

Off-diagonal approach to the exact solution ofquantum integrable systems

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SPECIAL TOPIC — Celebrating the 100th Anniversary of Physics Discipline of Northwest University

Off-diagonal approach to the exact solution of quantum integrable systems

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We investigate the t–W scheme for the anti-ferromagnetic XXX spin chain under both periodic and open boundary conditions. We propose a new parametrization of the eigenvalues of the transfer matrix. Based on it, we obtain the exact solution of the system. By analyzing the distribution of zero roots at the ground state, we obtain the explicit expressions of the eigenfunctions of the transfer matrix and the associated $\mathbb{W}$ operator (see Eqs. (10) and (70)) in the thermodynamic limit. We find that the ratio of the quantum determinant with the eigenvalue of $\mathbb{W}$ operator for the ground state exhibits exponential decay behavior. Thus this fact ensures that the so-called inversion relation (the t–W relation without the W-term) can be used to study the ground state properties of quantum integrable systems with/without $U(1)$-symmetry in the thermodynamic limit.

Keywords: quantum spin chain, bethe ansatz, Yang–Baxter equation

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1. Introduction

Quantum integrable systems, defined by the Yang–Baxter equation[1,2] or the Lax representation,[3] provide crucial insights in quantum field theory, condensed matter physics and statistical physics. They serve as reliable benchmarks for studying many-body effects, settling debates on fundamental concepts, and exhibiting phenomena such as thermodynamic phase transitions[4] and the generation of fractional charges.[5] These models also have many applications in different fields, such as cold atoms[6–10] and AdS/CFT correspondence.[11,12]

The eigenvalue problem of quantum integrable systems with $U(1)$ symmetry has been tackled by using the methods including coordinate Bethe ansatz,[13] T–Q relation[14,15] and algebraic Bethe ansatz.[16–20] It should be emphasized that there exist the integrable models which do not possess the $U(1)$ symmetry. Due to the $U(1)$ symmetry broken, it is very hard to construct the suitable reference state. Thus the exact solution of this kind of integrable systems is a challenging issue. Several techniques have been developed to address this problem, including gauge transformation,[21] fusion-based T–Q relation,[22,23] q-Onsager algebra,[24,25] separation of variables,[26,27] modified algebraic Bethe ansatz[28,29] and off-diagonal Bethe ansatz.[30,31] The eigenvalues of the transfer matrix of the quantum integrable systems without $U(1)$ symmetry are characterized by the inhomogeneous T–Q relations. However, the associated Bethe ansatz equations (BAEs) are inhomogeneous and the corresponding distributions of Bethe roots are very complicated. Consequently, the thermodynamic Bethe ansatz[32–36] does not work and it is very hard to calculate the exact physical properties in the thermodynamic limit, such as the ground state, elementary excitations and thermodynamic quantities including specific heat and magnetic susceptibility at the finite temperature.

Recently, a novel Bethe ansatz known as the t–W scheme has been proposed,[37] which effectively tackles the challenges posed by the inhomogeneous T–Q relations. Taking the XXZ spin chain with open boundary condition as an example, we showed the power of this approach with the help of inhomogeneous parameters.[38] The main advantage of this method is that the related BAEs are homogeneous and we can take the thermodynamic limit. We calculated the exact physical quantities such as ground state energy, elementary excitations and surface energy. We also extended the method to the twisted boundary situation.[39] Subsequently, we generalized the t–W method to the finite temperature. The thermodynamic quantities including the free energy of the XXX spin chain with periodic boundary condition were computed.[40] Later, we ap-

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plied this method to the supersymmetric t–J model,[41] Hubbard model[42] and integrable J_1–J_2 model with competition interactions,[43,44] among other notable achievements.[45,46]

The t–W relation can be used to determine the energy spectrum directly. Typically, the W operator can be neglected in the thermodynamic limit, resulting in the t–W relation becoming equivalent to the inversion relation,[2,47,48] but the exact proof is absent. Now, we focus on this issue. Our first investigation is as follows. By putting the inhomogeneous parameters into the transfer matrix, we prove that the t–W relations are closed at the inhomogeneous points, where the coefficients of the W terms are zero. The inhomogeneous parameters are utilized as the auxiliary functions to determine the distribution of zero roots in the thermodynamic limit.[38] In this paper, we analytically obtain the eigenfunction of the W operator in the thermodynamic limit at the ground state, and demonstrate that the leading term in the t–W relation is quantum determinant instead of the W term. We take the isotropic spin-1/2 chain as an example. The model Hamiltonian reads

$$H^{\mathrm{p}} = \sum_{n=1}^{N} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z), \tag{1}$$

for the periodic boundary condition (PBC) and

$$\begin{aligned} H^{\mathrm{o}} = & \sum_{n=1}^{N-1} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z) \\ & + \frac{\eta}{p}\sigma_1^z + \frac{\eta}{q}(\sigma_N^z + \xi \sigma_N^x), \end{aligned} \tag{2}$$

for the open boundary condition (OBC), where σ_n^α is the Pauli matrix along the α-direction at n-th site, $\alpha = (x, y, z)$, $\sigma_{N+1}^\alpha = \sigma_1^\alpha$, p, q, ξ are the boundary parameters associated with the boundary fields and $\eta = \mathrm{i}$.

The paper is organized as follows. In Section 2, we introduce the XXX spin chain with periodic boundary condition. We show the integrability, t–W solutions and the eigenfunction of the transfer matrix and W operator at the ground state in the thermodynamic limit. In Section 3, we generalized these results to the open boundary case. Section 4 includes the summary and further discussions. Appendix A gives the detailed derivation of the t–W relation and Appendix B shows the Hermitian property of the transfer matrix with PBC.

2. Closed chain

2.1. Integrability

Throughout this paper, V denotes a two-dimensional linear space and $\{|m\rangle, m = 0, 1\}$ are its orthogonal bases. We shall adopt the standard notations. For any matrix $A \in \mathrm{End}(V)$, A_j is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as A on the j-th space and as identity on the other factor spaces. For the matrix $B \in \mathrm{End}(V \otimes V)$, $B_{i,j}$ is an embedding operator of B in the tensor space, which acts as identity on the factor spaces except for the i-th and j-th ones.

Let us introduce the R-matrix $R_{0,j}(u) \in \mathrm{End}(V_0 \otimes V_j)$,

$$R_{0,j}(u) = \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix}, \tag{3}$$

where u is the spectral parameter and η is the crossing parameter. The R-matrix (3) has the following properties:

$$\begin{aligned} &\text{initial condition}: R_{0,j}(0) = \eta P_{0,j}, \\ &\text{unitary relation}: R_{0,j}(u) R_{j,0}(-u) = \phi(u) \times \mathrm{id}, \\ &\text{crossing relation}: R_{0,j}(u) = -\sigma_0^y R_{0,j}^{t_0}(-u-\eta)\sigma_0^y, \\ &\text{PT-symmetry}: R_{0,j}(u) = R_{j,0}(u) = R_{0,j}^{t_0 t_j}(u), \\ &Z_2\text{-symmetry}: \sigma_0^\alpha \sigma_j^\alpha R_{0,j}(u) = R_{0,j}(u)\sigma_0^\alpha \sigma_j^\alpha, \text{ for } \alpha = x, y, z, \\ &\text{fusion condition}: R_{0,j}(\pm\eta) = \eta(\pm 1 + P_{0,j}) = \pm 2\eta P_{0,j}^{(\pm)}, \end{aligned} \tag{4}$$

where $\phi(u) = \eta^2 - u^2$, t_0 (or t_j) denotes the transposition in the space V_0 (or V_j), $P_{0,j}$ is the permutation operator possessing the property

$$R_{j,k}(u) = P_{0,j} R_{0,k}(u) P_{0,j}, \tag{5}$$

$P_{0,j}^{(-)}$ is the one-dimensional antisymmetric project operator defined in the one-dimensional subspace spanned by $\frac{1}{\sqrt{2}}(|12\rangle_{0,j} - |21\rangle_{0,j})$, $P_{0,j}^{(+)}$ is the three-dimensional symmetric projector defined in the three-dimensional subspace spanned by the orthogonal bases $\{|11\rangle_{0,j}, \frac{1}{\sqrt{2}}(|12\rangle_{0,j} + |21\rangle_{0,j}), |22\rangle_{0,j}\}$, $P_{0,j}^{(-)} + P_{0,j}^{(+)} = \mathrm{id}$. The R-matrix (3) satisfies the Yang–Baxter equation (YBE)

$$\begin{aligned} & R_{1,2}(u_1 - u_2) R_{1,3}(u_1 - u_3) R_{2,3}(u_2 - u_3) \\ = \; & R_{2,3}(u_2 - u_3) R_{1,3}(u_1 - u_3) R_{1,2}(u_1 - u_2). \end{aligned} \tag{6}$$

Starting from the R-matrix, we construct the monodromy matrix as

$$T_0(u) = R_{0,N}(u - \theta_N) R_{0,N-1}(u - \theta_{N-1}) \cdots R_{0,1}(u - \theta_1), \tag{7}$$

where V_0 is the auxiliary space, $V_1 \otimes V_2 \otimes \cdots \otimes V_N$ is the physical or quantum space, N is the number of sites and $\{\theta_j | j = 1, \ldots, N\}$ are the inhomogeneous parameters. The transfer matrix $t(u)$ of the XXX spin-1/2 closed chain is given by

$$t(u) = \mathrm{tr}_0 T_0(u), \tag{8}$$

where tr_0 denotes trace over the auxiliary space V_0. The transfer matrix $t(u)$ is the generating function of conserved quantities in the system, and the Hamiltonian (1) is generated by $t(u)$ as

$$H^{\mathrm{p}} = 2\eta \left. \frac{\partial \ln t(u)}{\partial u} \right|_{u=0, \{\theta_j = 0\}} - N. \tag{9}$$

The transfer matrices with different spectral parameters commute mutually, i.e., $[t(u),t(v)]=0$, which ensures the integrability of the model (1).

