

Higher-order topological Anderson insulator on the Sierpiński lattice

Huan Chen(陈焕)1, Zheng-Rong Liu(刘峥嵘)1, Rui Chen(陈锐), and Bin Zhou(周斌)
Citation: Chin. Phys. B, 2024, 33 (**1**): 017202. DOI: 10.1088/1674-1056/ad09d4
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Doping tuned anomalous Hall effect in the van der Waals magnetic topological phases $Mn(Sb_{1-x}Bi_x)_4Te_7$

Xin Zhang(张鑫), Zhicheng Jiang(江志诚), Jian Yuan(袁健), Xiaofei Hou(侯骁飞), Xia Wang(王霞),Na Yu(余娜), Zhiqiang Zou(邹志强), Zhengtai Liu(刘正太), Wei Xia(夏威),Zhenhai Yu(于振海), Dawei Shen(沈大伟), and Yanfeng Guo(郭艳峰)
Chin. Phys. B, 2023, 32 (**9**): 097201. DOI: 10.1088/1674-1056/acd629

Long-range adsorbate interactions mediated by two-dimensional Dirac fermions

Xiaohui Wang(王晓慧), Zhen-Guo Fu(付振国), Zhigang Wang(王志刚), Feng Chi(迟锋), and Ping Zhang(张平)
Chin. Phys. B, 2023, 32 (**5**): 057201. DOI: 10.1088/1674-1056/acae77

Recent progress on the planar Hall effect in quantum materials

Jingyuan Zhong(钟景元), Jincheng Zhuang(庄金呈), and Yi Du(杜轶)
Chin. Phys. B, 2023, 32 (**4**): 047203. DOI: 10.1088/1674-1056/acb91a

Electron tunneling through double-electric barriers on HgTe/CdTe heterostructure interface

Liang-Zhong Lin(林亮中), Yi-Yun Ling(凌艺纭), Dong Zhang(张东), and Zhen-Hua Wu(吴振华)
Chin. Phys. B, 2022, 31 (**11**): 117201. DOI: 10.1088/1674-1056/ac7a0f

Electrical controllable spin valves in a zigzag silicene nanoribbon ferromagnetic junction

Lin Zhang(张林)
Chin. Phys. B, 2018, 27 (**6**): 067203. DOI: 10.1088/1674-1056/27/6/067203

RAPID COMMUNICATION

Higher-order topological Anderson insulator on the Sierpiński lattice

Huan Chen(陈焕)[1], Zheng-Rong Liu(刘峥嵘)[1], Rui Chen(陈锐)[1,†], and Bin Zhou(周斌)[1,2,‡]

[1] *Department of Physics, Hubei University, Wuhan* 430062, *China*

[2] *Key Laboratory of Intelligent Sensing System and Security of Ministry of Education, Hubei University, Wuhan* 430062, *China*

(Received 8 October 2023; revised manuscript received 6 November 2023; accepted manuscript online 6 November 2023)

Disorder effects on topological materials in integer dimensions have been extensively explored in recent years. However, its influence on topological systems in fractional dimensions remains unclear. Here, we investigate the disorder effects on a fractal system constructed on the Sierpiński lattice in fractional dimensions. The system supports the second-order topological insulator phase characterized by a quantized quadrupole moment and the normal insulator phase. We find that the second-order topological insulator phase on the Sierpiński lattice is robust against weak disorder but suppressed by strong disorder. Most interestingly, we find that disorder can transform the normal insulator phase to the second-order topological insulator phase with an emergent quantized quadrupole moment. Finally, the disorder-induced phase is further confirmed by calculating the energy spectrum and the corresponding probability distributions.

Keywords: fractal system, topological insulator

PACS: 72.15.Rn, 03.65.Vf

DOI: 10.1088/1674-1056/ad09d4

1. Introduction

Fractals are graphs with self-similarity, in which each constituent exhibits the same character as the whole.[1,2] In recent years, theoretical works concerning quantum effects on fractal lattices have been extensively studied, such as Anderson localization,[3–5] electronic[6,7] and optical conductivity,[8] plasmon dispersion relations,[9] and other related topics.[10–14] Despite being embedded in integer dimensional space, a fractal lattice is characterized by a non-integer Hausdorff dimension.[15] Due to its unique characteristics and motivated by the experimental developments,[16,17] the fractal lattices have attracted much attention in recent years.

