)}-b|c|\right)\geq-\frac{|c|}{b},\end{aligned}$$

which is exactly the HB (Eq. (19)). Here we have denoted $k=(\det\boldsymbol{A})/A_e+b^2$ and the minimum over $\gamma>0$, by derivative with respect to γ, is attained at

$$\gamma_* = \frac{|c|}{k}+\frac{b}{k}\frac{\sqrt{\det(\boldsymbol{C}-\tilde{\boldsymbol{J}^{-1}})}}{\sqrt{k-b^2}},$$

and the last inequality is due to $k-b^2\geq 0$.

In the case of $\tilde{\boldsymbol{S}}=0$, we have $s_W=0$ and $b=0$ so that the proof of explicit HB (Eq. (18)) is boiled down to show $\mathrm{Tr}\,(\boldsymbol{W}\boldsymbol{A}) \geq 2|c|$. This is true because $\mathrm{Tr}\,\boldsymbol{W}\boldsymbol{A} \geq 2\sqrt{\det\boldsymbol{A}}$ for arbitrary $\boldsymbol{W}\geq 0$ satisfying $\det\boldsymbol{W}=1$ (see Lemma 2 in Section 1 of the supplementary materials), while $\det(\boldsymbol{C}-\tilde{\boldsymbol{J}^{-1}}) = \det\boldsymbol{A}-|c|^2 \geq 0$ due to QCRB $\boldsymbol{C}\geq\tilde{\boldsymbol{J}^{-1}}$ and $\tilde{\boldsymbol{S}}=0$.

Some remarks are in order. First, according to the theorem proved above the HB is equivalent to two QCRBs arising from two kinds of QFI matrices. It should be emphasized however our main result does not imply that for a given weight matrix the Holevo bound is equivalent to the maximum of SLD bound and RLD corresponding to the same weight matrix. In fact, as shown in Section 2 of the supplementary materials, for some given weight matrix $\boldsymbol{W}$, the solution to the following optimization:

$$\begin{aligned}\min_{\boldsymbol{C}}\quad & \mathrm{Tr}\,\boldsymbol{W}\boldsymbol{C}\\ \text{subject to}\quad & \boldsymbol{C}\geq\boldsymbol{J}^{-1},\ \ \boldsymbol{C}\geq\tilde{\boldsymbol{J}^{-1}}\end{aligned} \tag{24}$$

is in general not equal to $\max\{C_{\mathrm{S}}[\boldsymbol{W}], C_{\mathrm{R}}[\boldsymbol{W}]\}$

$$\begin{aligned}C_{\mathrm{S}}[\boldsymbol{W}] &= \min_{\boldsymbol{C}}\{\mathrm{Tr}\,\boldsymbol{W}\boldsymbol{C}\mid\boldsymbol{C}\geq\boldsymbol{J}^{-1}\} = \mathrm{Tr}\,\boldsymbol{W}\boldsymbol{J}^{-1},\\ C_{\mathrm{R}}[\boldsymbol{W}] &= \min_{\boldsymbol{C}}\{\mathrm{Tr}\,\boldsymbol{W}\boldsymbol{C}\mid\boldsymbol{C}\geq\tilde{\boldsymbol{J}^{-1}}\},\end{aligned}$$

even though $\boldsymbol{C}\geq\boldsymbol{J}^{-1}$ is equivalent to $\mathrm{Tr}\,\boldsymbol{W}\boldsymbol{C}\geq C_{\mathrm{S}}[\boldsymbol{W}]$ while $\boldsymbol{C}\geq\tilde{\boldsymbol{J}^{-1}}$ is equivalent to $\mathrm{Tr}\,\boldsymbol{W}\boldsymbol{C}\geq C_{\mathrm{R}}[\boldsymbol{W}]$ for all weight matrices. Second, as HB is asymptotically attainable in a collective strategy,[18–21] the overlap of these two QCRBs is also asymptotically attainable. Third, as the joint constraint is independent of weight matrices, providing an intrinsic form of HB, it is possible to determine explicitly the precise range of two variances and one covariance in the MSE matrix $\boldsymbol{C}$.