2.2. The *t*–*W* scheme

By using the fusion technique,[49,50] we consider the product of transfer matrices $t(u)$ and $t(u-\eta)$

$$\begin{aligned} t(u)t(u-\eta) &= \mathrm{tr}_{1,2}\{T_2(u)T_1(u-\eta)\} \\ &= \mathrm{tr}_{1,2}\left\{T_2(u)T_1(u-\eta)(P^{(-)}_{1,2}+P^{(+)}_{1,2})\right\} \\ &= \mathrm{tr}_{1,2}\left\{P^{(-)}_{1,2}T_2(u)T_1(u-\eta)P^{(-)}_{1,2}\right\} \\ &\quad +\mathrm{tr}_{1,2}\left\{P^{(+)}_{1,2}T_2(u)T_1(u-\eta)P^{(+)}_{1,2}\right\} \\ &= a(u)d(u-\eta)\times \mathrm{id}+d(u)\mathbb{W}(u), \end{aligned} \tag{10}$$

where the functions $a(u)$ and $d(u)$ are given by

$$a(u)=\prod_{j=1}^{N}(u-\theta_j+\eta),\quad d(u)=a(u-\eta), \tag{11}$$

$a(u)d(u-\eta)$ is the quantum determinant and $\mathbb{W}(u)$ is a new operator. The detailed proof is given in Appendix A. At the points of $\{u=\theta_j\}$, the operator relation (10) can be simplified as

$$t(\theta_j)t(\theta_j-\eta)=a(\theta_j)d(\theta_j-\eta),\quad j=1,\ldots,N, \tag{12}$$

due to the fact $d(\theta_j)=0$.

From the definition, we know that both $t(u)$ and $\mathbb{W}(u)$ are the operator-valued polynomial of u with degree N. Moreover, the matrices $t(u)$ and $\mathbb{W}(u)$ commutate with each other, namely,

$$[t(u),t(v)]=[\mathbb{W}(u),\mathbb{W}(v)]=[t(u),\mathbb{W}(v)]=0. \tag{13}$$

Thus they have common eigenstates. Acting the operator identities (10) and (12) on the common eigenstate $|\Psi\rangle$, we obtain the *t*–*W* relation

$$\Lambda(u)\Lambda(u-\eta)=a(u)d(u-\eta)+d(u)W(u), \tag{14}$$

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta)=a(\theta_j)d(\theta_j-\eta),\quad j=1,\ldots,N, \tag{15}$$

where $\Lambda(u)$ and $W(u)$ are the eigenvalues of the transfer matrix $t(u)$ and $\mathbb{W}(u)$ operator, respectively,

$$t(u)|\Psi\rangle=\Lambda(u)|\Psi\rangle,\quad \mathbb{W}(u)|\Psi\rangle=W(u)|\Psi\rangle.$$

From the construction of transfer matrix (8), we conclude that

$$\Lambda(u),\text{ as a function of }u,\text{ is a polynomial of degree }N, \tag{16}$$

$$W(u),\text{ as a function of }u,\text{ is a polynomial of degree }N. \tag{17}$$

Meanwhile, $\Lambda(u)$ and $W(u)$ satisfy the asymptotic behaviors

$$\lim_{u\to\infty}\Lambda(u)=2u^N+\cdots,\quad \lim_{u\to\infty}W(u)=3u^N+\cdots. \tag{18}$$

2.3. Exact solution

In order to obtain the exact solution of the spin-1/2 XXX closed chain described by the Hamiltonian (1), let us take the homogeneous limit, i.e., $\{\theta_j=0\}$. Usually, the eigenvalues $\Lambda(u)$ and $W(u)$ are expressed by the T–Q relations with the help of Bethe roots. Here, we quantify $\Lambda(u)$ and $W(u)$ by their zero roots as

$$\Lambda(u)=2\prod_{j=1}^{N}(u-z_j+\frac{\eta}{2}),\quad W(u)=3\prod_{j=1}^{N}(u-w_j), \tag{19}$$

where $\{z_j|j=1,\ldots,N\}$ and $\{w_j|j=1,\ldots,N\}$ are the zero roots of $\Lambda(u)$ and $W(u)$, respectively.

Substituting Eq. (19) with $\{u=z_j-\frac{\eta}{2}\}$ and $\{u=w_j\}$ into the *t*–*W* relation (14) with $\{\theta_j=0\}$, we obtain that the zero roots $\{z_j\}$ and $\{w_j\}$ should satisfy the BAEs

$$(z_j+\frac{\eta}{2})^N(z_j-\frac{3}{2}\eta)^N=-(z_j-\frac{\eta}{2})^N W(z_j-\frac{\eta}{2}),\quad j=1,\ldots,N, \tag{20}$$

$$\Lambda(w_j)\Lambda(w_j-\eta)=(w_j+\eta)^N(w_j-\eta)^N,\quad j=1,\ldots,N. \tag{21}$$

Further analysis gives that the $2N$ roots satisfy the constrains

$$\{z_j^*\}=\{z_j\},\quad \{w_j^*\}=\{w_j\}. \tag{22}$$

The detailed proof can be found in Appendix B.

The eigenvalues of the Hamiltonian (1) can be expressed in terms of the zero roots $\{z_j\}$ as

$$E^{\mathrm{p}}=-2\eta\times\sum_{j=1}^{N}\frac{1}{z_j-\frac{\eta}{2}}-N. \tag{23}$$

For the finite system size N, one can solve the BAEs (20) and (21) numerically. Substituting the values of roots into Eq. (23), one obtains the energy of the system. The most interesting thing is the thermodynamic limit where N tends to infinity, which will be addressed in the next subsection.

2.4. Ground state eigenfunctions in the thermodynamic limit

We fist consider the distribution of zero roots. According to the exact numerical calculation with finite N and the analytic analysis with infinite N, we find that all the zero roots form the conjugate pairs around the line $\pm l\eta/2$ with a positive integer $l\geq 2$, which is called as the l-strings. At the ground state, both z-roots and w-roots form the 2-strings. For even N, the patterns of roots with $N=6,8,10,12$ are shown in Fig. 1. We can find that all the z-roots form conjugate pairs as $\{u_j^{(2)}\pm\eta|j=1,\ldots,N/2\}$ with real $u_j^{(2)}$ and all the w-roots form conjugate pairs as $\{\bar{u}_j^{(2)}\pm\frac{3\eta}{2}|j=1,\ldots,N/2\}$ with real

$\bar{u}_j^{(2)}$. Substituting the 2-strings into Eq. (19), we obtain the ground state eigenvalues $\Lambda_{\rm g}(u)$ and $W_{\rm g}(u)$ as

$$\Lambda_{\rm g}(u)=2\prod_{j=1}^{N/2}(u-u_j^{(2)}-\frac{\eta}{2})(u-u_j^{(2)}+\frac{3\eta}{2}), \tag{24}$$

$$W_{\rm g}(u)=3\prod_{j=1}^{N/2}(u-\bar{u}_j^{(2)}-\frac{3}{2}\eta)(u-\bar{u}_j^{(2)}+\frac{3}{2}\eta). \tag{25}$$

Now we analyze the leading terms of the eigenvalues. For this purpose, we define

$$\Lambda_{\rm g}(u)=\exp(N[\lambda_{\rm g}^{(0)}(u)+\frac{1}{N}\lambda_{\rm g}^{(1)}(u)+O(\frac{1}{N^2})]), \tag{26}$$

$$W_{\rm g}(u)=\exp(N[\omega_{\rm g}^{(0)}(u)+\frac{1}{N}\omega_{\rm g}^{(1)}(u)+O(\frac{1}{N^2})]), \tag{27}$$

where $\lambda_{\rm g}^{(0)}(u)$ and $\omega_{\rm g}^{(0)}(u)$ are the highest order terms of u, and $\lambda_{\rm g}^{(1)}(u)$ and $\omega_{\rm g}^{(1)}(u)$ are the second higher order terms. In the thermodynamic limit $N\to\infty$, the leading terms of $\Lambda_{\rm g}(u)$ can be obtained by taking the derivatives of logarithm of Eqs. (24) and (26)

$$\frac{\partial}{\partial u}\lambda_{\rm g}^{(\beta)}(u)=\int_{-\infty}^{\infty}\left(\frac{1}{u-\lambda-\frac{\eta}{2}}+\frac{1}{u-\lambda+\frac{3\eta}{2}}\right)\rho_{\rm g}^{(\beta)}(\lambda)\,{\rm d}\lambda,$$
$$\beta=0,1, \tag{28}$$

where $\rho_{\rm g}^{(\beta)}(\lambda)$ is the density of z-roots at the ground state. The leading terms of $W_{\rm g}(u)$ are determined by taking the derivatives of logarithm of Eqs. (25) and (27) as

$$\frac{\partial}{\partial u}w_{\rm g}^{(\beta)}(u)=\int_{-\infty}^{\infty}\left(\frac{1}{u-\lambda-\frac{3\eta}{2}}+\frac{1}{u-\lambda+\frac{3\eta}{2}}\right)\rho_w^{(\beta)}(\lambda)\,{\rm d}\lambda,$$
$$\beta=0,1, \tag{29}$$

where $\rho_w^{(\beta)}(\lambda)$ is the density of w-roots.

