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Quantum synchronization with correlated baths

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We study quantum synchronization under the nonequilibrium reservoirs. We consider a two-qubit XXZ chain coupled independently to their own reservoirs modeled by the collisional model. Two reservoir particles, initially prepared in a thermal state or a state with coherence, are correlated through a unitary transformation and afterward interact locally with the two quantum subsystems. We study the quantum effect of reservoir on synchronous dynamics of system. By preparing different reservoir initial states or manipulating the reservoir particles coupling and the temperature gradient, we find that quantum entanglement of reservoir is the key to control quantum synchronization of system qubits.

Keywords: quantum synchronization, entanglement, quantum coherence, nonequilibrium reservoir

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1. Introduction

Synchronization is a universal phenomenon that is widespread in natural and social sciences.[1,2] In physical systems, one of the synchronization emerges spontaneously without an external driver, is called spontaneous synchronization, instead, is forced synchronization, also known as entrainment. In classical systems, synchronization has been studied extensively.[3] And it has recently attracted much attention in the quantum regime,[4] where both spontaneous synchronization[5–19] and entrainment[20–27] have been explored in a variety of systems including spins, and harmonic and nonlinear oscillators. Significance of quantum synchronization have also been recently reported in experiments.[28,29]

Dissipation, either global or local, is known to induce synchronization among quantum oscillators and spins, either in the steady state or in the transient relaxation dynamics.[30] Quantum synchronization of spin that interacting collectively with a common dissipative environment have been carried out, recently.[5,12,18,31,32] However, only a very limited number of studies have been reported on the local dissipation induced quantum synchronization. Especially we have not seen any reports about the effect of locally nonequilibrium reservoir on synchronization.

Quantum correlation is one of the key resources for quantum information processing and quantum technologies.[33–36] Spontaneous mutual synchronization emerges as a result of some temporal correlation between the local dynamics of the subsystems of the system of interest. And certain well-known measures of quantum and total correlations contained in the global system, e.g., entanglement, quantum discord, and mutual information, have been explored to pinpoint their relation to the emergence of mutual synchronization between the local evolution of the subsystems.[4] Indeed, there exists no general connection between the onset of synchronization and the behavior of correlations in the global system. For example, recently Karpat *et al.* have shown that correlations in the global system play no relevant role for the dynamical synchronization of local observables by means of a collision model.[16] Lastly, the relationship between quantum coherence and synchronization has also been studied in a bioinspired vibronic dimer system.[37]

In this paper, we consider the effect of nonequilibrium reservoir on quantum synchronization of system spins, including the influences of quantum correlation and coherence of reservoir. We propose a framework based on collisional models,[38–59] which allows us to analyze the quantum effect of reservoir in the functioning of quantum synchronization of system. Our setup, depicted in Fig. 1, consists of two system spins S_A and S_B. Each of these is in contact with a reservoir modeled by the repeated interaction of ancillas. These ancillas are first prepared in a state with coherence at effective temperatures T_1 and T_2, respectively, and then undergo a unitary evolution U, which correlates them before their interact with the systems S_A and S_B. We study if, and how the synchronization of system can be affected by the quantum correlation and coherence of reservoir, and which one is the key to affect quantum synchronization.

The manuscript is organized as follows. In Section 2, model and figures of merit are introduced. In Section 3 we study the effect of nonequilibrium reservoir on intrasystem

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synchronization in detail, and the conclusions are given in Section 4.

2. Model and figures of merit

2.1. Model

We consider the system to be composed of two coupled qubits described by the XXZ Hamiltonian

$$H_S = J(\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2} + \Delta\sigma_{z1}\sigma_{z2}) + B_1\sigma_{z1} + B_2\sigma_{z2}, \quad (1)$$

where J, Δ, and B_i are the interaction strength, anisotropy and local magnetic field, respectively. The operators σ_{xi}, σ_{yi}, σ_{zi} are the Pauli matrices for the qubit i $(i = 1,2)$. As $[S_z, H_S] = 0$ $(S_z = \sigma_{z1} + \sigma_{z2})$, the total magnetization is a conserved quantity. Notice that we set $\hbar = k_B = 1$ throughout this paper.