On the other hand, tremendous efforts have been devoted to the study of topological systems in integer dimensions during the past two decades.[18–21] Nowadays, researchers have extended these topological phases to fractal systems, which allow for new topological phenomena that are impossible in crystals. For example, topological phases in fractals do not possess a well-defined bulk like their crystalline counterparts, but they are able to support topologically protected states on the boundary.[22] Topological phases have been widely investigated in different fractal systems,[23–25] such as the Chern insulator,[26–31] higher-order topological insulator,[32–34] non-Hermitian topological insulator,[35] and topological superconductor.[36,37] The second-order topological insulator (SOTI) in fractals exhibits unique inner corner modes.[32] The Chern insulator in fractals is protected by the robust mobility gap instead of the direct bandgap in conventional topological insulators.[33] In the meantime, topological phases in fractals have been experimentally reported in various systems, such as the Chern insulator[33] and higher-order topological insulator[34,38] in acoustic systems and Floquet topological insulator[22,39] and higher-order topological insulator[24] in photonic systems.

Moreover, the interplay between disorder and dimensionality plays a critical role in the research of topological systems. The disorder-induced topological phase, which is referred to as the topological Anderson insulator (TAI), was first proposed by Li *et al.*[40] Since then, the TAI phases have been investigated in numerous systems in integer dimensions.[41–58] So far, the TAI phase has been observed experimentally in one-dimensional disordered atomic wires,[59] photonic platforms,[60–62] and a quantum simulator on a superconducting-circuit device.[63] Recently, the TAI phase has been proposed to be realized in an electric circuit.[64] However, it is worth highlighting that the disorder effect on fractal systems remains an unexplored territory.

Here, we investigate the disorder-induced topological phase transition on the Sierpiński lattice, which is one of the best-known examples of a fractal system. Depending on the topological mass, the fractal system supports a normal insulator (NI) phase with a zero topological index and a SOTI phase with a quantized quadrupole moment. We find that the SOTI on the Sierpiński lattice is robust against weak disorder. Surprisingly, we reveal that disorder can induce a phase transition from the NI phase to the SOTI phase, indicating the occurrence of the higher-order TAI phase in the fractal system. Moreover,

[†]Corresponding author. E-mail: chenr@hubu.edu.cn

[‡]Corresponding author. E-mail: binzhou@hubu.edu.cn

© 2024 Chinese Physical Society and IOP Publishing Ltd

the disorder-induced phase is further confirmed by calculating the energy spectrum and the corresponding probability distribution.

This work is organized as follows. We give a brief description of the Sierpiński lattice in Subsection 2.1, introduce the tight-binding Hamiltonian in the fractal system in Subsection 2.2, and adopt the quadrupole moment to characterize the system in Subsection 2.3. Then we demonstrate that the topological nature of the system can be captured by the quadrupole moment in Section 3.1, discuss the disorder effects on the fractal system in Subsection 3.2, and show that the disorder-induced phase transitions can be further confirmed by calculating the energy spectrum and the probability distributions in Subsection 3.3. Finally, we make a summary and a brief discussion in Section 4.

2. Model and method

2.1. Sierpiński lattice

The Sierpiński lattice can be regarded as the lattice obtained after the manufacture of defects following certain laws on a square lattice.[26,27] The initial unit contains eight sites (the green sites in Fig. 1). This unit is replicated 8 times to obtain the structure of the next generation, which contains an inner hole without lattice site (the blue sites in Fig. 1). This operation is repeated to obtain the red sites in Fig. 1. Each iteration makes the number of lattice sites become eight times that of the previous generation by copying and moving. In this way, we can easily describe the fractal lattice of a Sierpiński carpet. For a certain generation f, the site number $N = 8^f$. For the iterations $f \gg 1$, we obtain an approximation of the Sierpiński carpet with fractional Hausdorff dimension $d_{\mathrm{H}} = \ln 8/\ln 3 \simeq 1.89$.

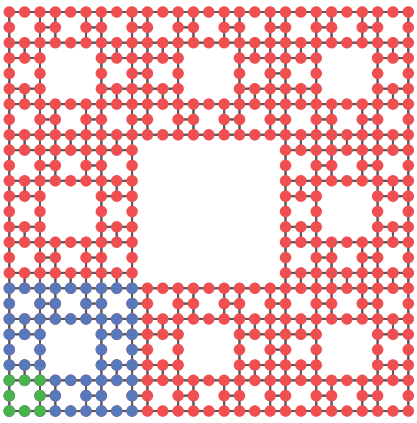


Fig. 1. Schematic illustration of the Sierpiński lattices. Here, the green, blue, and red sites make up the first, second, and third generations with the site numbers 8, 64, and 512, respectively.