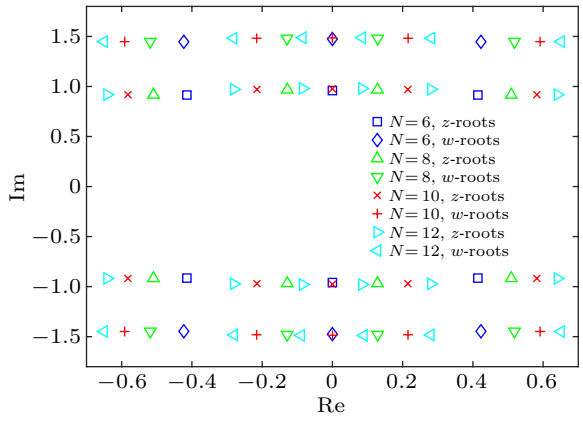


Fig. 1. Patterns of zero roots at the ground state with $N=6,8,10,12$. The data are obtained by using the exact numerical diagonalization with the inhomogeneous parameters $\{\theta_j=0\}$.

The role of inset inhomogeneous parameters is to help us to determine the density of z-roots. Take the difference of Eq. (15) at two nearest inhomogeneous points. In the thermodynamic limit, we set that the density of inhomogeneous parameters as the δ-function. Then we have

$$\frac{\partial}{\partial u}\ln[\Lambda_{\rm g}(u)\Lambda_{\rm g}(u-\eta)]$$
$$=\frac{\partial}{\partial u}\ln[(u+\eta)^N(u-\eta)^N]+O\left(\frac{1}{N}\right), \tag{30}$$

where $O(1/N)$ is the order-dependent correction. Substituting Eq. (26) into Eq. (30), we have

$$\frac{\partial}{\partial u}\left[\lambda_{\rm g}^{(0)}(u)+\lambda_{\rm g}^{(0)}(u-\eta)\right]=\frac{1}{u+\eta}+\frac{1}{u-\eta},\ \ \lambda_{\rm g}^{(0)}(0)=0,$$
$$\frac{\partial}{\partial u}\left[\lambda_{\rm g}^{(1)}(u)+\lambda_{\rm g}^{(1)}(u-\eta)\right]=0,\ \ \lambda_{\rm g}^{(1)}(0)=0. \tag{31}$$

Substituting Eq. (28) with $\beta=0$ into Eq. (31), we have

$$\int_{-\infty}^{\infty}\left(\frac{1}{u-\lambda-\frac{\eta}{2}}+\frac{1}{u-\lambda+\frac{\eta}{2}}+\frac{1}{u-\lambda-\frac{3\eta}{2}}+\frac{1}{u-\lambda+\frac{3\eta}{2}}\right)$$
$$\times\rho_{\rm g}^{(0)}(\lambda)\,{\rm d}\lambda=\frac{1}{u+\eta}+\frac{1}{u-\eta}. \tag{32}$$

Solving Eq. (32) by the Fourier transformation

$$\tilde{f}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\,{\rm e}^{-{\rm i}\omega t}\,{\rm d}t,\ \ f(t)=\int_{-\infty}^{\infty}\tilde{f}(\omega)\,{\rm e}^{{\rm i}\omega t}\,{\rm d}w, \tag{33}$$

we obtain the solution of densities of z-roots as

$$\rho_{\rm g}^{(0)}(\lambda)=\frac{1}{2\cosh(\pi\lambda)},\ \ \rho_{\rm g}^{(1)}(\lambda)=0. \tag{34}$$

By incorporating the 2-string structure into the energy expression (23) and integrating it with the densities (34) as the weighting factor, we can directly determine the ground state energy in the thermodynamic limit

$$\begin{aligned}E_{\rm g}&=-2N{\rm i}\int_{-\infty}^{\infty}\left(\frac{1}{\lambda+\frac{\rm i}{2}}+\frac{1}{\lambda-\frac{3{\rm i}}{2}}\right)\left(\rho_w^{(0)}(\lambda)+\rho_w^{(1)}(\lambda)\right){\rm d}\lambda\\&\quad-N\\&=(1-4\ln 2)N,\end{aligned} \tag{35}$$

which is consistent with the result obtained through algebraic Bethe ansatz.[31] Substituting Eq. (34) into Eq. (28) and taking the integral, we obtain

$$\lambda_{\rm g}^{(0)}(u)=\ln\frac{2\Gamma(1+\frac{{\rm i}u}{2})\Gamma(\frac{3}{2}-\frac{{\rm i}u}{2})}{\Gamma(\frac{1}{2}+\frac{{\rm i}u}{2})\Gamma(1-\frac{{\rm i}u}{2})},\ \ \lambda_{\rm g}^{(1)}(u)=0. \tag{36}$$

In the derivation, we have used the formula

$$\int_0^{\infty}\frac{{\rm e}^{-t}-{\rm e}^{-ut}}{1-{\rm e}^{-t}}\,{\rm d}t-\gamma=\frac{\rm d}{{\rm d}u}\ln\Gamma(u), \tag{37}$$

where γ is the Euler–Mascheroni constant. According to the formula

$$\begin{aligned}\ln\Gamma(x+1)&=\frac{1}{2}\ln(2\pi x)+x\ln(x)-x\\&\quad+\ln\left(1+\frac{1}{12x}+\frac{1}{288x^2}+\cdots\right),\end{aligned} \tag{38}$$

the $\lambda_{\rm g}^{(0)}(u)$ can be expanded with respect to u as

$$\lambda_{\rm g}^{(0)}(u)=\ln u+\ln\frac{\cosh\frac{\pi u}{2}}{\sinh\frac{\pi u}{2}}+\frac{\eta}{2u}+\frac{1}{2u^2}+O\left(\frac{1}{u^2}\right),\quad \mathrm{Im}(u)<1. \tag{39}$$

From Eqs. (26), (36) and (39), we obtain that the eigenvalue of the transfer matrix at the ground state in the thermodynamic limit is

$$\Lambda_{\rm g}(u)=\left(\frac{2\Gamma(1+\frac{\mathrm{i}u}{2})\Gamma(\frac{3}{2}-\frac{\mathrm{i}u}{2})}{\Gamma(\frac{1}{2}+\frac{\mathrm{i}u}{2})\Gamma(1-\frac{\mathrm{i}u}{2})}\right)^N \mathrm{e}^{O(\frac{1}{N})}. \tag{40}$$

Now, we calculate the leading terms of $W(u)$. Substituting $z_j=u_j^{(2)}+\frac{\eta}{2}$ into the BAEs (20), we obtain

$$(u_j^{(2)}+\frac{3\eta}{2})^N(u_j^{(2)}-\frac{\eta}{2})^N=-(u_j^{(2)}+\frac{\eta}{2})^N W_{\rm g}(u_j^{(2)}+\frac{\eta}{2}),\quad j=1,\cdots,\frac{N}{2}, \tag{41}$$

where $\{u_j^{(2)}\}$ are real. Taking the complex conjugation of the above relation, we have

$$(u_j^{(2)}+\frac{\eta}{2})^N(u_j^{(2)}-\frac{3\eta}{2})^N=-(u_j^{(2)}-\frac{\eta}{2})^N W_{\rm g}(u_j^{(2)}-\frac{\eta}{2}),\quad j=1,\ldots,\frac{N}{2}. \tag{42}$$

Multiplying Eq. (41) with Eq. (42), we obtain

$$(u_j^{(2)}+\frac{3\eta}{2})^N(u_j^{(2)}-\frac{3\eta}{2})^N=W_{\rm g}(u_j^{(2)}+\frac{\eta}{2})W_{\rm g}(u_j^{(2)}-\frac{\eta}{2}),\quad j=1,\ldots,\frac{N}{2}. \tag{43}$$

Taking the derivative of logarithm of the above equation, we arrive at

$$\frac{1}{N}\frac{\partial}{\partial u}\ln[W_{\rm g}(u+\frac{\eta}{2})W_{\rm g}(u-\frac{\eta}{2})]=\frac{1}{u+\frac{3\eta}{2}}+\frac{1}{u-\frac{3\eta}{2}}+O\left(\frac{1}{N^2}\right). \tag{44}$$

Substituting Eq. (27) into Eq. (44), we have

$$\frac{\partial}{\partial u}[w_{\rm g}^{(0)}(u+\frac{\eta}{2})+w_{\rm g}^{(0)}(u-\frac{\eta}{2})]=\frac{1}{u+\frac{3\eta}{2}}+\frac{1}{u-\frac{3\eta}{2}},$$
$$\frac{\partial}{\partial u}[w_{\rm g}^{(1)}(u+\frac{\eta}{2})+w_{\rm g}^{(1)}(u-\frac{\eta}{2})]=0. \tag{45}$$

Substituting Eq. (29) into the above equations and solving it by the Fourier transformation, we obtain the density of w-roots as

$$\rho_w^{(0)}(\lambda)=\frac{1}{2\cosh(\pi\lambda)},\quad \rho_w^{(1)}(\lambda)=0. \tag{46}$$