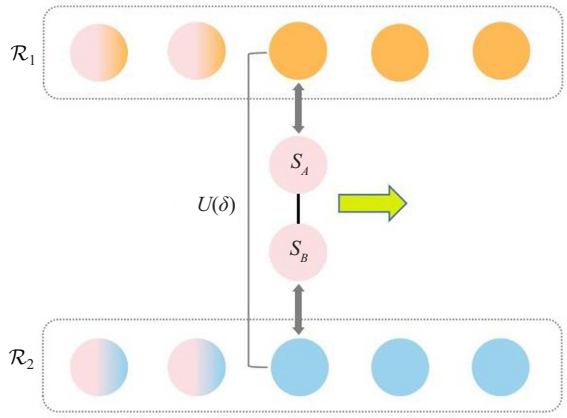


Fig. 1. Sketch of the protocol: a pair of ancillas come from reservoir $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively, collide to each other under a unitary operation $U(\delta)$. Then this correlated ancillas in the global state ρ_E (Eq. (5)) collide with the system's spins S_A and S_B.

We assume that each qubit of the system interacts with two reservoirs $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively, and each reservoir consisted of a stream of uncoupled qubits (we call ancillas) whose logical states are $\{|0\rangle, |1\rangle\}$. The interaction between the system qubit and a ancilla only lasts for a short time τ during which the interaction Hamiltonian is given by

$$H_{SE} = \sum_{i=1,2} \gamma(\sigma_{xi}\tilde{\sigma}_{xi} + \sigma_{yi}\tilde{\sigma}_{yi}), \quad (2)$$

where the operators $\tilde{\sigma}_{ji}$ $(j = x,y,z)$ are the Pauli matrices for a ancilla interacting with the system qubit i. The coefficient γ determines the strength of the interaction. Additionally each ancilla is subject to the local Hamiltonian

$$H_{E_i} = B_i\tilde{\sigma}_{zi}, \quad (3)$$

where $i = 1,2$ and $H_E = H_{E_1} + H_{E_2}$.

2.2. Quantum synchronization

Spontaneous quantum synchronization between a pair of quantum systems can be said to emerge through the establishment of coherent oscillations in the expectation values of their local observables, and it is generally possible to observe this behavior by just looking at the dynamics of these expectation values. We adopt the well-known Pearson correlation coefficient C_{12}, which is a standard tool in statistics for identifying correlations between two data sets, as our figure of merit for the detection of synchronous behavior.[4] Given two discrete variables x and y, linear correlation between them can be measured by the Pearson coefficient, which is given as

$$C_{12} = \frac{\sum_t (x_t - \bar{x})(y_t - \bar{y})}{\sqrt{\sum_t (x_t - \bar{x})^2}\sqrt{\sum_t (y_t - \bar{y})^2}}, \quad (4)$$

where $\bar{x}$ and $\bar{y}$ denote the averages of x and y over the data set t. C_{12} is a bounded function which satisfy $C_{12} \in [-1,1]$. While $C_{12} = 0$ indicates that the two variables are completely uncorrelated, $C_{12} = 1$ and $C_{12} = -1$ points out a full positive and negative linear correlation, respectively. In our work, the variables in question are the expectation values of the local observables of the system qubits S_A and S_B in the x direction, namely, $\langle\sigma^x_{S_A}\rangle$ and $\langle\sigma^x_{S_B}\rangle$. Consequently, based on the definition of the Pearson coefficient [Eq. (4)], completely positive and completely negative correlations imply fully synchronized and fully antisynchronized behaviors between the local expectation values, respectively. As we sample C_{12} over a sliding data window along the total evolution time, we can obtain a time-dependent Pearson coefficient to probe how the oscillations become phase locked over time. And to get a smooth behavior in the Pearson coefficient evolution, we allow the adjacent data windows to partially overlap for a certain interval.

