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RAPID COMMUNICATION

Sharing quantum nonlocality in the noisy scenario

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It was showed in [*Phys. Rev. Lett.* **125** 090401 (2020)] that there exist unbounded number of independent Bobs who can share quantum nonlocality with a single Alice by performing sequentially measurements on the Bob's half of the maximally entangled pure two-qubit state. However, from practical perspectives, errors in entanglement generation and noises in quantum measurements will result in the decay of nonlocality in the scenario. In this paper, we analyze the persistency and termination of sharing nonlocality in the noisy scenario. We first obtain the two sufficient conditions under which there exist n independent Bobs who can share nonlocality with a single Alice under noisy measurements and the noisy initial two qubit entangled state. Analyzing the two conditions, we find that the influences on persistency under different kinds of noises can cancel each other out. Furthermore, we describe the change patterns of the maximal nonlocality-sharing number under the influence of different noises. Finally, we extend our investigation to the case of arbitrary finite-dimensional systems.

Keywords: Bell nonlocality, quantum measurement, quantum noise

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1. Introduction

Quantum nonlocality is a kind of quantum resource distinct from entanglement and Einstein–Podolsky–Rosen (EPR) steering, and has broad applications in quantum information processing.[1–4] Nonlocal correlations offer a quantum advantage in communication complexity problems,[5] device-independent quantum cryptography,[6,7] randomness expansion,[8] and measurement-based quantum computation.[9,10] Quantum nonlocality can be demonstrated by violation of Bell inequalities (or Bell-type inequalities).[11–13] Furthermore, researchers have also developed various experimental techniques on quantum nonlocality.[14–17]

In previous researches, scientists explored the limitations of nonlocality by asking whether a pair of entangled qubits could produce a long sequence of nonlocal correlations in Refs. [18–21]. Silva *et al*. first introduced this sequential scenario (see Fig. 1) and demonstrated that at most two sequential observers (Bobs) could share nonlocality with a single Alice through weak measurements.[18] Since then, additional theoretical and experimental findings on nonlocality sharing have been obtained.[22–31] Brown and Colbeck found a measurement strategy such that there exists an unbounded number of independent Bobs who are able to violate the Clauser–Horne–Shimony–Holt (CHSH) inequality with a single Alice by sequentially measuring one half of a maximally entangled pure two-qubit state $\rho_{AB^{(1)}} = |\phi^+\rangle\langle\phi^+|$ with $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.[27] Foletto *et al*. implemented experimentally the nonlocality-sharing strategy.[32] Zhang and Fei extended the aforementioned results to the case of arbitrary finite-dimensional systems.[33] In Refs. [34,35], problems on sharing network nonlocality were also discussed. However, the above discussions are always limited to the ideal scenario, where the initial states and measurements are noiseless.

Recently the topics of noisy network nonlocality were focused on in Refs. [36,37]. Mukherjee studied error tolerance of nontrilocal correlations in noisy triangle networks,[36] where different sources of imperfections were considered, such as errors in entanglement generation, communications over noisy quantum channels and noises in measurements.[38–40] Furthermore, the similar problem on persistency of the non-n-local correlations in noisy linear networks was analyzed in Ref. [37]. The aforementioned researchers discovered that the influence of noises can result in the decay of nonlocality in quantum networks. So can it in the nonlocality-sharing scenario. In this paper, we analyze persistency of sharing nonlocality in the noisy scenario, where the initial states and measurements have errors in entanglement generation and white noises, respectively.

The study presents the conditions for nonlocality sharing termination and infinite nonlocality sharing, respectively, and uses images to reflect the impact of noise on the maximum number of independent Bobs who can share the nonlocality of the noisy initial state with the single Alice. Furthermore, we investigate the noisy nonlocality sharing in high-dimensional bipartite systems.

The paper is arranged as follows. In Section 2, we review some basic results on sharing nonlocality. In Section 3, we

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first focus on the case that the initial state is the two-qubit one. Here, noises from entanglement generation and measurements are introduced. Furthermore, we obtain two sufficient conditions of the persistency of sharing nonlocality noisily, and analyze the change patterns of the maximal number of Bobs who can share nonlocality with Alice under the influence of different noises. In Section 4, we devote to generalizing the persistency to arbitrary finite-dimensional systems.

2. Preliminaries

In this section, we review the results on the nonlocality-sharing scenario (see Fig. 1).

2.1. Detection of nonlocality

The quantum nonlocality of a bipartite state ρ_{AB} can be proved by violating Clauser–Horne–Shimony–Holt (CHSH) Bell inequality[41]

$$S_{\mathrm{CHSH}} := \langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \le 2, \quad (1)$$

where $\langle A_xB_y\rangle = \sum_{a,b}(-1)^{a+b}p(a,b|x,y)$ is the joint expectation value of the observations of Alice and Bob when their settings are x and y ($x,y\in\{0,1\}$), and binary outcomes are a and b ($a,b\in\{0,1\}$), respectively. A_i and B_i are observables with $r(A_i)\le 1$ and $r(B_i)\le 1$, $i=0,1$. Denote by $r(A)$ the spectral radius of the matrix A. The distributions are given by the Born rule

$$p(a,b|x,y) = \mathrm{tr}[(A_{a|x}\otimes B_{b|y})\rho_{AB}], \quad (2)$$

where $\sum_a(-1)^a A_{a|x} = A_x$ and $\sum_b(-1)^b B_{b|y} = B_y$ for Alice's measurement set $\{A_{a|x}\}_{a,x}$ and Bob's one $\{B_{b|y}\}_{b,y}$.

2.2. The sequential nonlocality-sharing scenario

Let H_A and H_B be finite-dimensional complex Hilbert spaces and $\rho_{AB^{(1)}}$ a bipartite state acting on $H_A\otimes H_B$ with $\dim(H_A)=s$ and $\dim(H_B)=t$. Denote by I_m the $m\times m$ identity matrix. In Ref. [18], the sequential nonlocality-sharing scenario was proposed (see Fig. 1). Here Alice and Bob$^{(1)}$ share an initial entangled bipartite state $\rho_{AB^{(1)}}$. Suppose Bob$^{(1)}$ performs the measurement according to binary input $Y^{(1)}=y$ with binary outcome $B^{(1)}=b$, the postmeasurement state shared between Alice and Bob$^{(2)}$ can be described by the Lüders rule,[27] i.e.,

$$\rho_{AB^{(2)}} = \frac{1}{2}\sum_{b,y}\left(I_s\otimes\sqrt{B^{(1)}_{b|y}}\right)\rho_{AB^{(1)}}\left(I_s\otimes\sqrt{B^{(1)}_{b|y}}\right), \quad (3)$$

where $B^{(1)}_{b|y}$ is the positive operator-valued measurement (POVM) corresponding to outcome b of Bob$^{(1)}$'s measurement for input y. Repeating this process, the state $\rho_{AB^{(k)}}$ shared by Alice and Bob$^{(k)}$ can be computed as

$$\rho_{AB^{(k)}} = \frac{1}{2}\sum_{b,y}\left(I_s\otimes\sqrt{B^{(k-1)}_{b|y}}\right)\rho_{AB^{(k-1)}}\left(I_s\otimes\sqrt{B^{(k-1)}_{b|y}}\right), \quad (4)$$

where $B^{(k-1)}_{b|y}$ is the POVM corresponding to outcomes b of Bob$^{(k-1)}$'s measurements for input y.