Substituting Eq. (46) into Eq. (29), we have

$$\frac{\partial}{\partial u}w_{\rm g}^{(0)}(u)=\frac{\partial}{\partial u}\ln\frac{\Gamma(\frac{3}{2}+\frac{\mathrm{i}u}{2})\Gamma(\frac{3}{2}-\frac{\mathrm{i}u}{2})}{\Gamma(1+\frac{\mathrm{i}u}{2})\Gamma(1-\frac{\mathrm{i}u}{2})}=\frac{\partial}{\partial u}\ln\left(\frac{(u+\eta)(u-\eta)}{2u}\tanh\frac{\pi u}{2}\right). \tag{47}$$

Taking the integration, we obtain

$$w_{\rm g}^{(0)}(u)=\ln\left(\frac{C_w^{(0)}}{2}\frac{(u+\eta)(u-\eta)}{u}\tanh\frac{\pi u}{2}\right), \tag{48}$$

where $C_w^{(0)}$ is the integration constant. Thus the ground state eigenvalue $W_{\rm g}(u)$ can be expressed as

$$W_{\rm g}(u)=\left(\frac{(u+\eta)(u-\eta)}{u}\right)^N\left(\frac{C_w^{(0)}}{2}\tanh\frac{\pi u}{2}\right)^N\times \mathrm{e}^{\omega_{\rm g}^{(1)}(0)}\,\mathrm{e}^{O(\frac{1}{N})}. \tag{49}$$

Substituting Eq. (49) into the t–W relation (14) with $\{\theta_j=0\}$, we obtain

$$\begin{aligned}&\Lambda_{\rm g}(u)\Lambda_{\rm g}(u-\eta)\\&=(u+\eta)^N(u-\eta)^N\mathrm{e}^{O(\frac{1}{N})}\\&=(u+\eta)^N(u-\eta)^N\left[1+\left(\frac{C_w^{(0)}}{2}\tanh\frac{\pi u}{2}\right)^N C_w^{(1)}\,\mathrm{e}^{O(\frac{1}{N})}\right].\end{aligned} \tag{50}$$

When $u\to\infty$, from the asymptotic behavior of $\Lambda_{\rm g}(u)$ or Eq. (40), we know that the coefficient of the highest order term of the left hand side of Eq. (50) is 4. In order to meet this constraint, the left hand side of Eq. (50) should also be 4, which gives that $C_w^{(0)}=2$ and $C_w^{(1)}=3$. Due to the fact that $\tanh\frac{\pi u}{2}<1$, the second term of the right hand side of Eq. (50) turns to zero when $N\to\infty$, which gives that the W function can be neglected in the thermodynamic limit. Then we conclude that the t–W relation (14) can be used to study the ground state physical properties.

3. Open chain

3.1. Integrability

Next, we consider the open boundary condition. The boundary reflections are characterized by the reflection matrices

$$K^-(u)=\begin{pmatrix}p+u & 0\\ 0 & p-u\end{pmatrix},\quad K^+(u)=\begin{pmatrix}q+u+\eta & \xi(u+\eta)\\ \xi(u+\eta) & q-u-\eta\end{pmatrix}, \tag{51}$$

which satisfy the reflection equation (RE)

$$\begin{aligned}&R_{1,2}(\lambda-u)K_1^-(\lambda)R_{2,1}(\lambda+u)K_2^-(u)\\&=K_2^-(u)R_{1,2}(\lambda+u)K_1^-(\lambda)R_{2,1}(\lambda-u),\end{aligned} \tag{52}$$

and the dual reflection equation

$$\begin{aligned}&R_{1,2}(-\lambda+u)K_1^+(\lambda)R_{2,1}(-\lambda-u-2\eta)K_2^+(u)\\&=K_2^+(u)R_{1,2}(-\lambda-u-2\eta)K_1^+(\lambda)R_{2,1}(-\lambda+u).\end{aligned} \tag{53}$$

Due to the boundary reflection, we should introduce the reflecting monodromy matrix

$$\hat{T}_0(u) = R_{0,1}(u+\theta_1)R_{0,2}(u+\theta_2)\cdots R_{0,N}(u+\theta_N). \quad (54)$$

The double-row monodromy matrix $\mathscr{U}_0(u)$ is

$$\mathscr{U}_0(u) = T_0(u)K_0^-(u)\hat{T}_0(u), \quad (55)$$

which satisfies the RE

$$\begin{aligned} & R_{1,2}(\lambda-u)\mathscr{U}_1(\lambda)R_{2,1}(\lambda+u)\mathscr{U}_2(u) \\ =\ & \mathscr{U}_2(u)R_{1,2}(\lambda+u)\mathscr{U}_1(\lambda)R_{2,1}(\lambda-u). \end{aligned} \quad (56)$$

The transfer matrix for the open boundary case is constructed as

$$t^{\mathrm{o}}(u) = \mathrm{tr}_0\{K_0^+(u)\mathscr{U}_0(u)\}. \quad (57)$$

Using the crossing symmetry of the R-matrix (4), we can demonstrate

$$t^{\mathrm{o}}(u) = t^{\mathrm{o}}(-u-\eta). \quad (58)$$

The YBE and RE lead to that the transfer matrices with different spectral parameters commutate mutually, i.e., $[t^{\mathrm{o}}(u), t^{\mathrm{o}}(v)] = 0$. Thus $t^{\mathrm{o}}(u)$ is the generating function of conserved quantities and the system is integrable. The Hamiltonian (59) is generated by the transfer matrix $t(u)$ as

$$H^{\mathrm{o}} = \eta\,\frac{\partial \ln t^{\mathrm{o}}(u)}{\partial u}\bigg|_{u=0,\{\theta_j=0\}} - N. \quad (59)$$

3.2. The *t*–*W* scheme

Following the idea of fusion, we still consider the product of two transfer matrices with certain shift of the spectral parameter

$$\begin{aligned} & t^{\mathrm{o}}(u)t^{\mathrm{o}}(u-\eta) \\ =\ & \mathrm{tr}_{1,2}\left\{K_2^+(u)\mathscr{U}_2(u)\,K_1^+(u-\eta)\mathscr{U}_1(u-\eta)\right\} \\ =\ & \mathrm{tr}_{1,2}\left\{K_2^{+t_2}(u)K_1^+(u-\eta)\,\mathscr{U}_2^{t_2}(u)\mathscr{U}_1(u-\eta)\right\} \\ \overset{(4)}{=}\ & \frac{1}{\rho_2(2u-\eta)}\mathrm{tr}_{1,2}\Big\{K_2^{+t_2}(u)K_1^+(u-\eta)R_{2,1}^{t_2}(-2u-\eta) \\ & \times R_{1,2}^{t_2}(2u-\eta)\mathscr{U}_2^{t_2}(u)\mathscr{U}_1(u-\eta)\Big\} \\ =\ & \frac{1}{\rho_2(2u-\eta)}\mathrm{tr}_{1,2}\Big\{\left(K_1^+(u-\eta)R_{2,1}(-2u-\eta)K_2^+(u)\right)^{t_2} \\ & \times\left(\mathscr{U}_2(u)R_{1,2}(2u-\eta)\mathscr{U}_1(u-\eta)\right)^{t_2}\Big\} \\ =\ & \frac{1}{\rho_2(2u-\eta)}\mathrm{tr}_{1,2}\Big\{\left(K_1^+(u-\eta)R_{2,1}(-2u-\eta)K_2^+(u)\right) \\ & \times\left(\mathscr{U}_2(u)R_{1,2}(2u-\eta)\mathscr{U}_1(u-\eta)\right)\left(P_{2,1}^{(-)}+P_{2,1}^{(+)}\right)\Big\} \\ =\ & \frac{1}{\rho_2(2u-\eta)}\mathrm{tr}_{1,2}\Big\{\left(K_1^+(u-\eta)R_{2,1}(2u-\eta)K_2^+(u)\right) \\ & \times\left(\mathscr{U}_2(u)R_{1,2}(2u-\eta)\mathscr{U}_1(u-\eta)\right)P_{2,1}^{(-)} \\ & +\left(K_1^+(u-\eta)R_{2,1}(-2u-\eta)K_2^+(u)\right) \\ & \times\left(\mathscr{U}_2(u)R_{1,2}(2u-\eta)\mathscr{U}_1(u-\eta)\right)P_{2,1}^{(+)}\Big\} \\ =\ & \Delta^{\mathrm{o}}(u)/\left((u+\frac{\eta}{2})(u-\frac{\eta}{2})\right)+t_2^{\mathrm{o}}(u), \end{aligned} \quad (60)$$

where $\rho_2(u) = -u(u+2\eta)$.