5. Range of variances and the covariance allowed by HB

In the case of two-parameter estimation, the MSE matrix $\boldsymbol{C}$, which is real and symmetric, has three independent variables, namely, two variances $x=C_{11}$, $z=C_{22}$ as diagonal entries and one covariance $y=C_{12}$ as the off-diagonal entry

$$\boldsymbol{C} = \begin{pmatrix} x & y\\ y & z\end{pmatrix}.$$

As an immediate consequence of our theorem, the possible values of x,y,z are determined explicitly by two matrix inequalities arising from two QCRBs. Since the relevant matrices are of dimension 2×2, those two QCRBs can be recast into the following scalar inequalities:

$$x \geq [[J^{-1}]]_{11},\ \ z \geq [[J^{-1}]]_{22},$$
$$\det(C - J^{-1}) \geq 0,\ \ \det(C - \tilde{J^{-1}}) \geq 0. \tag{25}$$

The constraints, e.g., $x \geq [[\tilde{J^{-1}}]]_{11}$ is ensured by $x \geq [[J^{-1}]]_{11}$ as it holds always $[[J^{-1}]]_{jj} \geq [[\tilde{J^{-1}}]]_{jj}$ as indicated by Eq. (16). It is clear that each determinant constraint gives rise to a constraint in the form of a paraboloid. If two determinant constraints intersect, they intersect at a plane given by

$$(1-s^2)s^2 = (\boldsymbol{s}_1\cdot\boldsymbol{s})^2 x + (\boldsymbol{s}_2\cdot\boldsymbol{s})^2 z + 2(\boldsymbol{s}_1\cdot\boldsymbol{s})(\boldsymbol{s}_2\cdot\boldsymbol{s})y. \tag{26}$$

The choice of the weight matrix is equivalent to choosing the normal directions of some planes.

5.1. Asymptotically classical models

A model is asymptotically classical if the Holevo bound is equal to the SLD bound. In the case of the two-parameter qubit model Suzuki derived the condition under which the Holeve bound coincides with the SLD bound.[22] Here we provide an alternative proof.

Corollary 1 (Suzuki). For any two-parameter qubit model, the following three conditions are equivalent:

(i) $c = 0$,

(ii) $\forall \boldsymbol{W}, C_{\mathrm{H}}[\boldsymbol{W}] = \mathrm{Tr}\,\boldsymbol{W}\boldsymbol{J}^{-1}$,

(iii) $\exists \boldsymbol{W} > 0, C_{\mathrm{H}}[\boldsymbol{W}] = \mathrm{Tr}\,\boldsymbol{W}\boldsymbol{J}^{-1}$.

Proof $1 \Rightarrow 2$: From Eq. (16), since $\tilde{\boldsymbol{S}}$ is a rank-1 projection, it follows that $\boldsymbol{J}^{-1} \geq \tilde{\boldsymbol{J}}^{-1}$ when $c = 0$ so that we have condition 2. $2 \Rightarrow 3$ is trivially true. As to $3 \Rightarrow 1$ we note that if $C_{\mathrm{H}}[\boldsymbol{W}_0] = \mathrm{Tr}\,\boldsymbol{W}_0\boldsymbol{J}^{-1}$ for a given weight matrix $\boldsymbol{W}_0 > 0$ then $\mathrm{Tr}\,\boldsymbol{W}_0\boldsymbol{J}^{-1}$ is the solution to the optimization problem Eq. (24), from which it follows, according to Section 2 of the supplementary materials, the optimal MSE matrix $\boldsymbol{C} = \boldsymbol{J}^{-1}$ so that $\boldsymbol{J}^{-1} \geq \tilde{\boldsymbol{J}^{-1}}$. However, from Eq. (16) it follows that $\det(\boldsymbol{J}^{-1} - \tilde{\boldsymbol{J}^{-1}}) = \det(\tilde{\boldsymbol{S}} + \mathrm{i}c\boldsymbol{\epsilon}) = -c^2$. Thus we must have $c = 0$.

5.2. Qubit model with two unitary parameters

We call a parameter is unitary model if the length of the Bloch vector $\boldsymbol{s}$ is independent of the given parameter, from which it follows that $\boldsymbol{s}_j \cdot \boldsymbol{s} = 0$. If both two parameters are D-invariant we have $S = 0$ so that $\boldsymbol{J}^{-1} = \mathrm{Re}\,\tilde{\boldsymbol{J}}^{-1}$ and the QCRB $\boldsymbol{C} \geq \tilde{\boldsymbol{J}}^{-1}$ arising from RLD is always tighter than the QCRB $\boldsymbol{C} \geq \boldsymbol{J}^{-1}$ arising from SLD. Thus there is no intersection of two determinant constraints. The range of x,y,z is completely determined by a single matrix inequality $\boldsymbol{C} \geq \tilde{\boldsymbol{J}}^{-1}$.