2.3. Repeated interactions

In order to investigate the effect of reservoir on quantum synchronization between two qubits of system, we correlate the ancillas with a unitary transformation U, and their state becomes

$$\rho_E = U\tilde{\rho}_E U^\dagger, \quad (5)$$

where $\tilde{\rho}_E = \tilde{\rho}_{E_1} \otimes \tilde{\rho}_{E_2}$, with $\tilde{\rho}_{E_1}$ and $\tilde{\rho}_{E_2}$ being the initial state of the ancillas of two reservoirs $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. After this initial preparation, the ancilla of $\mathcal{R}_1$ and $\mathcal{R}_2$ collides with system's qubit S_A and S_B, respectively (see Fig. 1). The collision, lasting for a time τ, is described by the unitary operator

$$U_{col} = e^{-iH_{tot}\tau}, \quad (6)$$

where $H_{tot} = H_S + H_E + H_{SE}$ is the total Hamiltonian. Thus the system is brought from time t to $t + \tau$ through the process

$$\rho_S(t) \otimes \rho_E \to \rho_{SE}(t+\tau) = U_{col}(\rho_S(t) \otimes \rho_E)U^\dagger_{col}, \quad (7)$$

where $\rho_S(t)$ is the state of the system at time t and $\rho_{SE}(t+\tau)$ is the state of the "system + two ancillas" at time $t + \tau$ after the collision.

We consider the initial state of each ancilla of two reservoirs as

$$\tilde{\rho}_{E_i} = p_i|\psi_i\rangle\langle\psi_i| + (1-p)\rho_{\beta_i}, \quad i = 1,2, \quad (8)$$

where $p_i \in [0,1]$,

$$|\psi_i\rangle = \frac{1}{\sqrt{Z_i}}\left(\exp -\frac{1}{4}B_i\beta_i|0\rangle + \exp \mathrm{i}\,\phi_i + \frac{1}{4}B_i\beta_i|1\rangle\right)$$

with a relative phase ϕ_i, and

$$\rho_{\beta_i} = \frac{1}{Z_i}\mathrm{e}^{-\beta_i H_{\mathrm{E}_i}}$$

is the thermal state. Here $\beta_i = 1/T_i$ and $Z_i = \mathrm{Tr}[\mathrm{e}^{-\beta_i H_{\mathrm{E}_i}}]$ are the inverse temperature and the partition function, respectively. Notice that the diagonal elements of states ρ_{E_i} and ρ_{β_i} are identical, and compared with the thermal state, the off-diagonal elements of state ρ_{E_i} are nonzero if $p_i \neq 0$. In this case the effective temperature of the reservoir is defined by its diagonal elements.

We assume that the unitary U in Eq. (5) correlating the ancillas of reservoir is a partial swap operation

$$U(\delta) = (\cos\delta)\mathbb{I} + \mathrm{i}(\sin\delta)S_{\mathrm{swa}}, \tag{9}$$

where $\delta \in [0, \pi/2]$ is a dimensionless interaction strength, $\mathbb{I}$ is the identity operator. S_{swa} is the two-particle swap operator, and in the ordered basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ it reads[60]

$$S_{\mathrm{swa}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

When $\delta = 0$ equation (9) is reduced into an identity operator and indicates that there is no interaction between two ancillas; and when $\delta = \pi/2$ equation (9) is reduced into a fully swap operator and represents a complete exchange of quantum state information between them.

3. Quantum synchronization

In the following, we investigate the roles of quantum correlation of reservoirs on synchronization behaviors between system spins. We suppose that the pair of system spins is uncorrelated initially, i.e.,

$$|\psi_S\rangle = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle).$$

We let $T_1 = 2$ and $T_2 = 1$ in Eq. (8) for two reservoirs $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. We use concurrence[61] to measure the degree of entanglement in the bipartite state of reservoir particles ρ_E, which can be expressed as

$$E(\rho_{\mathrm{E}}) = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\}, \tag{11}$$

where $\{\lambda_i\}$ are the eigenvalues of $\rho_{\mathrm{E}}\rho'_{\mathrm{E}}$ in decreasing order. $\rho'_{\mathrm{E}} = (\sigma_y \otimes \sigma_y)\rho^*_{\mathrm{E}}(\sigma_y \otimes \sigma_y)$, where ρ^*_{E} is the complex conjugation of ρ_{E}.