The first term of Eq. (60) give a number which is the quantum determinant

$$\Delta^{\mathrm{o}}(u) = a^{\mathrm{o}}(u)d^{\mathrm{o}}(u-\eta)(u+\frac{\eta}{2})(u-\frac{\eta}{2}), \quad (61)$$

where the functions $a^{\mathrm{o}}(u)$ and $d^{\mathrm{o}}(u)$ are

$$\begin{aligned} a^{\mathrm{o}}(u) =\ & \frac{u+\eta}{u+\frac{\eta}{2}}(u+p)(\sqrt{1+\xi^2}u+q) \\ & \times\prod_{j=1}^{N}(u-\theta_j+\eta)(u+\theta_j+\eta), \end{aligned} \quad (62)$$

$$\begin{aligned} d^{\mathrm{o}}(u) =\ & a^{\mathrm{o}}(-u-\eta) \\ =\ & \frac{u}{u+\frac{\eta}{2}}(u-p+\eta)(\sqrt{1+\xi^2}(u+\eta)-q) \\ & \times\prod_{j=1}^{N}(u-\theta_j)(u+\theta_j). \end{aligned} \quad (63)$$

The second term of Eq. (60) is a new operator which is the fused transfer matrix[31,51,52] up to a constant

$$\begin{aligned} t_2^{\mathrm{o}}(u) =\ & \frac{1}{\rho_2(2u-\eta)} \\ & \times\mathrm{tr}_{1,2}\left\{K_{\{1,2\}}^+(u)T_{\{1,2\}}(u)K_{\{1,2\}}^-(u)\hat{T}_{\{1,2\}}(u)\right\}. \end{aligned} \quad (64)$$

From the fusion of the reflection matrices and mondromy matrices

$$\begin{aligned} K_{\{1,2\}}^+(u) &= P_{2,1}^{(+)}K_1^+(u-\eta)R_{2,1}(-2u-\eta)K_2^+(u)P_{1,2}^{(+)} \\ &= 2uK_{\{1,2\}}^{(1)+}(u), \end{aligned} \quad (65)$$

$$\begin{aligned} K_{\{1,2\}}^-(u) &= P_{1,2}^{(+)}K_2^-(u)R_{1,2}(2u-\eta)K_1^-(u-\eta)P_{2,1}^{(+)} \\ &= 2uK_{\{1,2\}}^{(1)-}(u), \end{aligned} \quad (66)$$

$$\begin{aligned} T_{\{1,2\}}(u) &= P_{1,2}^{(+)}T_2(u)\,T_1(u-\eta)\,P_{1,2}^{(+)} \\ &= \prod_{l=1}^{N}(u-\theta_l)T_{\{1,2\}}^{(1,\frac{1}{2})}(u), \end{aligned} \quad (67)$$

$$\begin{aligned} \hat{T}_{\{1,2\}}(u) &= P_{2,1}^{(+)}\hat{T}_2(u)\,\hat{T}_1(u-\eta)\,P_{2,1}^{(+)} \\ &= \prod_{l=1}^{N}(u+\theta_l)\hat{T}_{\{1,2\}}^{(1,\frac{1}{2})}(u), \end{aligned} \quad (68)$$

we obtain

$$t_2^{\mathrm{o}}(u) = \frac{4u^2}{\rho_2(2u-\eta)}d^{\mathrm{o}}(u)\mathbb{W}^{\mathrm{o}}(u), \quad (69)$$

where

$$\mathbb{W}^{\rm o}(u)=\mathrm{tr}_{\{1,2\}}\left\{K^{(1)+}_{\{1,2\}}(u)T^{(1,\frac{1}{2})}_{\{1,2\}}(u)K^{(1)-}_{\{1,2\}}(u)\hat{T}^{(1,\frac{1}{2})}_{\{1,2\}}(u)\right\}. \tag{70}$$

Then the operator identity (60) can be expressed as

$$\begin{aligned}t^{\rm o}(u)t^{\rm o}(u-\eta) &= \Delta^{\rm o}(u)\times \mathrm{id}/((u+\frac{\eta}{2})(u-\frac{\eta}{2}))\\ &+\frac{4u^2}{\rho_2(2u-\eta)}d^{\rm o}(u)\mathbb{W}^{\rm o}(u).\end{aligned} \tag{71}$$

At the inhomogeneous points $\{u=\theta_j\}$, equation (71) reduces to

$$\begin{aligned}&(\theta_l+\frac{\eta}{2})(\theta_l-\frac{\eta}{2})t^{\rm o}(\theta_l)t^{\rm o}(\theta_l-\eta)=\Delta^{\rm o}(\theta_l),\\ &j=1,\ldots,N.\end{aligned} \tag{72}$$

The fusion does not break the integrability of the system, thus the transfer matrix and the fused transfer matrix commutate with each other. Thus they have common eigenstates. Acting the operator relation (71) on a common eigenstate, we obtain the t–W relation

$$\begin{aligned}&\Delta^{\rm o}(u)-(u+\frac{\eta}{2})(u-\frac{\eta}{2})\bar{\Lambda}(u)\bar{\Lambda}(u-\eta)\\ &=u^2\prod_{j=1}^{N}(u-\theta_j)(u+\theta_j)\bar{W}(u),\end{aligned} \tag{73}$$

where $\bar{\Lambda}(u)$ and $\bar{W}(u)$ are the eigenvalues of the transfer matrix $t^{\rm o}(u)$ and the fused one $\mathbb{W}^{\rm o}(u)$, respectively. At the points of inhomogeneous point, the eigenvalue $\bar{\Lambda}(u)$ satisfies

$$\begin{aligned}&(\theta_l+\frac{\eta}{2})(\theta_l-\frac{\eta}{2})\bar{\Lambda}(\theta_l)\bar{\Lambda}(\theta_l-\eta)=\Delta^{\rm o}(\theta_l),\\ &j=1,\ldots,N.\end{aligned} \tag{74}$$

3.3. Exact solution

The exact solution of the system does not depend on the inhomogeneous parameters. Thus we set them as zero. From the definitions, we know that the eigenvalue function $\bar{\Lambda}(u)$ is a polynomial of u with degree $2N+2$ and also satisfies the crossing symmetry and asymptotic behavior

$$\bar{\Lambda}(-u-\eta)=\bar{\Lambda}(u),\ \lim_{u\to\infty}\bar{\Lambda}(u)=2u^{2N+2}+\cdots. \tag{75}$$

The eigenvalue function $\bar{W}(u)$ is a polynomial of u with degree of $2N+4$ and has asymptotic behavior

$$\bar{W}(-u)=\bar{W}(u),\ \lim_{u\to\infty}\bar{W}(u)=(\xi^2-3)u^{2N+4}+\cdots. \tag{76}$$

Based on above analysis, we parameterize $\bar{\Lambda}(u)$ and $\bar{W}(u)$ as

$$\bar{\Lambda}(u)=2\prod_{j=1}^{N+1}(u-z_j+\frac{\eta}{2})(u+z_j+\frac{\eta}{2}), \tag{77}$$

$$\bar{W}(u)=(\xi^2-3)\prod_{k=1}^{N+2}(u-w_k)(u+w_k), \tag{78}$$

where $\{z_j|j=1,\ldots,N+1\}$ and $\{w_k|k=1,\ldots,N+2\}$ are the roots of the corresponding polynomials, which are completely determined by the BAEs

$$\begin{aligned}&\Delta(z_j-\frac{\eta}{2})=(z_j-\frac{\eta}{2})^{2N+2}\bar{W}(z_j-\frac{\eta}{2}),\\ &j=1,\ldots,N+1,\end{aligned} \tag{79}$$

$$\begin{aligned}&\Delta(w_k)=(w_k+\frac{\eta}{2})(w_k-\frac{\eta}{2})\bar{\Lambda}(w_k)\bar{\Lambda}(w_k-\eta),\\ &k=1,\ldots,N+2,\end{aligned} \tag{80}$$

where $\Delta(u)$ is the quantum determinant with homogeneous limit $\{\theta_j=0\}$

$$\begin{aligned}\Delta(u) &= (u-\eta)(u+\eta)(u-p)(u+p)(\sqrt{1+\xi^2}u+q)\\ &\times(\sqrt{1+\xi^2}u-q)(u+\eta)^{2N}(u-\eta)^{2N}.\end{aligned} \tag{81}$$

The eigenvalue of the Hamiltonian (2) is determined by the solutions of the above BAEs

$$E^{\rm o}=\sum_{j=1}^{N+1}\frac{\eta^2}{\frac{\eta^2}{4}-z_j^2}-N. \tag{82}$$

The Hermitian of the Hamiltonian (2) requires that the boundary parameters satisfy

$$p^*=-p,\ \ q^*=-q,\ \ \xi^*=\xi. \tag{83}$$

The above constrains imply

$$R^{\dagger}(u)=-R(-u^*),\ \ \left(K^{(\pm)}(u)\right)^{\dagger}=-K^{(\pm)}(-u^*), \tag{84}$$

which gives rise to the Hermitian properties of the transfer matrix and its eigenvalue

$$(t^{\rm o}(u))^{\dagger}=t^{\rm o}(-u^*),\ \ \bar{\Lambda}^*(u)=\bar{\Lambda}(-u^*). \tag{85}$$

Combining the expansions (77) and (78), the t–W relation (73) and the Hermitian relation (85), we conclude that if z_j is a root of $\bar{\Lambda}(u)$, then z_j^* must be the root and that if w_j is a root of $\bar{W}(u)$, then w_j^* must be the root.