5.3. Qubit model with one unitary parameter

Specifically we consider the following model with a Block vector given by

$$\boldsymbol{s} = (k\cos\phi, k\sin\phi, t)$$

with $s^2 = k^2 + t^2 < 1$, corresponding to a mixed state. Two parameters to be estimated are ϕ and t with $\boldsymbol{s}$ independent of ϕ. Then we have $\boldsymbol{s}_\phi \cdot \boldsymbol{s} = 0$ as well as $\boldsymbol{s}_\phi \cdot \boldsymbol{s}_t = 0$. As a result, we have two QFI matrices inverted

$$\boldsymbol{J}^{-1} = \begin{pmatrix} \frac{1-s^2}{1-k^2} & 0 \\ 0 & \frac{1}{k^2} \end{pmatrix},\ \ \tilde{\boldsymbol{J}}^{-1} = \frac{1-s^2}{1-k^2}\begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & \frac{1}{k^2} \end{pmatrix}. \tag{27}$$

It is clear that two paraboloids intersect at the plane given by $x = s^2(1-s^2)/t^2$ as in Eq. (26) and the explicit range of $x = \delta_t^2, y = \delta_{t\phi}, z = \delta_\phi^2$ is given by

$$\delta_t^2 \geq \frac{1-s^2}{1-k^2},\ \ \delta_\phi^2 \geq \frac{1}{k^2},$$

together with

$$\delta_{t\phi}^2 \leq \left(\delta_t^2 - \frac{1-s^2}{1-k^2}\right)\left(\delta_\phi^2 - \frac{1}{k^2}\right),$$

which corresponds to the blue paraboloid in Fig. 1 and

$$\frac{1-s^2}{k^2(1-k^2)}\left(\delta_t^2 + k^2\delta_\phi^2 + 1 - s^2\right) \leq \delta_t^2\delta_\phi^2 - \delta_{t\phi}^2,$$

which is plotted in Fig. 1 as the red paraboloid. We see that in this case, both QCRBs have to be taken into account.

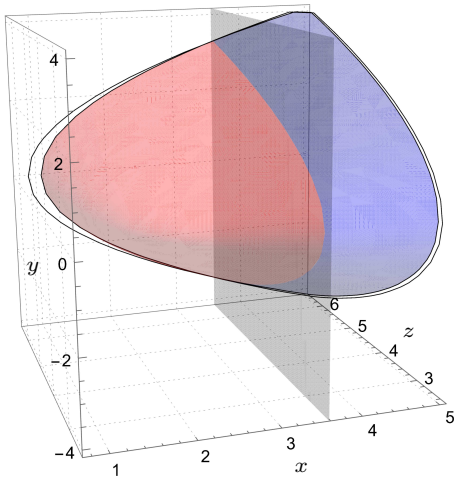


Fig. 1. The explicit range of the variances x,z of t,ϕ, respectively, and the covariance y between t and ϕ in a qubit mode with only one unitary model. Here $t = 0.264$ and $k = 0.640$. This is the common part of the red and blue paraboloids coming from QCRBs for RLD and SLD, respectively. The surfaces of these two paraboloids intersect at a plane parallel to the y,z plane.

5.4. Qubit model without unitary parameter

Now we consider a more general model with two parameters. In this case the Bloch vector of the state $\boldsymbol{s} = (\theta_1, \theta_2, \theta_3)$ has a length s that depends on both two parameters $\theta_{1,2}$ to be estimated, while θ_3 is known and fixed. We consider also a

mixed-state model with $|s|<1$. Obviously we have $\boldsymbol{s}_1\cdot\boldsymbol{s}_2=0$ and $\boldsymbol{s}\cdot\boldsymbol{s}_j=\theta_j$. As a result, we have the QFI matrix for SLD

$$\boldsymbol{J}^{-1}=\begin{pmatrix}1-\frac{\theta_1^2}{1-\theta_3^2} & -\frac{\theta_1\theta_2}{1-\theta_3^2}\\ -\frac{\theta_1\theta_2}{1-\theta_3^2} & 1-\frac{\theta_2^2}{1-\theta_3^2}\end{pmatrix} \tag{28}$$

and QFI matrix for RLD

$$\tilde{\boldsymbol{J}}^{-1}=\frac{1-s^2}{1-\theta_3^2}\begin{pmatrix}1 & -\mathrm{i}\theta_3\\ \mathrm{i}\theta_3 & 1\end{pmatrix}. \tag{29}$$