2.2. Hamiltonian

The tight-binding Hamiltonian of a quadrupole insulator on the Sierpiński lattice has the following form:[32,65]

$$H = H_1 + H_2 + H_W, \tag{1}$$

where

$$H_1 = -\sum_{j\neq k} \frac{l(r_{jk})}{2} c_j^\dagger [\mathrm{i}t_1(\sigma_3\tau_1\cos\phi_{jk} + \sigma_0\tau_2\sin\phi_{jk}) + t_2\sigma_0\tau_3]c_k + \sum_j c_j^\dagger (m+2t_2)\sigma_0\tau_3 c_j \tag{2}$$

describes a first-order topological insulator protected by time-reversal symmetry. Here, the long-range hopping between two different sites is considered and $l(r_{jk}) = \mathrm{e}^{1-r_{jk}/r_0}$ corresponds to the spatial decay factor of hopping amplitudes, where r_{jk} is the lattice spacing between sites j and k, and r_0 is the decay length. ϕ_{jk} is the polar angle between sites j and k. $\sigma_{1,2,3}$ and $\tau_{1,2,3}$ are the Pauli matrices acting on the spin and orbital spaces, respectively. σ_0 and τ_0 are identity matrices. t_1 and t_2 denote the hopping amplitudes. $c_j^\dagger = (c_{j\alpha\uparrow}^\dagger, c_{j\alpha\downarrow}^\dagger, c_{j\beta\uparrow}^\dagger, c_{j\beta\downarrow}^\dagger)$ is the creation operator at site j determined by spin direction $\uparrow,\downarrow$ and orbital index α, β. The second part

$$H_2 = g\sum_{j\neq k} \frac{l(r_{jk})}{2} c_j^\dagger (\cos 2\phi_{jk})\sigma_1\tau_1 c_k \tag{3}$$

is the Wilson mass term, which breaks the time-reversal symmetry. The Wilson mass term hybridizes and then gaps out the counter-propagating gapless edge modes of the system. Moreover, H_2 causes the sign of the domain-wall mass to flip four times under 2π rotation, which breaks four-fold rotational symmetry C_4.[66–68] In the clean limit, the Hamiltonian H respects the combined symmetry C_4T,[32,65] where $T = s_2\tau_0 K$ is the time-reversal symmetry and K is the complex conjugation. For the rest of the discussion, we set $r_0 = 1$, $t_1 = t_2 = 1$, $g = 1$. The last term

$$H_W = \sum_j c_j^\dagger U_j \sigma_0\tau_3 c_j \tag{4}$$

depicts the Anderson-type disorder. U_j is a set of uniform random numbers distributed within the range of $[-W/2, W/2]$, with W being the disorder strength.

2.3. Quadrupole

We adopt the quadrupole moment to characterize the fractal system, which is given by[32,65]

$$Q_{xy} = \frac{1}{2\pi}\mathrm{Im}\log\{\det[\Psi^\dagger \exp(2\pi\mathrm{i}\sum_j x_j y_j \hat{n}(\boldsymbol{r}))\Psi]\} - Q_0, \tag{5}$$

where $Q_0 = \frac{1}{2}\sum_j x_j y_j/L^2$ with linear dimension L. $\boldsymbol{r} = (x_j, y_j)$ is the coordinate of site j, and $\hat{n}(\boldsymbol{r})$ is the number operator. Ψ represents the eigenvectors of the occupied states.

It should be noticed that Q_{xy} is a gauge-dependent quantity for a finite-size fractal system and becomes gauge-invariant in the thermodynamic limit.[32] In this sense, strictly speaking, Q_{xy} is only available for an infinite system. However, we find that for a large-enough system with $f = 3$, Q_{xy} is able to capture the topology nature of the system (see Subsection 3.1 below). Therefore, in this work, we still employ Q_{xy} to characterize the topological properties of the finite-size system.