3.1. Thermal state

Firstly we consider initial thermal states of two reservoirs, i.e., Eq. (8) with $p_i = 0$, and the density matrix of two ancillas after the unitary $U(\delta)$ becomes Eq. (A2) in Appendix A. From the numerical calculation, we find that even though the Pearson coefficient can eventually approaches 1 with a long enough evolution time, the expectation values of the local observables become extremely small. In other words, the system has reached its steady state before the transient synchronization appears. Therefore in this case even though the Pearson coefficient can arrive at 1 finally, there is no visibility for this synchronization. Based on this, when the Pearson coefficient has not reached 1 before the expectation value of the local observable decays to 10^{-4} of the initial value, we say that the two qubits can not be synchronized. It is worth emphasizing that the magnitude of this value makes no qualitative difference to the following results in this paper. In order to investigate the roles of quantum correlation of reservoir on synchronization, we calculate the entanglement between the two ancillas after the unitary $U(\delta)$. It shows that the entanglement is always zero, and the full details are given in Appendix A.

3.2. State with coherence

Now we consider initial states of two reservoirs with coherence, i.e., Eq. (8) with $p_i \neq 0$, and the density matrix of two ancillas after the unitary $U(\delta)$ becomes Eq. (A1) in the Appendix A. In Fig. 2 we plot the Pearson coefficient C_{12} for the local spin observables $\langle\sigma^x_{S_A}\rangle$ and $\langle\sigma^x_{S_B}\rangle$, which has been calculated for data windows of 140 collisions with partial overlaps of 125. We consider different phase difference between each pair of ancillas, namely $\Delta\phi = \phi_1 - \phi_2 = \{0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3\}$. It can be seen that after a certain collision interval of incoherent oscillations, the spin expectation values become phase locked as witnessed by the behavior of the Pearson coefficient. Especially, the phase difference between two reservoirs can affect the degree of synchronization and the time arriving at synchronization.

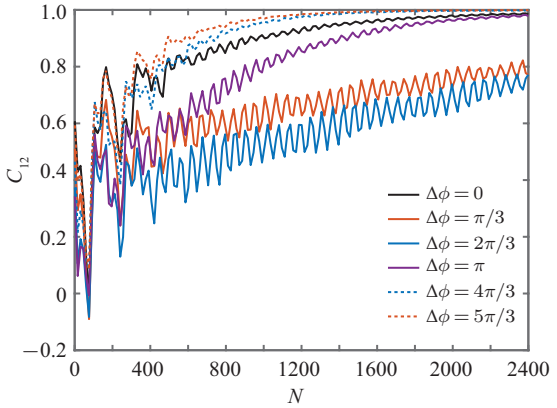


Fig. 2. Pearson coefficient C_{12} in terms of the number of collisions N for initial state (8) with $p_1 = p_2 = 0.9$ and different phase differences $\Delta\phi$ $[\Delta\phi = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3]$, which is plotted for data windows of 140 collisions with partial overlaps of 125. Parameters: $J = 0.1$, $\Delta = 1$, $B_1/B_2 = 1.2$, $T_1 = 2T_2 = 2$, $\gamma = 0.3$, and $\delta = \pi/32$.

In Figs. 3(a)–3(c), we display the final value of the Pearson coefficient C_{12} after 2400 collisions taking place between the system and the reservoir particles in terms of the phase difference $\Delta\phi$, figure 3(a) is in the weak coupling regime with $\delta = \pi/32$, figure 3(c) is in the strong coupling regime with $\delta = \pi/4$, and figure 3(b) is an intermediate case with $\delta = \pi/8$.