We plot the possible range (Eq. (25)) of variances x,z of $\theta_{1,2}$ and their covariance y in Fig. 2. The surfaces of two paraboloids intersect at the plane are determined by

$$\theta_1^2x+\theta_2^2z+2\theta_1\theta_2y=(1-s^2)s^2.$$

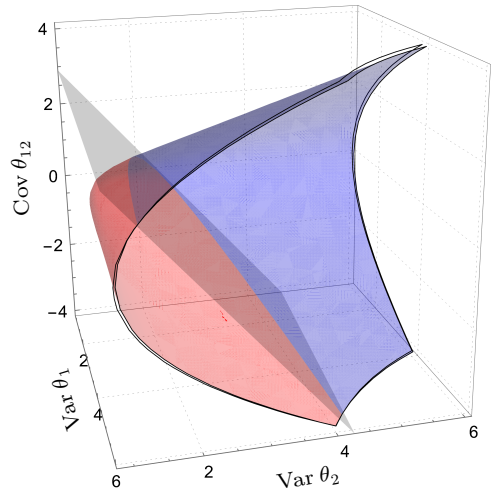


Fig. 2. The explicit range of the variances x,z of $\theta_{1,2}$, respectively, and the covariance y between $\theta_{1,2}$ in a more general model. Here $(\theta_1,\theta_2,\theta_3)=(0.087,0.315,0.540)$. The range is the common part of the red paraboloid coming from QCRB for RLD and the blue one coming from QCRB for SLD. The surfaces of these two paraboloids intersect at a general plane.

6. Weightless form of the maximum logarithmic derivative bound

Quite recently,[25] Yamagata introduced a particular one-parameter family of generalized logarithmic derivatives, each of which gives rise to a QCRB. Specifically, for each real $0\le\beta\le1$ a generalized logarithmic derivative $\boldsymbol{L}_j^\beta$, referred to as β-LD, can be defined by

$$\partial_j\rho=\frac{1+\beta}{2}\rho\boldsymbol{L}_j^\beta+\frac{1-\beta}{2}\boldsymbol{L}_j^\beta\rho, \tag{30}$$

from which it follows a QCRB

$$\boldsymbol{C}\ge\boldsymbol{J}_\beta^{-1},$$

with the corresponding QFI matrix $\boldsymbol{J}_\beta$ defined by matrix element $\mathrm{Tr}(\partial_j\rho\boldsymbol{L}_k^{\beta\dagger})$. It is clear that both SLD and RLD, corresponding to $\beta=0,1$, respectively, are special members of this family. In a two-parameter qubit model, we have explicitly (as shown in Section 3 of the supplementary materials)

$$[[J_\beta]]_{jk}=\frac{\boldsymbol{s}_j\cdot\boldsymbol{s}_k+\frac{1-\beta^2}{1-s^2}(\boldsymbol{s}_j\cdot\boldsymbol{s})(\boldsymbol{s}_k\cdot\boldsymbol{s})-\mathrm{i}\epsilon\beta(\boldsymbol{s}_1\times\boldsymbol{s}_2)\cdot\boldsymbol{s}}{1-\beta^2s^2},$$

from which it follows that

$$\det\boldsymbol{J}_\beta=\frac{\det\boldsymbol{J}}{1-\beta^2s^2}=\gamma\det\tilde{\boldsymbol{J}}+\bar\gamma\det\boldsymbol{J},$$

and using matrix elements Eqs. (13a) and (13b)

$$\boldsymbol{J}_\beta=\gamma\tilde{\boldsymbol{J}}+\bar\gamma\boldsymbol{J}-\mathrm{i}\frac{\beta(1-\beta)(\boldsymbol{s}_1\times\boldsymbol{s}_2)\cdot\boldsymbol{s}}{1-\beta^2s^2}\boldsymbol{\epsilon}, \tag{31}$$

in which $\bar\gamma=1-\gamma$ and

$$\gamma=\frac{\beta^2\left(1-s^2\right)}{1-\beta^2s^2}$$

are both non-negative.