3. Numerical investigation

In this section, we first show that the quadrupole moment Q_{xy} can indeed capture the topology nature of the fractal system. Subsequently, we investigate disorder effects on the fractal system. Moreover, all the numerical calculations are performed on the third generation of the Sierpiński lattices.

3.1. Clean limit

We start with the case of clean limit, i.e., $W = 0$. Figures 2(a)–2(d) show the energy spectrum and the corresponding probability distribution of the fractal system on the Sierpiński lattices with $m = 0$. The system hosts corner modes residing on both the outer corners [Fig. 2(b)] and the inner corners [Fig. 2(c)]. Moreover, it is found that the system is characterized by a quantized quadrupole moment with $Q_{xy} = 0.5$. The quantized topological index and the emergence of the corner models confirm that the system corresponds to a SOTI phase.

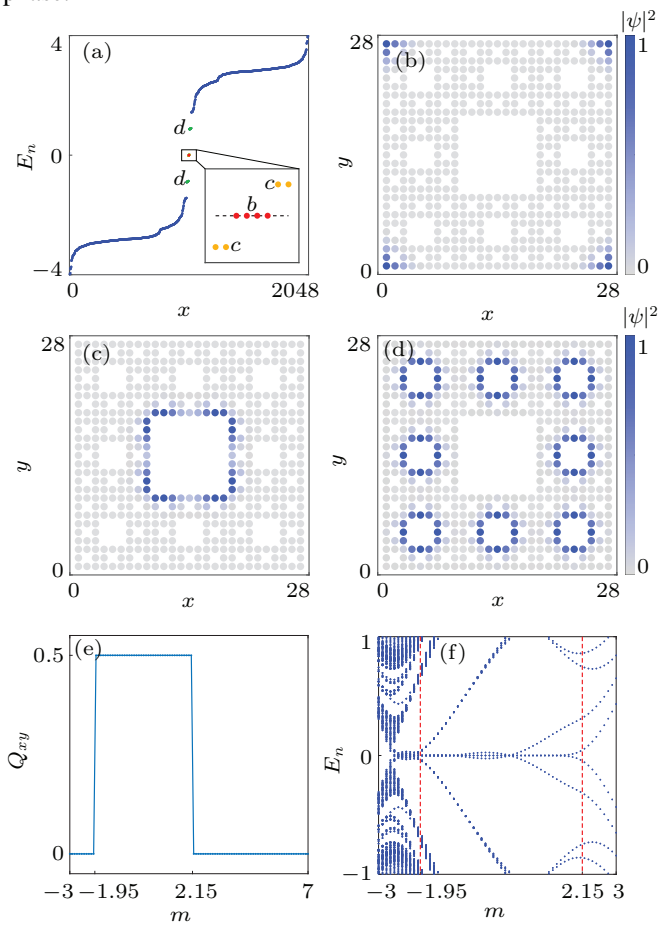


Fig. 2. (a) Energy spectrum of the fractal system constructed on the third generation of the Sierpiński lattice with $m = 0$. Panels (b), (c), and (d) show the probability distributions of the red, yellow, and green energy modes labeled in (a), respectively. (e) The quadrupole moment Q_{xy} and (f) the energy spectrum as functions of the parameter m.

Moreover, the energy of the four outer corner modes is nearly zero [the red points in Fig. 2(a)] and the four inner corner modes open a tiny energy gap in the spectrum [the yellow points in Fig. 2(a)]. In addition, there exist 32 energy modes inside the bulk gap [the green points in Fig. 2(a)], with their probability distributions localized around a series of secondary inner boundaries [Fig. 2(d)].

Figures 2(a) and 2(b) show the quadrupole moment Q_{xy} and the corresponding energy spectrum as functions of the parameter m on the fractal system, respectively. We find that the SOTI phase characterized by $Q_{xy} = 0.5$ always hosts four zero-energy inner corner modes. The other in-gap localized states are not that stable compared to the four outer corner modes. Therefore, we show that the quadrupole moment Q_{xy} can capture the topology nature of the fractal system.

3.2. Disorder effects

Now, we study the disorder effects on the fractal system. For the SOTI phase with $Q_{xy} = 0.5$, the results are as expected [Fig. 3(a)]. With increasing disorder strength W, Q_{xy} keeps the quantized value until W exceeds certain values. Therefore, similar to the previous studies on disordered topological systems,[40,41] the topological nature of the SOTI phase in the fractal system is also robust against weak disorder. Further increasing the disorder strength, the quantized Q_{xy} is suppressed by disorder, then gradually decreases and finally collapses to zero.