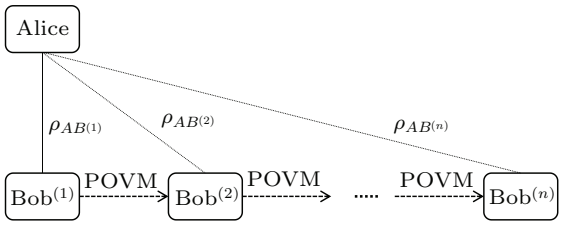


Fig. 1. A bipartite entangled state $\rho_{AB^{(1)}}$ is initially shared between Alice and Bob$^{(1)}$. Bob$^{(1)}$ performs a POVM on his part. Denote by $\rho_{AB^{(2)}}$ the post-measurement state. Bob$^{(2)}$ also performs a POVM, and this process continues until Bob$^{(n)}$. One want to ask how many Bobs can share the nonlocality with Alice at most.

It is dependent on the choice of POVMs in the scenario of Fig. 1 whether or not $\rho_{AB^{(i)}}$ is nonlocal, $i=1,2\ldots,n$. Denote by $\{A_{0|x},A_{1|x}\}$ the POVM of Alice, $x=0,1$, where $A_{0|x}+A_{1|x}=I_s$ and $A_{0|x}\ge 0$, $A_{1|x}\ge 0$. The corresponding observable is given by $A_x = A_{0|x}-A_{1|x}$. Similarly, the POVM $\{B^{(k)}_{0|y},B^{(k)}_{1|y}\}$ ($y=0,1$) is performed by Bob$^{(k)}$, $k=1,2,\ldots,n$. When $s=t=2$, the POVM strategies were introduced in Ref. [27] such that arbitrarily many independent Bobs can share the nonlocality of initial maximally entangled pure two-qubit state $\rho_{AB^{(1)}}=|\phi^-\rangle\langle\phi^-|$ ($|\phi^-\rangle=\frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)$) with the single Alice. These POVM strategies are of the following forms:

$$A_{0|0} := \frac{1}{2}(I_2+\cos(\theta)\sigma_3+\sin(\theta)\sigma_1), \quad (5)$$

$$A_{0|1} := \frac{1}{2}(I_2+\cos(\theta)\sigma_3-\sin(\theta)\sigma_1), \quad (6)$$

$$B^{(k)}_{0|0} := \frac{1}{2}(I_2+\sigma_3), \quad (7)$$

$$B^{(k)}_{0|1} := \frac{1}{2}(I_2+\eta_k\sigma_1), \quad (8)$$

where σ_j ($j=1,2,3$) are Pauli matrices, $\theta\in(0,\pi/4]$ and $\eta_k\in(0,1)$ for any $k=1,2,\ldots,n$.

In Ref. [33], Zhang and Fei extended the results in Ref. [27] to the arbitrary finite-dimensional case. There the initial state $\rho_{AB^{(1)}}=|\psi\rangle\langle\psi|$ is an arbitrary-dimensional bipartite entangled pure state, where $|\psi\rangle\in H_A\otimes H_B$ with Schmidt decomposition $|\psi\rangle=\sum_{i=1}^s c_i|i_A\rangle|i_B\rangle$ and $\dim(H_A)=s$, $\dim(H_B)=t$ ($s\le t$), with $c_i\in[0,1]$, $\sum_i^s c_i^2=1$, and $\{|i_A\rangle\}_1^s$ and $\{|i_B\rangle\}_1^t$ being the orthonormal bases of H_A and H_B, respectively. When both s and t are even, the measurement strategies (POVMs) are chosen as

$$A_{0|0} = \frac{1}{2}[I_s+(I_{\frac{s}{2}}\otimes(\cos(\theta)\sigma_3+\sin(\theta)\sigma_1))], \quad (9)$$

$$A_{0|1} = \frac{1}{2}[I_s+(I_{\frac{s}{2}}\otimes(\cos(\theta)\sigma_3-\sin(\theta)\sigma_1))], \quad (10)$$

$$B^{(k)}_{0|0} = \frac{1}{2}[I_t+(I_{\frac{t}{2}}\otimes\sigma_3)], \quad (11)$$

$$B_{0|1}^{(k)}=\frac{1}{2}[I_t+(I_{\frac{t}{2}}\otimes\eta_k\sigma_1)] \tag{12}$$

for $\theta\in(0,\pi/4]$ and $\eta_k\in(0,1)$, $k=1,2,\ldots,n$. When both s and t are odd, a set of POVMs can be given by

$$A_{0|0}=\frac{1}{2}\left[I_s+\begin{pmatrix} I_{[\frac{s}{2}]}\otimes(\cos(\theta)\sigma_3+\sin(\theta)\sigma_1) & 0\\ 0 & 1\end{pmatrix}\right], \tag{13}$$

$$A_{0|1}=\frac{1}{2}\left[I_s+\begin{pmatrix} I_{[\frac{s}{2}]}\otimes(\cos(\theta)\sigma_3-\sin(\theta)\sigma_1) & 0\\ 0 & 1\end{pmatrix}\right], \tag{14}$$

$$B_{0|0}^{(k)}=\frac{1}{2}\left[I_t+\begin{pmatrix} I_{[\frac{t}{2}]}\otimes\sigma_3 & 0\\ 0 & 1\end{pmatrix}\right], \tag{15}$$

$$B_{0|1}^{(k)}=\frac{1}{2}\left[I_t+\eta_k\begin{pmatrix} I_{[\frac{t}{2}]}\otimes\sigma_1 & 0\\ 0 & 1\end{pmatrix}\right] \tag{16}$$

for $\theta\in(0,\pi/4]$ and $\eta_k\in(0,1)$, $k=1,2,\ldots,n$.

3. Sharing nonlocality in noisy two-qubit systems

In this section, we study persistency and termination on sharing nonlocality noisily with initial two-qubit states.

Before that, we first introduce the noise generation.

3.1. Noise generation

The first kind of noises comes from entanglement generation of the initial state. In the experiment, a two-qubit entangled state $\rho_{AB^{(1)}}=|\phi^-\rangle\langle\phi^-|$ with $|\phi^-\rangle=\frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)$ is generated by the action of the Hadamard gate $\mathcal{H}$ and CNOT gate on $\zeta=|10\rangle\langle 10|$.