3.4. Eigenfunctions in the thermodynamic limit at the ground state

As shown in Fig. 2, at the ground state, the z-roots and w-roots form the 2-strings. Besides the bulk conjugate pairs around the $\pm\eta$ lines, there exist boundary conjugate pairs at the imaginary axis. We can conclude that the z-roots form conjugate pairs as $\{\pm z_1\eta, u_j^{(2)}\pm\eta\,|j=1,\ldots,N\}$ with real $z_1,u_j^{(2)}$, and w-roots form conjugate pairs as $\{\pm\chi_1\eta,\pm\chi_2\eta,w_j^{(2)}\pm\frac{3\eta}{2}|j=1,\ldots,N\}$ with real $\chi_1,\chi_2,w_j^{(2)}$. Substituting these 2-strings into Eqs. (77) and (78), we obtain

$$\bar{\Lambda}_{\rm g}(u) = 2\big(u-(z_1-\frac{1}{2})\eta\big)\big(u+(z_1+\frac{1}{2})\eta\big)$$

$$\times\prod_{j=1}^{N/2}(u-u_j^{(2)}-\frac{\eta}{2})(u+u_j^{(2)}-\frac{\eta}{2})$$
$$\times(u-u_j^{(2)}+\frac{3\eta}{2})(u+u_j^{(2)}+\frac{3\eta}{2})$$
$$=2\big(u-(z_1-\frac{1}{2})\eta\big)\big(u+(z_1+\frac{1}{2})\eta\big)$$
$$\times\exp(2N(\bar{\lambda}_{\rm g}^{(0)}(u)+\frac{1}{2N}\bar{\lambda}_{\rm g}^{(1)}(u)+O(\frac{1}{N^2}))), \quad (86)$$

$$\bar{W}_{\rm g}(u) = (\xi^2-3)(u-\chi_1\eta)(u+\chi_1\eta)(u-\chi_2\eta)(u+\chi_2\eta)$$
$$\times\prod_{j=1}^{N/2}(u-w_j^{(2)}-\frac{3}{2}\eta)(u+w_j^{(2)}-\frac{3}{2}\eta)$$
$$\times(u-w_j^{(2)}+\frac{3}{2}\eta)(u+w_j^{(2)}+\frac{3}{2}\eta)$$
$$=(\xi^2-3)(u-\chi_1\eta)(u+\chi_1\eta)(u-\chi_2\eta)(u+\chi_2\eta)$$
$$\times\exp(2N(\bar{\omega}_{\rm g}^{(0)}(u)+\frac{1}{2N}\bar{\omega}_{\rm g}^{(1)}(u)+O(\frac{1}{N^2}))), \quad (87)$$

$$\bar{\Lambda}_{\rm g}(0) = 2(z_1-\frac{1}{2})(z_1+\frac{1}{2})$$
$$\times\exp(2N(\bar{\lambda}_{\rm g}^{(0)}(u)+\frac{1}{2N}\bar{\lambda}_{\rm g}^{(1)}(u)+O(\frac{1}{N^2})))$$
$$=2pq\equiv 2p\bar{q}\sqrt{1+\xi^2}, \quad (88)$$

where the leading terms are determined by

$$\frac{\partial}{\partial u}\bar{\lambda}_{\rm g}^{(\beta)}(u)=\int_{-\infty}^{\infty}\left(\frac{1}{u-\lambda-\frac{\eta}{2}}+\frac{1}{u-\lambda+\frac{3\eta}{2}}\right)\rho_{\rm g}^{(\beta)}(\lambda)\,{\rm d}\lambda,$$
$$\beta=0,1, \quad (89)$$

$$\frac{\partial}{\partial u}\bar{\omega}_{\rm g}^{(\beta)}(u)=\int_{-\infty}^{\infty}\left(\frac{1}{u-\lambda-\eta}+\frac{1}{u-\lambda+\eta}+\frac{1}{u-\lambda-2\eta}\right.$$
$$\left.+\frac{1}{u-\lambda+2\eta}\right)\rho_w^{(\beta)}(\lambda)\,{\rm d}\lambda,\ \ \beta=0,1. \quad (90)$$

Taking the difference of Eq. (74) at two inhomogeneous points and considering the continuum limit, we have

$$\frac{\partial}{\partial u}\ln\big[(u+\frac{\eta}{2})(u-\frac{\eta}{2})\bar{\Lambda}(u)\bar{\Lambda}(u-\eta)\big]$$
$$=\frac{\partial}{\partial u}\ln\Delta(u)+O(\frac{1}{N}). \quad (91)$$

Similarly with the periodic boundary case, substituting Eq. (86) into Eq. (91), we have

$$\frac{\partial}{\partial u}[\bar{\lambda}_{\rm g}^{(0)}(u)+\bar{\lambda}_{\rm g}^{(0)}(u-\eta)]=\frac{1}{u+\eta}+\frac{1}{u-\eta},\ \ \bar{\lambda}_{\rm g}^{(0)}(0)=0,$$
$$\frac{\partial}{\partial u}[\bar{\lambda}_{\rm g}^{(1)}(u)+\bar{\lambda}_{\rm g}^{(1)}(u-\eta)]$$
$$=\frac{1}{u+\eta}+\frac{1}{u-\eta}+\frac{1}{u+p\eta}+\frac{1}{u-p\eta}+\frac{1}{u+\bar{q}\eta}+\frac{1}{u-\bar{q}\eta}$$
$$-\frac{1}{u+\frac{\eta}{2}}-\frac{1}{u-\frac{\eta}{2}}-\frac{1}{u+\frac{\eta}{2}}-\frac{1}{u-\frac{\eta}{2}}-\frac{1}{u+(z_1-\frac{1}{2})\eta}$$
$$-\frac{1}{u-(z_1-\frac{1}{2})\eta}-\frac{1}{u+(z_1+\frac{1}{2})\eta}-\frac{1}{u-(z_1+\frac{1}{2})\eta},$$

$$\bar{\lambda}_{\rm g}^{(1)}(0)=\ln(2p\bar{q}\sqrt{1+\xi^2})-\ln\big[2(z_1+\frac{1}{2})(z_1-\frac{1}{2})\big]. \quad (92)$$

Substituting Eq. (89) into Eq. (92) and solving it by the Fourier transformation, we obtain the densities of z-roots at the ground state as

$$\tilde{\rho}_{\rm g}^{(0)}(w)=\frac{a_2(w)}{2\pi\big[a_1(w)+a_3(w)\big]},$$
$$\tilde{\rho}_{\rm g}^{(1)}(w)=\frac{1}{2\pi\big[a_1(w)+a_3(w)\big]}\big[a_2(w)+a_{2p}(w)+a_{2\bar{q}}(w)$$
$$-a_1(w)-a_{2z_1-1}(w)-a_{2z_1+1}(w)\big], \quad (93)$$

where $\tilde{\rho}_{\rm g}^{(\beta)}(w)$ is the Fourier transformation of $\rho_{\rm g}^{(\beta)}(\lambda)$ and $a_n(w)={\rm e}^{-n|w|/2}$. By incorporating the 2-string structure into the energy expression (82) and integrating it with the densities (93) as the weighting factor, we obtain the ground state energy

$$E_{\rm g}^{\rm o} = -N{\rm i}\int_{-\infty}^{\infty}\left(\frac{1}{z+\frac{\rm i}{2}}+\frac{1}{z-\frac{3{\rm i}}{2}}\right)$$
$$\times\int_{-\infty}^{\infty}{\rm e}^{{\rm i}\omega z}\big(\tilde{\rho}_{\rm g}^{(0)}(\omega)+\tilde{\rho}_{\rm g}^{(1)}(\omega)\big)\,{\rm d}\omega\,{\rm d}z+\frac{1}{z_1^2-\frac{1}{4}}-N$$
$$=(1-4\ln 2)N-1+\pi-2\ln 2+\frac{1}{|p|}+\frac{\sqrt{1+\xi^2}}{|q|}$$
$$-2{\rm i}\int_0^{\infty}\frac{{\rm e}^{-|p|w}+{\rm e}^{-|q|w/\sqrt{1+\xi^2}}}{1+{\rm e}^{-w}}. \quad (94)$$

Substituting the densities (93) into Eq. (89), we arrive at

$$\bar{\lambda}_{\rm g}^{(0)}(u) = \ln\frac{2\Gamma(1+\frac{{\rm i}u}{2})\Gamma(\frac{3}{2}-\frac{{\rm i}u}{2})}{\Gamma(\frac{1}{2}+\frac{{\rm i}u}{2})\Gamma(1-\frac{{\rm i}u}{2})}, \quad (95)$$

$$\bar{\lambda}_{\rm g}^{(1)}(u) = \ln\left[\frac{4\sqrt{1+\xi^2}}{\big(u-(z_1-\frac{1}{2})\eta\big)\big(u+(z_1+\frac{1}{2})\eta\big)(u+\frac{\eta}{2})}\right.$$
$$\times\frac{\cosh(\frac{\pi u}{2}-\frac{{\rm i}\pi}{4})}{\sinh(\frac{\pi u}{2}-\frac{{\rm i}\pi}{4})}\frac{\Gamma(1+\frac{{\rm i}u}{2})\Gamma(\frac{3}{2}-\frac{{\rm i}u}{2})}{\Gamma(\frac{1}{2}+\frac{{\rm i}u}{2})\Gamma(1-\frac{{\rm i}u}{2})}$$
$$\times\frac{\Gamma(\frac{p+1}{2}+\frac{{\rm i}u}{2})\Gamma(\frac{p+2}{2}-\frac{{\rm i}u}{2})}{\Gamma(\frac{p}{2}+\frac{{\rm i}u}{2})\Gamma(\frac{p+1}{2}-\frac{{\rm i}u}{2})}$$
$$\left.\times\frac{\Gamma(\frac{\bar{q}+1}{2}+\frac{{\rm i}u}{2})\Gamma(\frac{\bar{q}+2}{2}-\frac{{\rm i}u}{2})}{\Gamma(\frac{\bar{q}}{2}+\frac{{\rm i}u}{2})\Gamma(\frac{\bar{q}+1}{2}-\frac{{\rm i}u}{2})}\right], \quad (96)$$