For an arbitrary 2×2 Hermitian matrix $\boldsymbol{A}$ with eigenvalues $\lambda_{1,2}$, we have identity $\det(\boldsymbol{A}-1)=(\lambda_1-1)(\lambda_2-1)=\lambda_1\lambda_2-\lambda_1-\lambda_2+1=1+\det\boldsymbol{A}-\mathrm{Tr}\,\boldsymbol{A}$, we calculate

$$\begin{aligned}&\det\left(\sqrt{\boldsymbol{J}_\beta}\boldsymbol{C}\sqrt{\boldsymbol{J}_\beta}-1\right)\\ =\;&1+\det\boldsymbol{C}\cdot\det\boldsymbol{J}_\beta-\mathrm{Tr}\boldsymbol{C}\boldsymbol{J}_\beta\\ =\;&1+\det\boldsymbol{C}\cdot(\gamma\det\tilde{\boldsymbol{J}}+\bar\gamma\det\boldsymbol{J})-\mathrm{Tr}\boldsymbol{C}(\gamma\tilde{\boldsymbol{J}}+\bar\gamma\boldsymbol{J})\\ =\;&\gamma\det(\sqrt{\tilde{\boldsymbol{J}}}\boldsymbol{C}\sqrt{\tilde{\boldsymbol{J}}}-1)+\bar\gamma\det(\sqrt{\boldsymbol{J}}\boldsymbol{C}\sqrt{\boldsymbol{J}}-1)\ge0,\end{aligned}$$

in which the second equality is due the fact that $\boldsymbol{C}$ is symemtric while the last inequality is because $\boldsymbol{C}\ge\boldsymbol{J}^{-1}$ and $\boldsymbol{C}\ge\tilde{\boldsymbol{J}}^{-1}$. As $\det\boldsymbol{J}_\beta>0$ we have $\det(\boldsymbol{C}-\boldsymbol{J}_\beta^{-1})\ge0$. Furthermore, we have

$$\boldsymbol{J}_\beta^{-1}=\frac{\boldsymbol{\epsilon}\boldsymbol{J}_\beta\boldsymbol{\epsilon}^{\mathrm{T}}}{\det\boldsymbol{J}_\beta}=\boldsymbol{J}^{-1}-\beta^2\tilde{\boldsymbol{S}}-\mathrm{i}\beta c\boldsymbol{\epsilon}, \tag{32}$$

from which it follows that $[[C]]_{jj}\ge[[J^{-1}]]_{jj}\ge[[J_\beta^{-1}]]_{jj}$. As a result we have derived the whole family of QCRBs $\boldsymbol{C}\ge\boldsymbol{J}_\beta^{-1}$ for all $|\beta|\le1$ from only two QCRBs $\boldsymbol{C}\ge\boldsymbol{J}^{-1}$ and $\boldsymbol{C}\ge\tilde{\boldsymbol{J}}^{-1}$. To summarize, we have the following corollaries.

Corollary 2 In a two-parameter qubit model, the following bounds for MSE matrix $\boldsymbol{C}$ coincide the following equations.

(i) Holevo bound Eq. (9)

$$\mathrm{Tr}\boldsymbol{W}\boldsymbol{C}\ge C_{\mathrm{H}}[\boldsymbol{W}]\quad(\forall\boldsymbol{W}\ge0).$$

(ii) Suzuki's explicit bound Eq. (18).

(iii) Yamagata's maximum logarithmic bound

$$\mathrm{Tr}\boldsymbol{W}\boldsymbol{C}\ge\max_\beta\left\{\mathrm{Tr}\boldsymbol{W}\boldsymbol{J}_\beta^{-1}+\mathrm{Tr}|\sqrt{\boldsymbol{W}}(\mathrm{Im}\boldsymbol{J}_\beta^{-1})\sqrt{\boldsymbol{W}}|\right\},$$

for all $\boldsymbol{W}\ge0$, where $\mathrm{Im}\,\boldsymbol{Z}=(1/2)(\boldsymbol{Z}-\boldsymbol{Z}^{\mathrm{T}})$ is the imaginary part of Hermitian $\boldsymbol{Z}$ which is antisymmetric.

(iv) Joint QCRBs from SLD and RLD

$$\boldsymbol{C}\ge\boldsymbol{J}^{-1},\quad\boldsymbol{C}\ge\tilde{\boldsymbol{J}}^{-1}.$$

(v) Joint QCRBs from all β-LD

$$\boldsymbol{C} \geq \boldsymbol{J}_{\beta}^{-1} \quad (\forall 0 \leq \beta \leq 1).$$