Let α and δ denote the noisy parameters characterizing the $\mathcal{H}$ gate and CNOT gate, respectively. Starting from $\zeta=|10\rangle\langle 10|$, and the noisy Hadamard gate generates[38]

$$\begin{aligned}\zeta_1' &= \alpha(\mathcal{H}\otimes I_2\zeta\mathcal{H}^\dagger\otimes I_2)+\frac{1-\alpha}{2}I_2\otimes\mathrm{tr}_1(\zeta)\\ &= \frac{1}{2}(|00\rangle\langle 00|+|10\rangle\langle 10|)\\ &\quad-\frac{\alpha}{2}(|00\rangle\langle 10|+|10\rangle\langle 00|).\end{aligned} \tag{17}$$

Subjection of ζ_1' to noisy CNOT gives

$$\begin{aligned}\bar{\rho}_{AB^{(1)}} &= \delta(\mathrm{CNOT}\zeta_1'(\mathrm{CNOT})^\dagger)+\frac{1-\delta}{4}I_2\otimes I_2\\ &= \frac{1}{4}\Bigg[\sum_{i,j=0}^{1}(1+(-1)^{i+j}\delta)|ij\rangle\langle ij|\\ &\quad-2\alpha\delta(|11\rangle\langle 00|+|00\rangle\langle 11|)\Bigg].\end{aligned} \tag{18}$$

Denote by $T(\rho)=(w_{i,j})_{3\times 3}$ the correlation matrix whose entries are given by $w_{i,j}=\mathrm{tr}[\rho(\sigma_i\otimes\sigma_j)]$. Then $T(\bar{\rho}_{AB^{(1)}})=\mathrm{diag}(-\alpha\delta,\alpha\delta,\delta)$.

Secondly, the white noises are fixed on POVMs $\{A_{0|i},A_{1|i}\}_{i=0,1}$ and $\{B_{0|i}^{(k)},B_{1|i}^{(k)}\}_{i=0,1}$ of parties Alice and Bob$^{(k)}$, $k=1,2,\ldots,n$. Let $\beta_1\in[0,1]$ characterize the noisy measurement set $\{A_{0|0},A_{1|0}\}$ in the sense that it fails to detect with probability $1-\beta_1$, i.e., β_1 parametrizes a faulty measurement device. Similarly, the noisy parameters for measurements $\{A_{0|1},A_{1|1}\}$, $\{B_{0|0}^{(k)},B_{1|0}^{(k)}\}$ and $\{B_{0|1}^{(k)},B_{1|1}^{(k)}\}$ are denoted by β_2, γ_k and μ_k, $k=1,2,\ldots,n$, respectively, and $\beta_2\in[0,1]$, $\gamma_k\in[0,1]$, $\mu_k\in[0,1]$. If $\dim(H_A)=s$ and $\dim(H_B)=t$, then noisy POVMs can be represented as

$$\bar{A}_{0|0}=\beta_1A_{0|0}+\frac{1-\beta_1}{2}I_s, \tag{19}$$

$$\bar{A}_{0|1}=\beta_2A_{0|1}+\frac{1-\beta_2}{2}I_s, \tag{20}$$

$$\bar{B}_{0|0}^{(k)}=\gamma_kB_{0|0}^{(k)}+\frac{1-\gamma_k}{2}I_t, \tag{21}$$

$$\bar{B}_{0|1}^{(k)}=\mu_kB_{0|1}^{(k)}+\frac{1-\mu_k}{2}I_t. \tag{22}$$

We take $s=t=2$ when we consider the case of the initial state being a two-qubit state. Then the corresponding noisy observables are

$$\bar{A}_0=\bar{A}_{0|0}-\bar{A}_{1|0}=\beta_1A_0, \tag{23}$$

$$\bar{A}_1=\bar{A}_{0|1}-\bar{A}_{1|1}=\beta_2A_1, \tag{24}$$

$$\bar{B}_0^{(k)}=\bar{B}_{0|0}^{(k)}-\bar{B}_{1|0}^{(k)}=\gamma_kB_0^{(k)}, \tag{25}$$

$$\bar{B}_1^{(k)}=\bar{B}_{0|1}^{(k)}-\bar{B}_{1|1}^{(k)}=\mu_kB_1^{(k)}. \tag{26}$$

Combing Eqs. (5)–(8) with the form of noisy POVMs (see Eqs. (19)–(22)), when the initial sharing state is a two-qubit state we write the noisy measurement strategy as follows:

$$\bar{A}_{0|0}:=\frac{1}{2}[I_2+\beta_1(\cos(\theta)\sigma_3+\sin(\theta)\sigma_1)], \tag{27}$$

$$\bar{A}_{1|0}:=\frac{1}{2}[I_2-\beta_1(\cos(\theta)\sigma_3+\sin(\theta)\sigma_1)], \tag{28}$$

$$\bar{A}_{0|1}:=\frac{1}{2}[I_2+\beta_2(\cos(\theta)\sigma_3-\sin(\theta)\sigma_1)], \tag{29}$$

$$\bar{A}_{1|1}:=\frac{1}{2}[I_2-\beta_2(\cos(\theta)\sigma_3-\sin(\theta)\sigma_1)], \tag{30}$$

$$\bar{B}_{0|0}^{(k)}:=\frac{1}{2}(I_2+\gamma_k\sigma_3),\quad \bar{B}_{1|0}^{(k)}:=\frac{1}{2}(I_2-\gamma_k\sigma_3), \tag{31}$$

$$\bar{B}_{0|1}^{(k)}:=\frac{1}{2}(I_2+\mu_k\eta_k\sigma_1),\quad \bar{B}_{1|1}^{(k)}:=\frac{1}{2}(I_2-\mu_k\eta_k\sigma_1). \tag{32}$$

3.2. Persistency of the noisy nonlocality-sharing scenario

In order to analyze the persistency, we first calculate the expected CHSH value $S_{\mathrm{CHSH}}^{(k)\mathrm{noisy}}$ with the noisy initial state and noisy POVMs. The detail of the proofs for our theorems will be presented in the appendix.

Theorem 1 For the noisy initial state $\bar{\rho}_{AB^{(1)}}$ in Eq. (18), if Alice and Bobs perform noisy measurements in Eqs. (27)–(30) and Eqs. (31)–(32), respectively, then the expected CHSH value of $\rho_{AB^{(k)}}$ is given by

$$\begin{aligned}S_{\mathrm{CHSH}}^{(k)\mathrm{noisy}} &= 2^{1-k}(\beta_1+\beta_2)\delta\\ &\quad\times\left[\gamma_k\cos(\theta)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2}\right)\right.\end{aligned}$$

$$+\alpha\mu_k\eta_k\sin(\theta)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right)\Big], \quad (33)$$

where $\theta\in(0,\pi/4]$, $\delta,\alpha,\beta_i,\eta_k,\gamma_k$ and $\mu_k\in[0,1]$ for $i=1,2$, $k=1,2,\ldots,n$.

In the following, we provide two sufficient conditions of persistency of sharing nonlocality noisily.