which implies

$$\bar{\Lambda}_{\rm g}(u) = \frac{8\sqrt{1+\xi^2}}{u+\frac{\eta}{2}}\frac{\cosh(\frac{\pi u}{2}-\frac{{\rm i}\pi}{4})}{\sinh(\frac{\pi u}{2}-\frac{{\rm i}\pi}{4})}\frac{\Gamma(1+\frac{{\rm i}u}{2})\Gamma(\frac{3}{2}-\frac{{\rm i}u}{2})}{\Gamma(\frac{1}{2}+\frac{{\rm i}u}{2})\Gamma(1-\frac{{\rm i}u}{2})}$$
$$\times\frac{\Gamma(\frac{p+1}{2}+\frac{{\rm i}u}{2})\Gamma(\frac{p+2}{2}-\frac{{\rm i}u}{2})}{\Gamma(\frac{p}{2}+\frac{{\rm i}u}{2})\Gamma(\frac{p+1}{2}-\frac{{\rm i}u}{2})}$$
$$\times\frac{\Gamma(\frac{\bar{q}+1}{2}+\frac{{\rm i}u}{2})\Gamma(\frac{\bar{q}+2}{2}-\frac{{\rm i}u}{2})}{\Gamma(\frac{\bar{q}}{2}+\frac{{\rm i}u}{2})\Gamma(\frac{\bar{q}+1}{2}-\frac{{\rm i}u}{2})}$$
$$\times\left(\frac{2\Gamma(1+\frac{{\rm i}u}{2})\Gamma(\frac{3}{2}-\frac{{\rm i}u}{2})}{\Gamma(\frac{1}{2}+\frac{{\rm i}u}{2})\Gamma(1-\frac{{\rm i}u}{2})}\right)^{2N}{\rm e}^{O(\frac{1}{N})}. \quad (97)$$

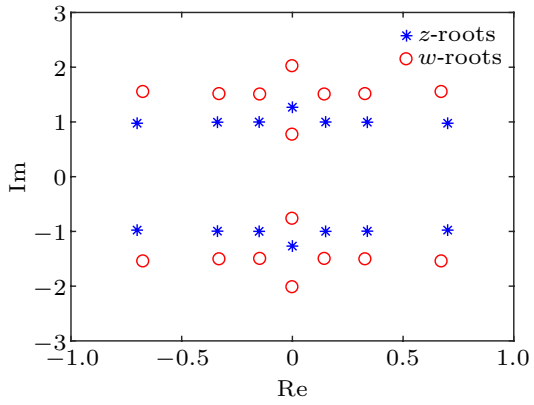


Fig. 2. The patterns of z-roots and w-roots in complex plane at the ground state with $N=6, \eta=\mathrm{i}, p=-1.2\mathrm{i}, \bar{q}=0.8\mathrm{i}, \xi=1$. The data are obtained by the exact numerical diagonalization. The blue asterisks indicate z-roots $\{z_j+\frac{\eta}{2}\}$ and red circles represent the w-roots $\{w_j\}$ with the inhomogeneous parameters $\{\theta_j=0\}$.

Now, we consider the leading terms of $\bar{W}_{\mathrm{g}}(u)$ in the thermodynamic limit. Analogous to the case of the periodic boundary condition, we set the variable $z_j=u_j^{(2)}+\eta$ in BAEs (79) and multiply it by its conjugate counterpart. As a result, we obtain

$$\begin{aligned}&\Delta(u_j^{(2)}+\frac{\eta}{2})\Delta(u_j^{(2)}-\frac{\eta}{2})\\ =\;&(u_j^{(2)}+\frac{\eta}{2})^{2N+2}(u_j^{(2)}-\frac{\eta}{2})^{2N+2}\\ &\times\bar{W}(u_j^{(2)}+\frac{\eta}{2})\bar{W}(u_j^{(2)}-\frac{\eta}{2}).\end{aligned}\tag{98}$$

By taking the derivative of the logarithm of Eq. (98), we obtain

$$\begin{aligned}&\frac{1}{2N}\frac{\partial}{\partial u}\ln[W(u+\frac{\eta}{2})W(u-\frac{\eta}{2})]\\ =\;&\frac{1}{2N}\frac{\partial}{\partial u}\Big(\ln\big[\Delta(u+\frac{\eta}{2})\Delta(u-\frac{\eta}{2})\big]\\ &-\ln\big[(u+\frac{\eta}{2})^{2N+2}(u-\frac{\eta}{2})^{2N+2}\big]\Big)+O(\frac{1}{N^2}).\end{aligned}\tag{99}$$

Substituting Eq. (87) into Eq. (99), we have

$$\begin{aligned}&\frac{\partial}{\partial u}\big(\bar{\omega}_{\mathrm{g}}^{(0)}(u+\frac{\eta}{2})+\bar{\omega}_{\mathrm{g}}^{(0)}(u-\frac{\eta}{2})\big)=\frac{1}{u+\frac{3\eta}{2}}+\frac{1}{u-\frac{3\eta}{2}},\\ &\frac{\partial}{\partial u}\big(\bar{\omega}_{\mathrm{g}}^{(1)}(u+\frac{\eta}{2})+\bar{\omega}_{\mathrm{g}}^{(1)}(u-\frac{\eta}{2})\big)\\ =\;&\frac{1}{u+\frac{3}{2}\eta}+\frac{1}{u-\frac{3}{2}\eta}+\frac{1}{u+(p+\frac{1}{2})\eta}+\frac{1}{u-(p+\frac{1}{2})\eta}\\ &+\frac{1}{u+(p-\frac{1}{2})\eta}+\frac{1}{u-(p-\frac{1}{2})\eta}+\frac{1}{u+(\bar{q}+\frac{1}{2})\eta}\\ &+\frac{1}{u-(\bar{q}+\frac{1}{2})\eta}+\frac{1}{u+(\bar{q}-\frac{1}{2})\eta}+\frac{1}{u-(\bar{q}-\frac{1}{2})\eta}\\ &-\Big(\frac{1}{u+\frac{1}{2}\eta}+\frac{1}{u-\frac{1}{2}\eta}+\frac{1}{u+(\chi_1+\frac{1}{2})\eta}+\frac{1}{u-(\chi_1+\frac{1}{2})\eta}\\ &+\frac{1}{u+(\chi_1-\frac{1}{2})\eta}+\frac{1}{u-(\chi_1-\frac{1}{2})\eta}+\frac{1}{u+(\chi_2+\frac{1}{2})\eta}\\ &+\frac{1}{u-(\chi_2+\frac{1}{2})\eta}+\frac{1}{u+(\chi_2-\frac{1}{2})\eta}+\frac{1}{u-(\chi_2-\frac{1}{2})\eta}\Big).\end{aligned}\tag{100}$$

Substituting Eq. (90) into Eq. (100) and solving it by the Fourier transformation, we obtain the densities of w-roots at the ground state

$$\tilde{\rho}_w^{(0)}(w)=\frac{a_2(w)}{2\pi\big[a_1(w)+a_3(w)\big]},\tag{101}$$

$$\begin{aligned}\tilde{\rho}_w^{(1)}(w)=\;&\frac{1}{2\pi\big[a_1(w)+a_3(w)\big]}\big[a_3(w)+a_{2p+1}(w)+a_{2p-1}(w)\\ &+a_{2\bar{q}+1}(w)+a_{2\bar{q}-1}(w)-a_1(w)-a_{2\chi_1-1}(w)\\ &-a_{2\chi_1+1}(w)-a_{2\chi_2-1}(w)-a_{2\chi_2+1}(w)\big].\end{aligned}\tag{102}$$

Substituting the densities (101) and (102) into Eq. (90), we arrive at

$$\begin{aligned}\frac{\partial}{\partial u}\bar{\omega}_{\mathrm{g}}^{(0)}(u)=\;&\frac{\partial}{\partial u}\ln\frac{\Gamma(\frac{3}{2}+\frac{\mathrm{i}u}{2})\Gamma(\frac{3}{2}-\frac{\mathrm{i}u}{2})}{\Gamma(1+\frac{\mathrm{i}u}{2})\Gamma(1-\frac{\mathrm{i}u}{2})}\\ =\;&\frac{\partial}{\partial u}\ln\Big[\frac{(u+\eta)(u-\eta)}{2u}\tanh\frac{\pi u}{2}\Big],\end{aligned}\tag{103}$$

$$\begin{aligned}\frac{\partial}{\partial u}\bar{\omega}_{\mathrm{g}}^{(1)}(u)=\;&\frac{\partial}{\partial u}\ln\Big(\frac{(u-p\eta)(u+p\eta)(u-\bar{q}\eta)(u+\bar{q}\eta)}{(u-\chi_1\eta)(u+\chi_1\eta)(u-\chi_2\eta)(u+\chi_2\eta)}\\ &\times\frac{\Gamma(\frac{3}{2}+\frac{\mathrm{i}u}{2})\Gamma(\frac{3}{2}-\frac{\mathrm{i}u}{2})}{\Gamma(1+\frac{\mathrm{i}u}{2})\Gamma(1-\frac{\mathrm{i}u}{2})}\\ &\times\frac{\Gamma(\frac{1}{2}+\frac{\mathrm{i}u}{2})\Gamma(\frac{1}{2}-\frac{\mathrm{i}u}{2})}{\Gamma(1+\frac{\mathrm{i}u}{2})\Gamma(1-\frac{\mathrm{i}u}{2})}\Big).\end{aligned}\tag{104}$$