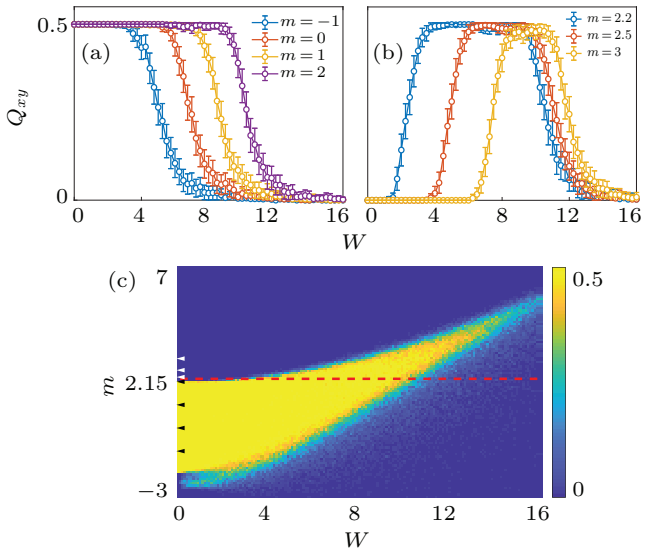


Fig. 3. The quadrupole moment Q_{xy} as a function of the disorder strength W for (b) $m = -1$, 0, 1, 2 and (c) $m = 2.2$, 2.5, 3, respectively. The error bar represents the standard deviation of 500 samples. (c) The quadrupole moment Q_{xy} calculated in the (W, m) plane. In (c), each data point is averaged on 50 independent disorder configurations.

In the clean limit, the NI phase is characterized by $Q_{xy} = 0$ [Fig. 3(a)]. With increasing disorder strength, the disorder-averaged quadrupole moment Q_{xy} increases and then forms a quantized plateau with $Q_{xy} = 0.5$. The quantized plateau is observed for a certain range of disorder strength, and it decreases and finally disappears with increasing the disorder strength. The zero fluctuation of the quantized plateau indicates that it corresponds to a disorder-induced SOTI phase. Moreover, we

plot the diagrams of the system as a function of m and W in Fig. 3(c). The disorder-induced topological phase transitions can be observed more clearly.

3.3. Disorder-averaged energy spectrum and probability

Here, we show that the above disorder-induced phenomenon can be further confirmed by checking the disorder-averaged spectrum and the corresponding probability distribution.

Figure 4(a) shows the disorder-averaged energy spectrum as a function of the disorder strength W for $m = 0$. In the clean limit ($W = 0$), the system corresponds to a SOTI and hosts eight nearly-zero-energy modes with their probability distributions localized on the four outer corners and the four inner corners [Fig. 4(b)]. With the increasing disorder strength, the bulk gap diminishes and vanishes for about $W = 6$. This is in accordance with the results shown in Fig. 3(a), where the quantized Q_{xy} also starts to collapse when $W = 6$. Further increasing the disorder strength, the four inner corner modes first disappear [Fig. 4(c)], followed by the disappearance of all corner modes [Fig. 4(d)].

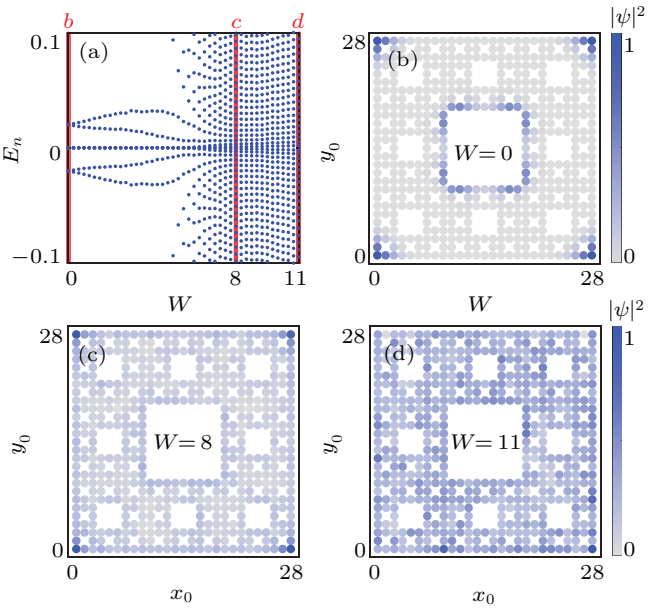


Fig. 4. (a) The energy spectrum as a function of the disorder strength W for the SOTI phase with $m = 0$. (b)–(d) Disorder-averaged probability distribution of the middle eight energy modes with disorder strength $W = 0$, 8, and 11, respectively. Here, each data is obtained after averaging on 500 random disorder configurations.