Theorem 2 For the noisy initial state $\bar{\rho}_{AB^{(1)}}$ in Eq. (18), assume Alice and Bobs perform noisy measurements in Eqs. (27)–(30) and Eqs. (31)–(32), respectively, then there exist n Bobs sharing the nonlocality of $\bar{\rho}_{AB^{(1)}}$ with the single Alice if there exists $\varepsilon>0$ such that

$$\max_{k\in\{1,2,\ldots,n\}}\arctan[x_{k2}(\varepsilon)]\le\min_{k\in\{1,2,\ldots,n\}}\arctan[x_{k1}(\varepsilon)], \quad (34)$$

where

$$x_{k1}(\varepsilon)=\frac{\frac{a_kb_k}{1+\varepsilon}+2^k\sqrt{a_k^2+(\frac{b_k}{1+\varepsilon})^2-2^{2k}}}{2^{2k}-(\frac{b_k}{1+\varepsilon})^2},$$
$$x_{k2}(\varepsilon)=\frac{\frac{a_kb_k}{1+\varepsilon}-2^k\sqrt{a_k^2+(\frac{b_k}{1+\varepsilon})^2-2^{2k}}}{2^{2k}-(\frac{b_k}{1+\varepsilon})^2},$$

with

$$a_k=(\beta_1+\beta_2)\delta\gamma_k\prod_{j=1}^{k-1}(1+\sqrt{1-\mu_j^2}),$$
$$b_k=(\beta_1+\beta_2)\delta\alpha\mu_k\prod_{j=1}^{k-1}(1+\sqrt{1-\gamma_j^2}),$$
$$k=1,2,\ldots,n.$$

Theorem 3 For the noisy initial state $\bar{\rho}_{AB^{(1)}}$ in Eq. (18), assume Alice and Bobs perform noisy measurements in Eqs. (27)–(30) and Eqs. (31)–(32), respectively, if $\gamma_n(\beta_1+\beta_2)\delta=2$, then there exist n Bobs who can share the nonlocality of the noisy initial state $\bar{\rho}_{AB^{(1)}}$ with the single Alice for arbitrary given noisy parameters $\alpha,\gamma_k,\mu_j\in(0,1]$, $k=1,2,\ldots,n-1$ and $j=1,2,\ldots,n$.

The set $\{\alpha,\delta,\beta_1,\beta_2,\mu_k,\gamma_k\}_{k=1}^n$ consists of all noisy parameters. The condition in Theorem 2 reveals how the persistency of sharing nonlocality in the noisy scenario depends on all noisy parameters. If, for some n such that Eq. (34) holds for some $\varepsilon>0$, but there is no $\varepsilon>0$ so that Eq. (34) holds for $n+1$, then Alice can at most share the nonlocality with n Bobs, and the sharing process stops at nth step. The persistency condition in Theorem 3 is more pragmatic and some what surprising. Note that $\gamma_n(\beta_1+\beta_2)\delta=2$ means $\gamma_n=\beta_1=\beta_2=\delta=1$, i.e., their corresponding noises vanish. Theorem 3 reveals that, if the noises parameters $\beta_1=\beta_2=\delta=1$ and the noise parameter $\gamma_n=1$ for $n\ge1$, then Alice can share nonlocality at least with n Bobs whatever the noise parameters $\alpha,\{\gamma_k\}_{k=1}^{n-1},\{\mu_j\}_{j=1}^n$ take any values in $(0,1]$. Why the phenomenon in Theorem 3 happens? One can find from the proof of Theorem 3 that when θ is small enough, each index of persistency $\eta_k=\eta_k(\theta)$ always can lie in (0,1) for arbitrary nonzero noisy parameters $\alpha,\{\gamma_k\}_{k=1}^{n-1},\{\mu_j\}_{j=1}^n$. Recall the parameter θ comes from the Alice's measurement. This means Alice's appropriate measurement device can counteract the influence on persistency under noises from the Hadamard gate and Bobs' measurements.

In the aforementioned theorems, it is always assumed that the initial state is a kind of special noisy maximal entangled states. Finally, we consider the nonlocality-sharing problem on the general case that the initial state is an arbitrary two-qubit state. For an arbitrary two-qubit state $\rho_{AB^{(1)}}$, assume that Alice and Bobs perform noisy measurements in Eqs. (27)–(30) and Eqs. (31)–(32), respectively, it follows that the CHSH value is

$$\begin{aligned}S_{\mathrm{CHSH}}^{(k)\mathrm{noisy}} &= \gamma_k2^{1-k}\prod_{j=1}^{k-1}(1+\sqrt{1-\mu_j^2\eta_j^2})\\ &\quad\times[(\beta_1+\beta_2)\cos(\theta)(\boldsymbol{c}_0,T(\rho_{AB^{(1)}})\boldsymbol{b}_0)\\ &\quad+(\beta_1-\beta_2)\sin(\theta)(\boldsymbol{c}_1,T(\rho_{AB^{(1)}})\boldsymbol{b}_0)]\\ &\quad+\mu_k\eta_k2^{1-k}\prod_{j=1}^{k-1}(1+\sqrt{1-\gamma_j^2})\\ &\quad\times[(\beta_1+\beta_2)\cos(\theta)(\boldsymbol{c}_0,T(\rho_{AB^{(1)}})\boldsymbol{b}_1)\\ &\quad+(\beta_1+\beta_2)\sin(\theta)(\boldsymbol{c}_1,T(\rho_{AB^{(1)}})\boldsymbol{b}_1)], \quad (35)\end{aligned}$$

where $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$ are the two orthogonal eigenvectors corresponding to two largest eigenvalues of $T(\rho_{AB^{(1)}})T^{\mathrm{T}}(\rho_{AB^{(1)}})$, respectively, $\boldsymbol{b}_i=\frac{T^{\mathrm{T}}(\rho_{AB^{(1)}})\boldsymbol{c}_i}{\|T^{\mathrm{T}}(\rho_{AB^{(1)}})\boldsymbol{c}_i\|}$, $i=0,1$, and $\sigma_{\boldsymbol{r}}=r_1\sigma_1+r_2\sigma_2+r_3\sigma_3$ for $\boldsymbol{r}=(r_1,r_2,r_3)$, where $\{\sigma_1,\sigma_2,\sigma_3\}$ are Pauli operators. It is worth noting that for two vectors $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$, $\mathrm{tr}[\rho(\sigma_{\boldsymbol{a}}\otimes\sigma_{\boldsymbol{b}})]=(\boldsymbol{a},T(\rho)\boldsymbol{b})$, where $(\cdot,\cdot)$ is the Euclidean inner product for the real vector space $\mathbb{R}^3$. The proof of Eq. (35) can be found in Appendix A. Equation (35) can help us deal with more cases of different initial states. For instance, assume that $\rho_{AB^{(1)}}$ is a Werner state, which can be written as $\rho_{AB^{(1)}}=\frac{(1-p)}{4}I_4+p|\phi^-\rangle\langle\phi^-|$ with $p\in(\frac{1}{3},1]$. By the calculation of Eq. (A12), we get

$$\begin{aligned}S_{\mathrm{CHSH}}^{(k)\mathrm{noisy}} &= 2^{-1-k}(\beta_1+\beta_2)p\\ &\quad\times\Big[\gamma_k\cos(\theta)\prod_{j=1}^{k-1}(1+\sqrt{1-\mu_j^2\eta_j^2})\\ &\quad+\mu_k\eta_k\sin(\theta)\prod_{j=1}^{k-1}(1+\sqrt{1-\gamma_j^2})\Big], \quad (36)\end{aligned}$$

where $\theta\in(0,\pi/4]$, $\delta,\alpha,\beta_i,\eta_k,\gamma_k$ and $\mu_k\in[0,1]$ for $i=1,2$, $k=1,2,\ldots,n$. Similar to the proof of Theorem 3, we can claim that if $\gamma_n(\beta_1+\beta_2)p=2$, then there exist n Bobs who can share the nonlocality of the initial state $\rho_{AB^{(1)}}$ with the single Alice for arbitrary given noisy parameters $\gamma_k,\mu_j\in(0,1]$, $k=1,2,\ldots,n-1$ and $j=1,2,\ldots,n$.