Then the leading terms can be obtained by taking the integration of the above equation and the finial results are

$$\bar{\omega}_{\mathrm{g}}^{(0)}(u)=\ln\Big(\frac{\bar{C}_w^{(0)}}{2}\frac{(u+\eta)(u-\eta)}{u}\tanh\frac{\pi u}{2}\Big),\tag{105}$$

$$\begin{aligned}\bar{\omega}_{\mathrm{g}}^{(1)}(u)=\;&\ln\Big(\frac{(u-p\eta)(u+p\eta)(u-\bar{q}\eta)(u+\bar{q}\eta)}{(u-\chi_1\eta)(u+\chi_1\eta)(u-\chi_2\eta)(u+\chi_2\eta)}\\ &\times\frac{\Gamma(\frac{3}{2}+\frac{\mathrm{i}u}{2})\Gamma(\frac{3}{2}-\frac{\mathrm{i}u}{2})}{\Gamma(1+\frac{\mathrm{i}u}{2})\Gamma(1-\frac{\mathrm{i}u}{2})}\frac{\Gamma(\frac{1}{2}+\frac{\mathrm{i}u}{2})\Gamma(\frac{1}{2}-\frac{\mathrm{i}u}{2})}{\Gamma(1+\frac{\mathrm{i}u}{2})\Gamma(1-\frac{\mathrm{i}u}{2})}\Big)\\ &+\ln\bar{C}_w^{(1)}.\end{aligned}\tag{106}$$

Thus the the ground state eigenfunction $\bar{W}_{\mathrm{g}}(u)$ in the thermodynamic limit can be expressed as

$$\begin{aligned}\bar{W}_{\mathrm{g}}(u)=\;&4\bar{C}_w^{(1)}(\xi^2-3)(u-p\eta)(u+p\eta)(u-\bar{q}\eta)\\ &\times(u+\bar{q}\eta)\tanh^2\frac{\pi u}{2}\\ &\times\frac{(u+\eta)^{2N+1}(u-\eta)^{2N+1}}{u^{2N+2}}\\ &\times\Big(\frac{\bar{C}_w^{(0)}}{2}\tanh\frac{\pi u}{2}\Big)^{2N}.\end{aligned}\tag{107}$$

Substituting Eq. (107) into the t–W relation (73), we have

$$\begin{aligned}&(u+\frac{\eta}{2})(u-\frac{\eta}{2})\bar{\Lambda}_{\mathrm{g}}(u)\bar{\Lambda}_{\mathrm{g}}(u-\eta)\\ =\;&(1+\xi^2)(u-p\eta)(u+p\eta)(u-\bar{q}\eta)\\ &\times(u+\bar{q}\eta)(u-\eta)^{2N+1}(u+\eta)^{2N+1}\end{aligned}$$

$$\times\left\{1-\frac{4(\xi^2-3)}{1+\xi^2}\bar{C}_w^{(1)}\tanh^2\frac{\pi u}{2}\left(\frac{\bar{C}_w^{(0)}}{2}\tanh\frac{\pi u}{2}\right)^{2N}\mathrm{e}^{O(\frac{1}{N})}\right\}. \tag{108}$$

When u tends to infinity, the coefficient of the highest order term of the left hand side of Eq. (108) is 4. Thus the corresponding coefficient of the right hand side of Eq. (108) should also be 4, which gives $\bar{C}_w^{(0)}=2$ and $\bar{C}_w^{(1)}=1/4$. Because $\tanh\frac{\pi u}{2}<1$, the second term of the right hand side of Eq. (108) is negligible in the thermodynamic limit.

4. Conclusion

In this paper, we take the XXX spin chain as an example to study the t–W scheme for the quantum integrable systems. We present the exact solutions of the model with periodic and generic open boundary conditions. We also obtain the analytical expressions of the ground state eigenfunctions of the transfer matrix and $\mathbb{W}$ operator in the thermodynamic limit. By analyzing these expressions, we find that the ratio of the quantum determinant with the W function converges to zero when the number of size tends to infinity. Thus the main contribution in the t–W relation comes from the quantum determinant. This finding serves as a compelling proof of the validity of the extensively applied inversion relation in the field of integrability.

Appendix A: Proof of the t–W relation

Starting from the YBE (6) with certain shift of spectral parameter and using the fusion technique,[49,50] we obtain

$$\begin{aligned}R_{2,3}(u)R_{1,3}(u-\eta)P_{1,2}^{(-)} &= P_{1,2}^{(-)}R_{2,3}(u)R_{1,3}(u-\eta)P_{1,2}^{(-)}\\ &= (u+\eta)(u-\eta)\times\mathrm{id}.\end{aligned} \tag{A1}$$

We see that the fusion result of R-matrices with one-dimensional antisymmetric projector $P_{1,2}^{(-)}$ is a number. The fusion of monodromy matrices with $P_{1,2}^{(-)}$ gives

$$P_{1,2}^{(-)}T_2(u)\,T_1(u-\eta)P_{1,2}^{(-)}=a(u)\,d(u-\eta)\times\mathrm{id}. \tag{A2}$$

Then we obtain the first term in the very relation (10).

The fusion of R-matrices with the three-dimensional symmetric projector $P_{1,2}^{(+)}$ gives

$$P_{1,2}^{(+)}\,R_{2,3}(u)\,R_{1,3}(u-\eta)\,P_{1,2}^{(+)}=u\times R_{\{1,2\},3}^{(1,\frac{1}{2})}(u), \tag{A3}$$

where $R_{\{1,2\},3}^{(1,\frac{1}{2})}(u)$ is the 6×6 fused R-matrix with the form of

$$R_{\{1,2\},3}^{(1,\frac{1}{2})}(u)=\begin{pmatrix}u+\eta & & & & & \\ & u-\eta & \sqrt{2}\eta & & & \\ & \sqrt{2}\eta & u & & & \\ & & & u & \sqrt{2}\eta & \\ & & & \sqrt{2}\eta & u-\eta & \\ & & & & & u+\eta\end{pmatrix}. \tag{A4}$$

Then the fusion of monodromy matrices reads

$$P_{1,2}^{(+)}T_2(u)\,T_1(u-\eta)P_{1,2}^{(+)}=d(u)T_{\{1,2\}}^{(1,\frac{1}{2})}(u), \tag{A5}$$

where $T_{\{1,2\}}^{(1,\frac{1}{2})}(u)$ is the fused monodromy matrix which is constructed by the fused R-matrix $R_{\{1,2\},j}^{(1,\frac{1}{2})}(u)$ as

$$T_{\{1,2\}}^{(1,\frac{1}{2})}(u)=R_{\{1,2\},N}^{(1,\frac{1}{2})}(u-\theta_N)\cdots R_{\{1,2\},1}^{(1,\frac{1}{2})}(u-\theta_1). \tag{A6}$$

Then we arrive at that the $\mathbb{W}(u)$ operator in Eq. (10) is the fused transfer matrix as

$$\mathbb{W}(u)=\mathrm{tr}_{\{1,2\}}T_{\{1,2\}}^{(1,\frac{1}{2})}(u). \tag{A7}$$

From the constructions (A4) and (A6), we know that the $\mathbb{W}(u)$ operator is an operator polynomial of u with the degree N.

Appendix B: Hermitian property of the transfer matrix

The R-matrix (3) has the Hermitian relation

$$R_{0,j}^{\dagger}(u)=-R_{0,j}(-u^*). \tag{B1}$$

Based on it, we obtain the Hermitian conjugation of the transfer matrix in the homogeneous limit as

$$\begin{aligned}t^{\dagger}(u) &= \mathrm{tr}_0\left\{[R_{0,N}(u)\cdots R_{0,1}(u)]^{\dagger}\right\}\\ &= \mathrm{tr}_0\left\{R_{0,1}^{\dagger}(u)\cdots R_{0,N}^{\dagger}(u)\right\}\\ &\overset{(B1)}{=} (-1)^N\mathrm{tr}_0\left\{R_{0,1}(-u^*)\cdots R_{0,N}(-u^*)\right\}\\ &= (-1)^N\mathrm{tr}_0\left\{R_{0,N}^{t_0}(-u^*)\cdots R_{0,1}^{t_0}(-u^*)\right\}\\ &\overset{(4)}{=} \mathrm{tr}_0\left\{\sigma_0^y R_{0,N}(u^*-\eta)\cdots R_{0,1}(u^*-\eta)\,\sigma_0^y\right\}\\ &= \mathrm{tr}_0\left\{R_{0,N}(u^*-\eta)\cdots R_{0,1}(u^*-\eta)\right\}\\ &= t(u^*-\eta).\end{aligned} \tag{B2}$$

With the help of the t–W relation (10), we find that the $\mathbb{W}(u)$ operator satisfies

$$\mathbb{W}^{\dagger}(u)=\mathbb{W}(u^*). \tag{B3}$$

The relations (B2) and (B3) imply that the corresponding eigenvalues have the properties

$$\Lambda^*(u)=\Lambda(u^*-\eta),\quad W^*(u)=W(u^*). \tag{B4}$$

Combining the relations (B4) and (14)–(19), we conclude that if z_j is the solution of $\Lambda(u)$, its complex conjugation z_j^* must be the solution, and if w_j is a solution of $W(u)$, the w_j^* must be the solution.

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