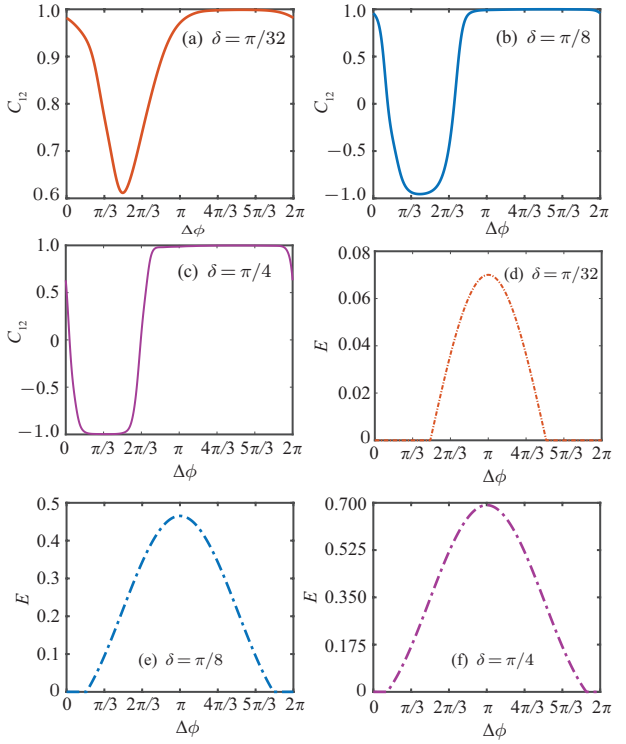


Fig. 3. (a)–(c) Synchronization diagram displaying the final value of the Pearson coefficient C_{12} after a time interval of $N = 2400$, as a function of the phase difference between two ancillas [$\Delta\phi$], in the weak ($\delta = \pi/32$)- and strong ($\delta = \pi/4$)-coupling regimes for panels (a) and (c) respectively, and $\delta = \pi/8$ which results in an intermediate case for panel (b). (d)–(f) Quantum entanglement between the two ancillas [Eq. (A1)] as a function of $\Delta\phi$, in the weak ($\delta = \pi/32$)- and strong ($\delta = \pi/4$)-coupling regimes for panels (d) and (f) respectively, also $\delta = \pi/8$ is an intermediate case of panel (e). For all plots the other parameters are the same as those in Fig. 2.

Also the phase difference can influence the time for system spins to get synchronized. And the synchronization–antisynchronization transition between system spins can be realized by manipulating $\Delta\phi$ in the strong coupling regime (Fig. 3(c)). Physically, this can be easily understood as follows. As all four qubits (system + two ancillas) are interacted after state preparation, thus if one scans the phase difference of the ancillas would change the phase difference of the system, and two system qubits are in synchronization (the phase difference between two system qubits is 0) or anti-synchronization (the phase difference between two system qubits is π). As mentioned above, the synchronization behaviors show a much stronger response in the strong coupling regime in contrast to the weak coupling regime. In order to investigate this in detail, we calculate the entanglement E between the two ancillas after the unitary $U(\delta)$ as a function of $\Delta\phi$. Figure 3(d) ($\delta = \pi/32$) and figure 3(f) ($\delta = \pi/4$) are the weak- and strong-coupling regimes respectively, and figure 3(e) is an intermediate case with $\delta = \pi/8$. It can be seen that the range of synchronization and the speed of the establishment of synchronization can be improved with the increase of entanglement. Thus, we clearly demonstrates an example of the dynamical establishment of reservoir entanglement-induced spontaneous mutual synchronization in spin chain locally coupled to two independent environments. In all, we can control the synchronization behaviors of system spins by manipulating the phase difference between two reservoirs and the quantum entanglement of reservoir.

In Fig. 4(a), the final value of C_{12} after 3500 collisions between the system and the reservoir particles is shown as a function of left bath temperature T_1, at $T_2 = 1$ for different phase differences ($\Delta\phi = 2\pi/3, \pi, 4\pi/3$). We show that increasing the temperature gradient, $T_1 - T_2$, speeds up the emergence of synchronization between system spins for fixed $\Delta\phi$. In Fig. 4(b), we present the entanglement between the two ancillas of reservoir (Eq. (A1)) as a function of T_1. As expected, the entanglement is increased with the increase of temperature gradient. In other words, the entanglement of reservoir has an essential impact on the establishment of synchronization between system spins.