3.3. Termination of sharing nonlocality in noisy scenarios

As we see from Theorems 2 and 3, different kinds of noises result in different influences on the decay of nonlocality. In this section, we discuss the question how the noises influence the persistency of sharing nonlocality. We divide the discussion to two cases: state noises and measurement noises. It is assumed that the noisy parameters are always nonzero, the given noisy initial state $\bar{\rho}_{AB^{(1)}}$ is presented in Eq. (18) and the noisy measurements are presented in Eqs. (27)–(32).

3.3.1. Errors on entanglement generation

Let $\delta=\beta_1=\beta_2=\mu_k=\gamma_k=1$ for all $k=1,2,\ldots,n$, we investigate the influence of the noisy parameter α on the persistency. By Theorem 1, we have in this case that

$$S_{\mathrm{CHSH}}^{(k)\mathrm{noisy}}=2^{2-k}[\cos(\theta)\prod_{j=1}^{k-1}(1+\sqrt{1-\eta_j^2})+\alpha\eta_k\sin(\theta)].$$

For any $\varepsilon>0$, define a sequence $(\eta_i(\theta,\alpha,\varepsilon))_i$ recursively as $\eta_1(\theta,\alpha,\varepsilon)=\frac{1-\cos(\theta)}{\alpha\sin(\theta)}$ and

$$\eta_k(\theta,\alpha,\varepsilon)=\frac{(1+\varepsilon)\left[2^{k-1}-\cos(\theta)\prod_{j=1}^{k-1}(1+\sqrt{1-\eta_j^2(\theta,\alpha,\varepsilon)})\right]}{\alpha\sin(\theta)},\quad(37)$$

whenever $0<\eta_{k-1}(\theta,\alpha,\varepsilon)<1$, where $\theta\in(0,\pi/4]$ and $\alpha\in[0,1]$. One can check that $0<\eta_m(\theta,\alpha,\varepsilon)<1$ for all $m\leq k$ if and only if there exist k Bobs who can share nonlocality with Alice, and the nonlocality-sharing behavior terminates once if there is l such that $\eta_l(\theta,\alpha,\varepsilon)$ equals to or lager than 1. As we know from Ref. [27], if $\alpha=1$ (this means all the corresponding noise vanishes), then for any given positive integer k, there exists θ closed to 0 enough such that one can chose $\eta_j(\theta,\alpha)\in(0,1)$ for all $j=1,2,\ldots,k$. However, by Theorem 3, the similar can be done for any noise parameter $\alpha\in(0,1]$.

We plot the functional values of $\eta_k(\theta,\alpha)$ as a function of α in the image (see Fig. 2(a)) for $k=1,2,3,4$ when we take $\varepsilon=10^{-5}$. In Fig. 2(b), the corresponding $\eta_k(\theta,\alpha)$ falls into the interval $(0,1)$ when $(\theta,\ \alpha)$ lies in the lower region of each curve for $k=1,2,3,4$. When α moves from 1 to 0 and θ moves from 0 to 1 respectively, the maximal number $n^B_{\max}$ of Bobs who can share the nonlocality with Alice will decrease and can be calculated by determining how many $\eta_k(\theta,\alpha)$s lie in the interval $(0,1)$. From Fig. 2(b), one can see clearly the value of $n^B_{\max}$ when (θ,α) lies in the given region $[0.05,\pi/4]\times(0,0.95]$. For instance, there are at most three Bobs sharing the nonlocality with Alice when $(\theta,\alpha)=(0.1,0.9)$.

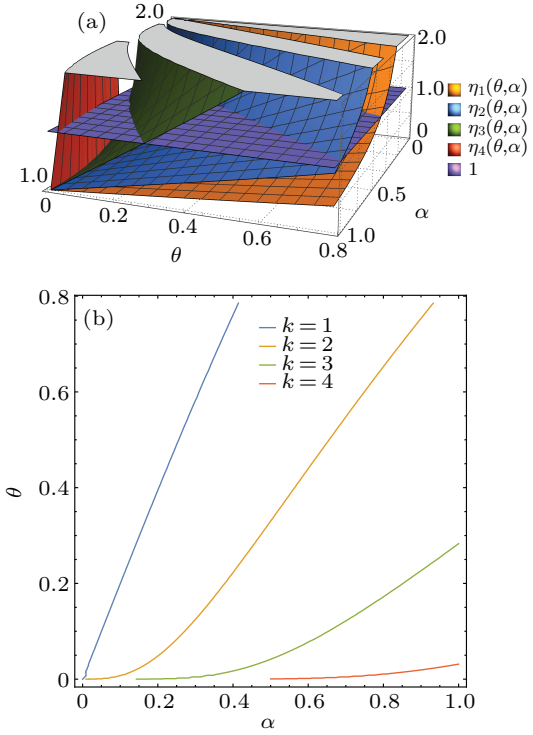


Fig. 2. (a) The yellow, blue, green and red surfaces represent the values of $\eta_1(\theta,\alpha)$, $\eta_2(\theta,\alpha)$, $\eta_3(\theta,\alpha)$ and $\eta_4(\theta,\alpha)$, respectively, as functions of parameters θ and α. The purple plane denotes the unit plane. Due to the limit of accuracy of the computer calculation, the images with $k\geq 5$ cannot be plotted. (b) The blue, orange, green and red curves represent $\eta_1(\theta,\alpha)=1$, $\eta_2(\theta,\alpha)=1$, $\eta_3(\theta,\alpha)=1$ and $\eta_4(\theta,\alpha)=1$, respectively.

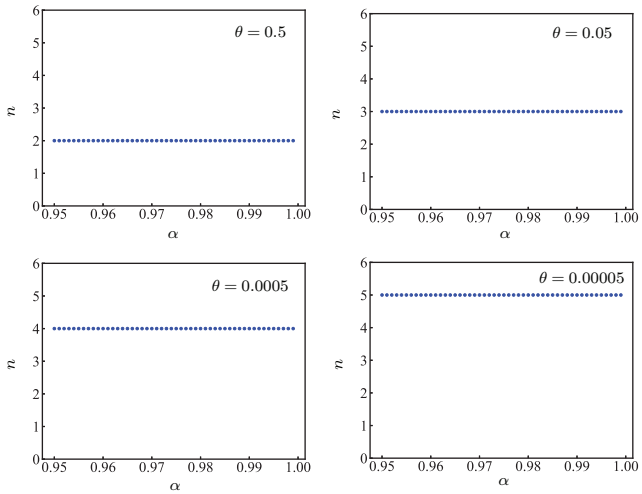


Fig. 3. The value of the vertical axis $n=n^B_{\max}$ is the maximal number of Bobs who can share nonlocality with Alice.