Figure 5(a) shows the disorder-averaged energy spectrum as a function of the disorder strength W for $m = 2.2$. In the clean limit ($W = 0$), the system corresponds to an NI and no corner modes appear [Fig. 5(b)]. With the increasing disorder strength, the bulk gap decreases and vanishes for about $W = 4$. This is in accordance with the results shown in Fig. 3(b), where the quantized Q_{xy} emerges when $W = 4$. Further increasing the disorder strength, the eight corner modes appear [Fig. 5(c)] and finally are suppressed by strong disorder [Fig. 5(d)].

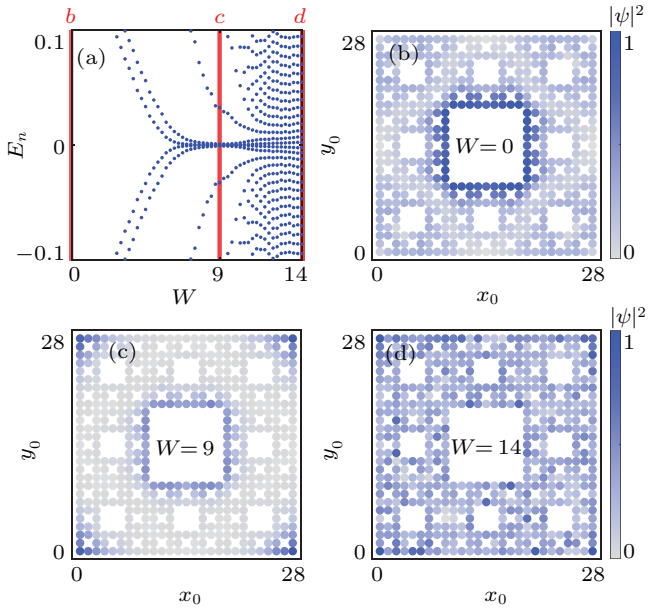


Fig. 5. (a) The energy spectrum as a function of the disorder strength W for the NI phase with $m = 2.2$. (b)–(d) Disorder-averaged probability distribution of the middle eight energy modes with disorder strength $W = 0$, 9, and 14, respectively. Here, each data is obtained after averaging on 500 random disorder configurations.

4. Conclusion

In this work, we study the disorder effects on a fractal system constructed on the Sierpiński lattice. We show that SOTI phase on the fractal system is robust against weak disorder. Moreover, we reveal a disorder-induced SOTI phase characterized by an emergent quantized quadrupole moment with $Q_{xy} = 0.5$. Finally, the disorder induced phenomena on the fractal system are further confirmed by checking the disorder-averaged energy spectra and the corresponding probability distributions. In crystals, disordered-induced higher-order topological phase manifests localized states at external corners.[57,58] However, in fractal lattices, the disorder-induced phases manifest localized states at both internal and external corners. This property represents a unique characteristic of the disorder-induced second-order topological phase in the fractal system.

We expect that the disorder-induced SOTI phase in the fractal system can be experimentally realized in some metamaterials in the future. The SOTI phases in fractal systems have been experimentally observed in both the acoustic[34,38] and the photonic[24] systems. On the other hand, the disorder effects have been introduced in both the acoustic[69,70] and the photonic[60–62] systems. The above experiments offer the possibility of future experimental realization of our proposal.

Acknowledgements

R.C. acknowledges the support of the National Natural Science Foundation of China (Grant No. 12304195) and the Chutian Scholars Program in Hubei Province. B.Z. was

supported by the National Natural Science Foundation of China (Grant No. 12074107), the program of outstanding young and middle-aged scientific and technological innovation team of colleges and universities in Hubei Province (Grant No. T2020001), and the innovation group project of the Natural Science Foundation of Hubei Province of China (Grant No. 2022CFA012). Z.-R.L. was supported by the Postdoctoral Innovation Research Program in Hubei Province (Grant No. 351342).