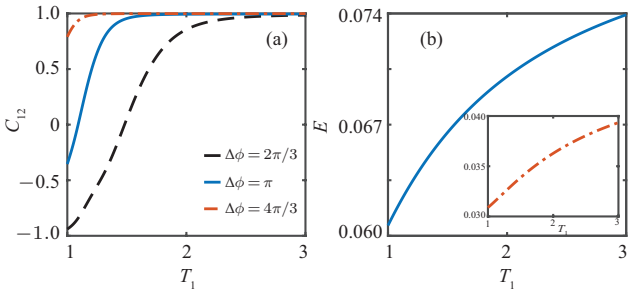


Fig. 4. (a) Synchronization diagram displaying the final value of the Pearson coefficient C_{12} after a time interval of $N = 3500$, as a function of left bath temperature T_1, at $T_2 = 1$ for different $\Delta\phi$. (b) Quantum entanglement between the two ancillas [Eq. (A1)] as a function of T_1, with $\Delta\phi = \pi$ (the blue solid line) and $\Delta\phi = 2\pi/3, 4\pi/3$ (the red dot–dashed line). For all plots the other parameters are the same as those in Fig. 2.

4. Conclusion

In this paper, we have investigated the nonequilibrium effects of reservoir on quantum synchronization of the system. We have considered two-qubit XXZ chain coupled independently to their own reservoirs, and the reservoirs are modeled by the so-called collisional model. In our framework, two reservoir particles, initially prepared in a thermal state or a state with coherence. In order to investigate the roles of quantum entanglement of reservoir, we have correlated two reservoir particles through a unitary transformation and afterward interact locally with the two quantum subsystems. For initial thermal states of two reservoirs, it has no synchronization and the quantum entanglement of two reservoir particles is zero. However, for the initial states of two reservoirs with coherence, the synchronization between system spins can be con-

trolled by manipulating the phase difference and the quantum entanglement between two reservoir particles. And the degree of synchronization and the speed of the establishment of synchronization have be improved with the increase of entanglement. It means quantum entanglement of reservoir is the key of controlling quantum synchronization of system qubits. We expect that these properties revealed in this paper can help one to gain some insight into the connections between entanglement and quantum synchronization.

Appendix A

For the initial states of two reservoir particles with coherence (Eq. (8)), the density matrix after the unitary $U(\delta)$ can be expressed as

$$\rho_{\mathrm{E}} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}, \tag{A1}$$