To see what happens when (θ,α) lies in the region $(0,0.05)\times(0.95,1)$, we exhibit the change pattern of $n^B_{\max}$ dependent on the noise parameter $\alpha\in(0.95,1)$ when θ is fixed as 0.5, 0.05, 0.0005 and 0.00005 respectively (see Fig. 3). It is surprising that when θ is given, the value of $n^B_{\max}$ keeps invariant if α moves from 0.95 to 1. Also, in Fig. 3, as θ decreases, the $n^B_{\max}$ increases, i.e., the persistency becomes better.

Moreover, when we observe the influence of the noise parameters δ and $\beta_1+\beta_2$ in Eq. (33), it is not difficult to conclude that the change pattern of $n^B_{\max}$ on δ and that on $\beta_1+\beta_2$

are the same. Therefore, we will merge the discussions of the above two cases to Case 2 in Subsection 3.3.2 hereinafter.

3.3.2. Noises come from measurements

Case 1. For any $n \geq 1$, cosider the case that $\gamma_1 = \gamma_2 = \cdots = \gamma_k = \gamma$ and $\alpha = \delta = \beta_1 = \beta_2 = \mu_k = 1$ for all $k = 1,2,\ldots,n$. By Theorem 1 we have

$$S_{\mathrm{CHSH}}^{(k)\mathrm{noisy}} = 2^{2-k}\Big[\gamma\cos(\theta)\prod_{j=1}^{k-1}(1+\sqrt{1-\eta_j^2}) + \eta_k\sin(\theta)(1+\sqrt{1-\gamma^2})^{k-1}\Big]. \quad (38)$$

Take a small $\varepsilon > 0$ and define a sequence $\{\eta_i(\theta,\gamma,\varepsilon)\}_{i=1}^n$ by

$$\eta_k(\theta,\gamma,\varepsilon) = (1+\varepsilon)\frac{2^{k-1} - \gamma\cos(\theta)\prod_{j=1}^{k-1}(1+\sqrt{1-\eta_j^2(\theta,\gamma,\varepsilon)})}{\sin(\theta)(1+\sqrt{1-\gamma^2})^{k-1}} \quad (39)$$

whenever $0 < \eta_{k-1}(\theta,\gamma,\varepsilon) < 1$ with $\eta_1(\theta,\gamma,\varepsilon) = (1+\varepsilon)\frac{1-\gamma\cos(\theta)}{\sin(\theta)}$ for $\theta \in (0,\pi/4]$ and $\gamma \in [0,1]$. If $\eta_k(\theta,\gamma,\varepsilon) < 1$, $k = 1,2,\ldots,n$, for some given $\varepsilon > 0$ and there exists no $\varepsilon > 0$ so that we still have $\eta_{n+1}(\theta,\gamma,\varepsilon) < 1$, then there are at most n Bobs sharing the nonlocality with Alice. Similarly, we plot the function values of $\eta_k(\theta,\gamma)$ again when $\varepsilon = 10^{-5}$ in Fig. 4(a), and show the ranges of $0 < \eta_k(\theta,\gamma) < 1$ in Fig. 4(b) for $k = 1,2,3$, i.e., when (θ,γ) lies in the right-hand sides of each curves, the corresponding $0 < \eta_k(\theta,\gamma) < 1$.

Similarly, the maximal number $n_{\max}^B$ of Bobs who can share the nonlocality with Alice can be calculated by determining how many $\eta_k(\theta,\gamma)$s lie in the interval $(0,1)$. Also from Fig. 4(b), the value of $n_{\max}^B$ can be straightforward investigated when (θ,γ) lies in the given region $[0.4,\pi/4]\times(0,0.9]$. For instance, $n_{\max}^B = 2$ when $(\theta,\gamma) = (0.4,0.9)$.

How $n_{\max}^B$ will change in the region $(0,0.4)\times(0.9,1)$? We describe the change pattern of $n_{\max}^B$ dependent on the noise parameter $\gamma \in (0.9,1)$ when θ is fixed at 0.4, 0.32, 0.27 and 0.1 respectively (see Fig. 5). We see that when θ is given, $n_{\max}^B$ increases if γ moves from 0.9 to 1. On the other hand, in the four plots of Fig. 5, the best persistency (here the better persistency is defined as the greater expectation of $n_{\max}^B$ in each figure) occurs when $\theta = 0.27$. Moreover, in Fig. 5(b), the unique blue dot such that $n_{\max}^B = 3$ occurs since the corresponding (θ,γ) lies in the cusp range of the green curve in Fig. 4(b).

Finally it is similar to analyze the change pattern of $n_{\max}^B$ depending on noisy parameters μ_ks. We do not give the elaboration on the case to avoid repetition.

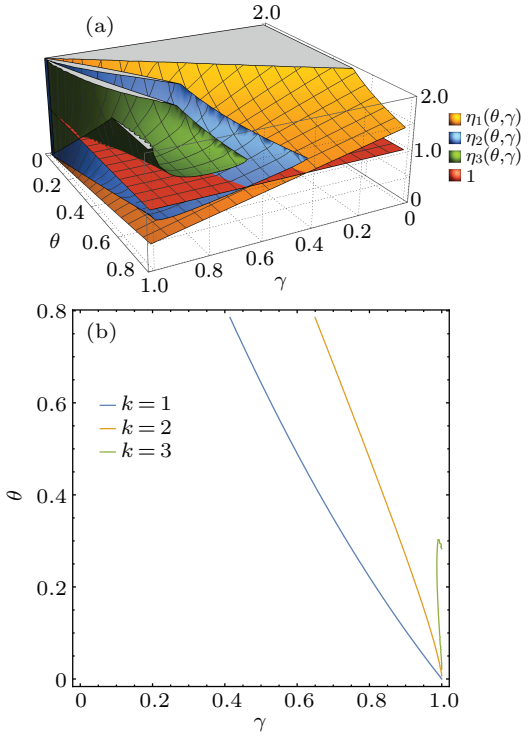


Fig. 4. (a) The yellow, blue and green surfaces represent the values of $\eta_1(\theta,\gamma)$, $\eta_2(\theta,\gamma)$ and $\eta_3(\theta,\gamma)$, respectively, as functions of parameters θ and γ. The red plane denotes $z = 1$. Due to the limit of accuracy of the computer calculation, the images with $k \geq 4$ cannot be plotted. (b) The blue, orange, and green curves represent $\eta_1(\theta,\gamma) = 1$, $\eta_2(\theta,\gamma) = 1$ and $\eta_3(\theta,\gamma) = 1$, respectively.