References

[1] Bunde A and Havlin S 1994 *Fractals in Science* (Springer)
[2] Nakayama T and Yakubo K 2003 *Fractal concepts in condensed matter physics* (Springer Science & Business Media)
[3] Nakayama T, Yakubo K and Orbach R L 1994 *Rev. Mod. Phys.* **66** 381
[4] Sticlet D and Akhmerov A 2016 *Phys. Rev. B* **94** 161115
[5] Kosior A and Sacha K 2017 *Phys. Rev. B* **95** 104206
[6] Van Veen E, Yuan S, Katsnelson M I, Polini M and Tomadin A 2016 *Phys. Rev. B* **93** 115428
[7] Liu Y, Hou Z, Hui P M and Sritrakool W 1999 *Phys. Rev. B* **60** 13444
[8] Van Veen E, Tomadin A, Polini M, Katsnelson M I and Yuan S 2017 *Phys. Rev. B* **96** 235438
[9] Westerhout T, Van Veen E, Katsnelson M I and Yuan S 2018 *Phys. Rev. B* **97** 205434
[10] Khalili Golmankhaneh A 2019 *Fractal and Fract.* **3** 20
[11] Akal I 2018 *Phys. Rev. D* **98** 106003
[12] Pal B and Saha K 2018 *Phys. Rev. B* **97** 195101
[13] Nandy A and Chakrabarti A 2016 *Phys. Rev. A* **93** 013807
[14] Nandy A, Pal B and Chakrabarti A 2015 *J. Phys.: Condens. Matter* **27** 125501
[15] Li J, Mo Q, Jiang J H and Yang Z 2022 *Sci. Bull.* **67** 2040
[16] Shang J, Wang Y, Chen M, Dai J, Zhou X, Kuttner J, Hilt G, Shao X, Gottfried J M and Wu K 2015 *Nat. Chem.* **7** 389
[17] Kempkes S N, Slot M R, Freeney S E, Zevenhuizen S J M, Vanmaekelbergh D, Swart I and Smith C M 2019 *Nat. Phys.* **15** 127
[18] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
[19] Qi X L and Zhang S C 2011 *Rev. Mod. Phys.* **83** 1057
[20] Shen S Q 2017 *Topological Insulators* (Springer Singapore)
[21] Bernevig B A and Hughes T L 2013 *Topological Insulators and Topological Superconductors* (Princeton University Press)
[22] Biesenthal T, Maczewsky L J, Yang Z, Kremer M, Segev M, Szameit A and Heinrich M 2022 *Science* **376** 1114
[23] Yang Y B, Wang J H, Li K and Xu Y 2023 arXiv:2309.03688
[24] Ren B, Kartashov Y V, Maczewsky L J, Kirsch M S, Wang H, Szameit A, Heinrich M and Zhang Y 2023 *Nanophotonics* **12** 3829
[25] Li X, Jha M C and Nielsen A E B 2022 *Phys. Rev. B* **105** 085152
[26] Iliasov A A, Katsnelson M I and Yuan S 2020 *Phys. Rev. B* **101** 045413
[27] Brzezińska M, Cook A M and Neupert T 2018 *Phys. Rev. B* **98** 205116
[28] Fischer S, van Hooft M, van der Meijden T, Smith C M, Fritz L and Fremling M 2021 *Phys. Rev. Res.* **3** 043103
[29] Song Z G, Zhang Y Y and Li S S 2014 *Appl. Phys. Lett.* **104** 233106
[30] Fremling M, van Hooft M, Smith C M and Fritz L 2020 *Phys. Rev. Res.* **2** 013044
[31] Ivaki M N, Sahlberg I, Pöyhönen K and Ojanen T 2022 *Commun. Phys.* **5** 327
[32] Manna S, Nandy S and Roy B 2022 *Phys. Rev. B* **105** L201301
[33] Li J, Sun Y, Mo Q, Ruan Z and Yang Z 2023 *Phys. Rev. Res.* **5** 023189
[34] Zheng S, Man X, Kong Z L, Lin Z K, Duan G, Chen N, Yu D, Jiang J H and Xia B 2022 *Sci. Bull.* **67** 2069