The equivalence between bounds 1 and 2 is due to Suzuki, between bounds 2 and 4 is established in our theorem, and between bounds 4 and 5 has been established in the discussions above. The equivalence between bounds 2 and 3, due to Yamagata, can be directly established by substituting Eq. (32) into Yamagata's bound, which turns out to be a quadratic function of $|\beta|$

$$\begin{aligned} &\operatorname{Tr} \boldsymbol{W}\boldsymbol{J}^{-1} - \beta^2 s_W + 2|\beta c| \\ =\ &\operatorname{Tr} \boldsymbol{W}\boldsymbol{J}^{-1} + \frac{|c|^2}{s_W} - \left(|\beta|\sqrt{s_W} - \frac{|c|}{\sqrt{s_W}}\right)^2. \end{aligned}$$

After taking the maximum over $|\beta| \leq 1$ we can obtain Suzuki's explicit bound Eq. (18). We note that the first three bounds involve the weight matrices while the last two bounds are expressed intrinsically, i.e., without referring to any weight matrices. We note that bound 5 is a general bound for all possible estimation models.

7. Summary

We have derived the first instance of intrinsic Holevo bound without involving any weight matrice in a nontrivial case, namely, for estimating two parameters of a qubit, based on Suzuki's analytical Holevo bound. It turns out that HB is equivalent to the joint constraint imposed by two QCRBs coming from RLD and SLD. The relation to Yamagata's maximum logarithmic bound is clarified. With the help of this explicit form of HB in terms of pure matrix inequalities, we are able to explicitly determine the range of two variances and a covariance in the MSE matrix characterizing the precision, allowed by HB. As examples, we plot these ranges in the cases of qubit models with different numbers of unitary parameters. Finally, we point out the particularity of SLD and RLD QFI matrices among all monotone QFI matrices in this kind of models. Due to the pivotal role played by the HB in quantum multiparameter estimations, our result sheds new insight into the relations between Holevo bound and quantum Cramér–Rao bounds. We conjecture that there are similar weightless forms for Holevo bounds, (e.g., bound 5 in Corollary 2) in more general cases.

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References

[1] Holevo A 1982 *Probabilistic and Statistical Aspects of Quantum Theory* (Amsterdam: North-Holland Publishing Company)
[2] Helstrom C W 1976 *Quantum Detection and Estimation Theory* (New York: Academic Press)
[3] Braunstein S L and Caves C M 1994 *Phys. Rev. Lett.* **72** 3439
[4] Tóth G and Apellaniz I 2014 *J. Phys. A: Math. Theor.* **47** 424006
[5] Szczykulska M, Baumgratz T and Datta A 2016 *Adv. Phys. X* **1:4** 621
[6] Carollo A, Spagnolo B, Dubkov A A, *et al.* 2019 *J. Stat. Mech.: Theory Exp.* **2019** 094010
[7] Demkowicz-Dobrzaski R, Grecki W and Gu M 2020 *J. Phys. A: Math. Theor.* **53** 363001
[8] Meyer J J, Borregaard J and Eisert J A 2021 *NPJ Quantum Inf.* **7** 89
[9] Grecki W, Zhou S, Jiang L, *et al.* 2020 *Quantum* **4** 288
[10] Caves C M 1981 *Phys. Rev. D* **23** 1693
[11] Adhikari R X 2014 *Rev. Mod. Phys.* **86** 121
[12] Abbott B P, Abbott R, Abbott T D, *et al.* 2016 *Phys. Rev. Lett.* **116** 061102
[13] Taylor M A and Bowen W P 2014 arXiv: 1409.0950 [quant-ph]
[14] Helstrom C W 1967 *Phys. Lett. A* **25** 101
[15] Helstrom C W 1968 *IEEE Trans. Inf. Theory* **14** 234
[16] Belavkin V P 1976 *Theor. Math. Phys.* **26** 213
[17] Yuen H P and Lax M 1973 *IEEE Trans. Inf. Theory* **19** 740
[18] Gu M and Kahn J 2006 *Phys. Rev. A* **73** 052108
[19] Hayashi M and Matsumoto K 2008 *J. Math. Phys.* **49** 102101
[20] Kahn J and Gu M 2009 *Commun. Math. Phys.* **289** 597
[21] Yamagata K, Fujiwara A and Gill R D 2013 *Ann. Stat.* **41** 2197
[22] Suzuki J 2016 *J. Math. Phys.* **57** 042201
[23] Hayashi M 2005 *Asymptotic Theory of Quantum Statistical Inference* (Singapore: World Scientific Publishing)
[24] Suzuki J 2020 *J. Phys. A: Math. Theor.* **53** 264001
[25] Yamagata K 2021 *J. Math. Phys.* **62** 062203