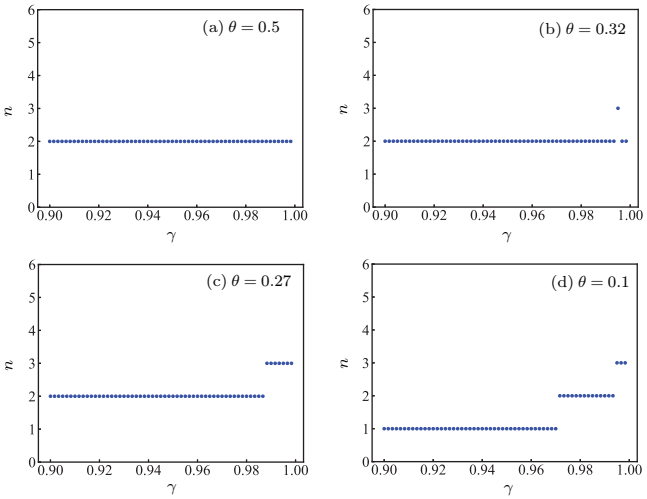


Fig. 5. The value of the vertical axis $n = n_{\max}^B$ is the maximal number of Bobs who can share nonlocality with Alice.

Case 2. $0 < \beta_1 + \beta_2 < 2$ and $\alpha = \delta = \gamma_k = \mu_k = 1$ for all $k = 1,2,\ldots,n$. In this case, we have

$$S_{\mathrm{CHSH}}^{(k)\mathrm{noisy}} = 2^{1-k}(\beta_1+\beta_2)\Big[\cos(\theta)\prod_{j=1}^{k-1}(1+\sqrt{1-\eta_j^2}) + \eta_k\sin(\theta)\Big], \quad (40)$$

and obtain the corresponding sequence $\{\eta_i(\theta,\beta_1+\beta_2,\varepsilon)\}_{i=1}^n$ as

$$\eta_k(\theta,\beta_1+\beta_2,\varepsilon)$$

$$= (1+\varepsilon)\frac{\frac{2^k}{\beta_1+\beta_2} - \cos(\theta)\prod_{j=1}^{k-1}(1+\sqrt{1-\eta_j^2(\theta,\beta_1+\beta_2,\varepsilon)})}{\sin(\theta)}, \tag{41}$$

where $0 < \eta_{k-1}(\theta,\beta_1+\beta_2,\varepsilon) < 1$ with $\eta_1(\theta,\beta_1+\beta_2,\varepsilon) = (1+\varepsilon)\frac{\frac{2}{\beta_1+\beta_2}-\cos(\theta)}{\sin(\theta)}$ for $\varepsilon > 0$, $\theta \in (0,\pi/4]$ and $\beta_1+\beta_2 \in [0,2]$. Let $x = \beta_1+\beta_2$. If $\eta_k(\theta,x,\varepsilon) < 1$, $k = 1,2,\ldots,n$, and $\eta_{n+1}(\theta,x,\varepsilon) \geq 1$, then there are at most n Bobs sharing the nonlocality with Alice. We also plot the function values of $\eta_k(\theta,x)$ again when $\varepsilon = 10^{-5}$ in Fig. 6(a), and show the ranges of $0 < \eta_k(\theta,x) < 1$ in Fig. 6(b), i.e., (θ, x) lies in the right-hand sides of each curves for $k = 1,2,3$.

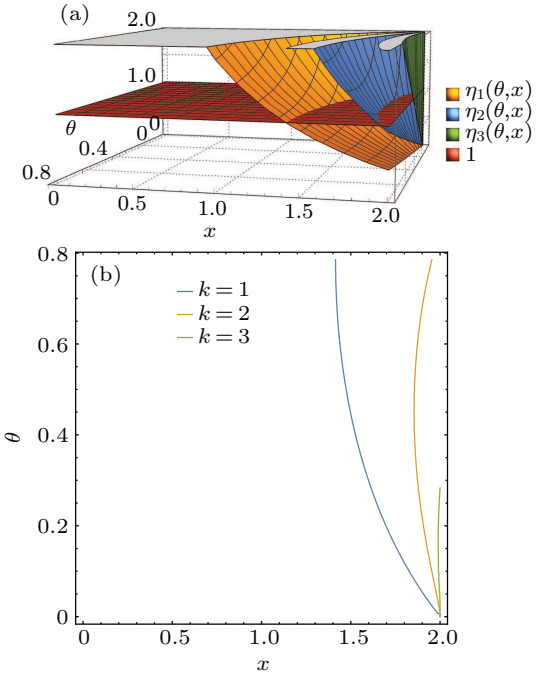


Fig. 6. (a) The yellow, blue and green surfaces represent the values of $\eta_1(\theta,x)$, $\eta_2(\theta,x)$ and $\eta_3(\theta,x)$, respectively, as functions of parameters θ and x. The red plane denotes $z = 1$. Due to the limit of accuracy of the computer calculation, the images with $k \geq 4$ cannot be plotted. (b) The blue, orange, and green curves represent the functions $\eta_1(\theta,x) = 1$, $\eta_2(\theta,x) = 1$ and $\eta_3(\theta,x) = 1$, respectively.

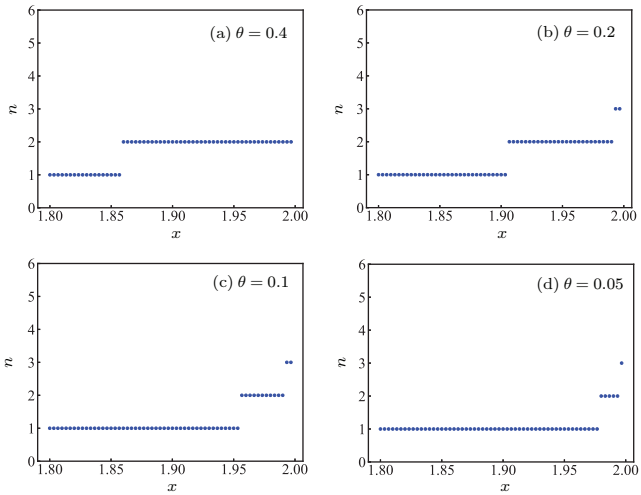


Fig. 7. The value of the vertical axis $n = n^B_{\max}$ is the maximal number of Bobs who can share nonlocality with Alice dependent on the noisy parameter $x = \beta_1+\beta_2$. When θ is fixed, n increases if x moves to 1.

Similarly, from Fig. 6(b), the value of $n^B_{\max}$ can be straightforward investigated when (θ,x) lies in the given region $[0.4,\pi/4]\times(0,1.8]$. For instance, $n^B_{\max} = 1$ when $(\theta,x) = (0.4,1.5)$. In the region $(0,0.4)\times(1.8,2)$, we describe the change pattern of $n^B_{\max}$ depending on the noise parameter $x \in (1.8,2)$ when θ is fixed as 0.4, 0.2, 0.1 and 0.05 respectively (see Fig. 7). We see that when θ is given, $n^B_{\max}$ increases if x moves from 1.8 to 2. On the other hand, in the four plots of Fig. 7, the best persistency occurs when $\theta = 0.2$.

4. The high-dimensional systems

In this section, we consider the persistency on sharing nonlocality noisily in the case of arbitrary finite-dimensional systems. The initial state with white noises is

$$\tilde{\rho}_{AB^{(1)}} = v|\psi\rangle\langle\psi| + \frac{(1-v)}{st}I_{st} \tag{42}$$

with the noisy parameter $v \in [0,1]$.