where

$$\begin{aligned}
\rho_{11} &= \frac{1}{1+\mathrm{e}+\mathrm{e}^{\beta_1}+\mathrm{e}^{1+\beta_1}}, \\
\rho_{12} &= \rho_{21}^{*} = \frac{\mathrm{e}^{\mathrm{i}\delta} p_1 (\sqrt{\mathrm{e}}\cos\delta - \mathrm{i}\,\mathrm{e}^{\beta_1/2-\mathrm{i}\phi_1}\sin\delta)}{(1+\mathrm{e})(1+\mathrm{e}^{\beta_1})}, \\
\rho_{13} &= \rho_{31}^{*} = \frac{\mathrm{e}^{\mathrm{i}\delta} p_1 (\mathrm{e}^{\beta_1/2-\mathrm{i}\phi_1}\cos\delta - \mathrm{i}\sqrt{\mathrm{e}}\sin\delta)}{(1+\mathrm{e})(1+\mathrm{e}^{\beta_1})}, \\
\rho_{14} &= \rho_{41}^{*} = \frac{\mathrm{e}^{(1+\beta_1-2\mathrm{i}\phi_1)/2} p_1^2}{(1+\mathrm{e})(1+\mathrm{e}^{\beta_1})}, \\
\rho_{22} &= \frac{\mathrm{e}^{-\mathrm{i}\phi_1}\left(\mathrm{i}\,\mathrm{e}^{(1+\beta_1)/2}(-1+\mathrm{e}^{2\mathrm{i}\phi_1}) p_1^2 \cos\delta\sin\delta + \mathrm{e}^{\mathrm{i}\phi_1}(\mathrm{e}\cos\delta^2 + \mathrm{e}^{\beta_1}\sin\delta^2)\right)}{(1+\mathrm{e})(1+\mathrm{e}^{\beta_1})}, \\
\rho_{23} &= \rho_{32}^{*} = \frac{\mathrm{e}^{(1+\beta_1-2\mathrm{i}\phi_1)/2} p_1^2 \cos\delta^2 + \mathrm{i}\left(-\mathrm{e}+\mathrm{e}^{\beta_1}\right)\cos\delta\sin\delta + \mathrm{e}^{(1+\beta_1+2\mathrm{i}\phi_1)/2} p_1^2 \sin\delta^2}{(1+\mathrm{e})(1+\mathrm{e}^{\beta_1})}, \\
\rho_{24} &= \rho_{42}^{*} = \frac{\mathrm{e}^{1+\beta_1/2-\mathrm{i}\phi_1-\mathrm{i}\delta} p_1 \cos\delta + \mathrm{i}\,\mathrm{e}^{1/2+\beta_1-\mathrm{i}\delta} p_1 \sin\delta}{1+\mathrm{e}+\mathrm{e}^{\beta_1}+\mathrm{e}^{1+\beta_1}}, \\
\rho_{33} &= \frac{\mathrm{e}^{-\mathrm{i}\phi_1}\left(-\mathrm{i}\,\mathrm{e}^{(1+\beta_1)/2}(-1+\mathrm{e}^{2\mathrm{i}\phi_1}) p_1^2 \cos\delta\sin\delta + \mathrm{e}^{\mathrm{i}\phi_1}(\mathrm{e}^{\beta_1}\cos\delta^2 + \mathrm{e}\sin\delta^2)\right)}{(1+\mathrm{e})(1+\mathrm{e}^{\beta_1})}, \\
\rho_{34} &= \rho_{43}^{*} = \frac{\mathrm{e}^{1/2+\beta_1-\mathrm{i}\delta} p_1 \cos\delta + \mathrm{i}\,\mathrm{e}^{1+\beta_1/2-\mathrm{i}\phi_1-\mathrm{i}\delta} p_1 \sin\delta}{1+\mathrm{e}+\mathrm{e}^{\beta_1}+\mathrm{e}^{1+\beta_1}}, \\
\rho_{44} &= \frac{\mathrm{e}^{1+\beta_1}}{(1+\mathrm{e})(1+\mathrm{e}^{\beta_1})}.
\end{aligned}$$

Here we let $\beta_2 = 1$, $\phi_2 = 0$, $B_1/B_2 = 1.2$, and $p_1 = p_2$. For initial thermal states of two reservoirs ($p_1 = p_2 = 0$) with $\beta_1 = \beta_2/2 = 1/2$, equation (A1) becomes

$$\rho_E = \begin{pmatrix} \dfrac{1}{1+\sqrt{\mathrm{e}}+\mathrm{e}+\mathrm{e}^{3/2}} & 0 & 0 & 0 \\ 0 & \dfrac{\mathrm{e}\cos\delta^2+\sqrt{\mathrm{e}}\sin\delta^2}{\left(1+\sqrt{\mathrm{e}}\right)(1+\mathrm{e})} & -\mathrm{i}\left(\dfrac{1}{1+\sqrt{\mathrm{e}}}-\dfrac{1}{1+\mathrm{e}}\right)\cos\delta\sin\delta & 0 \\ 0 & \mathrm{i}\left(\dfrac{1}{1+\sqrt{\mathrm{e}}}-\dfrac{1}{1+\mathrm{e}}\right)\cos\delta\sin\delta & \dfrac{\sqrt{\mathrm{e}}\cos\delta^2+\mathrm{e}\sin\delta^2}{\left(1+\sqrt{\mathrm{e}}\right)(1+\mathrm{e})} & 0 \\ 0 & 0 & 0 & \dfrac{1}{1+\dfrac{1}{\mathrm{e}^{3/2}}+\dfrac{1}{\mathrm{e}}+\dfrac{1}{\sqrt{\mathrm{e}}}} \end{pmatrix}. \tag{A2}$$

Acknowledgments

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