[35] Manna S and Roy B 2023 *Commun. Phys.* **6** 10
[36] Pai S and Prem A 2019 *Phys. Rev. B* **100** 155135
[37] Manna S, Das S K and Roy B 2022 arXiv:2207.02203
[38] Li J, Mo Q, Jiang J H and Yang Z 2022 arXiv:2205.05298
[39] Yang Z, Lustig E, Lumer Y and Segev M 2020 *Light Sci. Appl.* **9** 128
[40] Li J, Chu R L, Jain J K and Shen S Q 2009 *Phys. Rev. Lett.* **102** 136806
[41] Jiang H, Wang L, Sun Q F and Xie X C 2009 *Phys. Rev. B* **80** 165316
[42] Groth C W, Wimmer M, Akhmerov A R, Tworzyd lo J and Beenakker C W J 2009 *Phys. Rev. Lett.* **103** 196805
[43] Wu B, Song J, Zhou J and Jiang H 2016 *Chin. Phys. B* **25** 117311
[44] Xing Y, Zhang L and Wang J 2011 *Phys. Rev. B* **84** 035110
[45] Orth C P, Sekera T, Bruder C and Schmidt T L 2016 *Sci. Rep.* **6** 24007
[46] Guo H M, Rosenberg G, Refael G and Franz M 2010 *Phys. Rev. Lett.* **105** 216601
[47] Guo H, Feng S and Shen S Q 2011 *Phys. Rev. B* **83** 045114
[48] Chen R, Xu D H and Zhou B 2017 *Phys. Rev. B* **95** 245305
[49] Chen R, Xu D H and Zhou B 2017 *Phys. Rev. B* **96** 205304
[50] Chen R, Chen C Z, Sun J H, Zhou B and Xu D H 2018 *Phys. Rev. B* **97** 235109
[51] Chen R, Xu D H and Zhou B 2018 *Phys. Rev. B* **98** 235159
[52] Su Y, Avishai Y and Wang X R 2016 *Phys. Rev. B* **93** 214206
[53] Kimme L and Hyart T 2016 *Phys. Rev. B* **93** 035134
[54] Chen C Z, Song J T, Jiang H, Sun Q F, Wang Z and Xie X 2015 *Phys. Rev. Lett.* **115** 246603
[55] Chen C Z, Liu H, Jiang H, Sun Q F, Wang Z and Xie X C 2015 *Phys. Rev. B* **91** 214202
[56] Shapourian H and Hughes T L 2016 *Phys. Rev. B* **93** 075108
[57] Yang Y B, Li K, Duan L M and Xu Y 2021 *Phys. Rev. B* **103** 085408
[58] Li C A, Fu B, Hu Z A, Li J and Shen S Q 2020 *Phys. Rev. Lett.* **125** 166801
[59] Meier E J, An F A, Dauphin A, Maffei M, Massignan P, Hughes T L and Gadway B 2018 *Science* **362** 929
[60] Sttzer S, Plotnik Y, Lumer Y, Titum P, Lindner N H, Segev M, Rechtsman M C and Szameit A 2018 *Nature* **560** 461
[61] Liu G G, Yang Y, Ren X, Xue H, Lin X, Hu Y H, Sun H X, Peng B, Zhou P, Chong Y and Zhang B 2020 *Phys. Rev. Lett.* **125** 133603
[62] Cui X, Zhang R Y, Zhang Z Q and Chan C T 2022 *Phys. Rev. Lett.* **129** 043902
[63] Li X G, Xu H K, Wang J H, Tang L Z, Zhang D W, Yang C H, Su T, Wang C L, Mi Z Y, Sun W J, *et al.* 2023 arXiv:2301.12138
[64] Zhang Z Q, Wu B L, Song J and Jiang H 2019 *Phys. Rev. B* **100** 184202
[65] Agarwala A, Juričić V and Roy B 2020 *Phys. Rev. Res.* **2** 012067
[66] Tao Y L and Xu Y 2022 arXiv:2209.02262
[67] Wang C, Liu F and Huang H 2022 *Phys. Rev. Lett.* **129** 056403
[68] Chen R, Chen C Z, Gao J H, Zhou B and Xu D H 2020 *Phys. Rev. Lett.* **124** 036803
[69] Liu H, Xie B, Wang H, Liu W, Li Z, Cheng H, Tian J, Liu Z and Chen S 2021 arXiv:2111.06520
[70] Zangeneh-Nejad F and Fleury R 2020 *Adv. Mater.* **32** 2001034