We also affix white noises on measurements Eqs. (9)–(16). Then, when both s and t are even, the noisy measurement strategies are defined as

$$\bar{A}_{0|0} = \frac{1}{2}[I_s + \beta_1(I_{\frac{s}{2}} \otimes (\cos(\theta)\sigma_3 + \sin(\theta)\sigma_1))], \tag{43}$$

$$\bar{A}_{0|1} = \frac{1}{2}[I_s + \beta_2(I_{\frac{s}{2}} \otimes (\cos(\theta)\sigma_3 - \sin(\theta)\sigma_1))], \tag{44}$$

$$\bar{B}^{(k)}_{0|0} = \frac{1}{2}[I_t + \gamma_k(I_{\frac{t}{2}} \otimes \sigma_3)], \tag{45}$$

$$\bar{B}^{(k)}_{0|1} = \frac{1}{2}[I_t + \mu_k(I_{\frac{t}{2}} \otimes \eta_k\sigma_1)] \tag{46}$$

for $\theta \in (0,\pi/4]$ and $k = 1,2,\ldots,n$. When both s and t are odd, a set of noisy POVMs are chosen to be

$$\bar{A}_{0|0} = \frac{1}{2}\left[I_s + \beta_1\begin{pmatrix} I_{[\frac{s}{2}]} \otimes (\cos(\theta)\sigma_3 + \sin(\theta)\sigma_1) & 0 \\ 0 & 1 \end{pmatrix}\right], \tag{47}$$

$$\bar{A}_{0|1} = \frac{1}{2}\left[I_s + \beta_2\begin{pmatrix} I_{[\frac{s}{2}]} \otimes (\cos(\theta)\sigma_3 - \sin(\theta)\sigma_1) & 0 \\ 0 & 1 \end{pmatrix}\right], \tag{48}$$

$$\bar{B}^{(k)}_{0|0} = \frac{1}{2}\left[I_t + \gamma_k\begin{pmatrix} I_{[\frac{t}{2}]} \otimes \sigma_3 & 0 \\ 0 & 1 \end{pmatrix}\right], \tag{49}$$

$$\bar{B}^{(k)}_{0|1} = \frac{1}{2}\left[I_t + \mu_k\eta_k\begin{pmatrix} I_{[\frac{t}{2}]} \otimes \sigma_1 & 0 \\ 0 & 1 \end{pmatrix}\right] \tag{50}$$

for $\theta \in (0,\pi/4]$ and $k = 1,2,\ldots,n$.

Theorem 4 For the noisy initial state $\tilde{\rho}_{AB^{(1)}}$ in Eq. (42), assume that the Alice and Bobs perform noisy measurements Eqs. (43), (44) or (47), (48) and Eqs. (45), (46) or (49), (50), respectively. If $\beta_1 \geq \beta_2$, then the expected CHSH value of $\rho_{AB^{(k)}}$ satisfies the inequality

$$\tilde{S}^{(k)\text{noisy}}_{\text{CHSH}} \geq 2^{1-k}(\beta_1+\beta_2)v\left[\gamma_k\cos(\theta)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2}\right)\right.$$
$$\left. + L\mu_k\eta_k\sin(\theta)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right)\right], \tag{51}$$

where $\theta \in (0,\pi/4]$, ν, β_i, γ_k, η_k and $\mu_k \in [0,1]$ for $i=1,2$, $k=1,2,\ldots,n$, $L=2(c_1c_2+c_3c_4+\cdots+c_{2[\frac{s}{2}]-1}c_{2[\frac{s}{2}]})$ and c_j is the Schmidt coefficient of $|\psi\rangle$ for $j=1,2,\ldots,c_{2[\frac{s}{2}]}$.

A proof of Theorem 4 will be given in Appendix D.

By Theorem 4, we get a sufficient condition of persistency of sharing nonlocality noisily in the case of the arbitrary-finite dimension.

Theorem 5 For the noisy initial state $\tilde{\rho}_{AB^{(1)}}$ in Eq. (42), assume the Alice and Bobs perform noisy measurements as in Eqs. (43), (44) or (47), (48) and Eqs. (45), (46) or (49), (50), respectively, and $\beta_1 \geq \beta_2$. Then there exist n Bobs sharing the nonlocality of the noisy initial state $\tilde{\rho}_{AB^{(1)}}$ with Alice if there exists $\varepsilon > 0$ such that

$$\max_{k\in\{1,2,\ldots,n\}} \arctan[y_{k2}(\varepsilon)] \leq \min_{k\in\{1,2,\ldots,n\}} \arctan[y_{k1}(\varepsilon)], \quad (52)$$

where

$$y_{k1}(\varepsilon) = \frac{\frac{e_k d_k}{1+\varepsilon} + 2^k\sqrt{e_k^2+(\frac{d_k}{1+\varepsilon})^2-2^{2k}}}{2^{2k}-(\frac{d_k}{1+\varepsilon})^2},$$
$$y_{k2}(\varepsilon) = \frac{\frac{e_k d_k}{1+\varepsilon} - 2^k\sqrt{e_k^2+(\frac{d_k}{1+\varepsilon})^2-2^{2k}}}{2^{2k}-(\frac{d_k}{1+\varepsilon})^2},$$

with

$$e_k = (\beta_1+\beta_2)\nu\gamma_k\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2}\right),$$
$$d_k = (\beta_1+\beta_2)\nu\mu_k L\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right)$$

for any $k=1,2,\ldots,n$, $L=2(c_1c_2+c_3c_4+\cdots+c_{2[\frac{s}{2}]-1}c_{2[\frac{s}{2}]})$ and c_i is the Schmidt coefficient of $|\psi\rangle$, $i=1,2,\ldots,c_{2[\frac{s}{2}]}$.

Furthermore, we get another sufficient condition for the persistency of sharing nonlocality when part of the noises disappears, which extends the sufficient persistency condition for the two-qubit case to the arbitrary finite-dimension case.

Theorem 6 For the noisy initial state $\tilde{\rho}_{AB^{(1)}}$ in Eq. (42), let the Alice and Bobs perform noisy measurements in Eqs. (43), (44) or (47), (48) and Eqs. (45), (46) or (49), (50), respectively. If $\gamma_n(\beta_1+\beta_2)\nu = 2$ for some n, then there exist at least n independent Bobs who can share the nonlocality of the noisy initial state $\tilde{\rho}_{AB^{(1)}}$ with the single Alice for arbitrary given noisy parameters $\gamma_k,\mu_j \in (0,1]$, $k=1,2,\ldots,n-1$ and $j=1,2,\ldots,n$.

5. Discussion and conclusion

The aim of the paper is to determine how many Bobs can share the nonlocality with Alice at most in a noisy environment. For a given measurement strategy, two kinds of noises are considered on the initial states and measurements respectively. We establish a CHSH type inequality and obtain two persistency conditions of sharing nonlocality noisily. We also analyze the influence of the noises to persistency, that is, the maximal number of Bobs who can share with Alice. It is significant to find other anti-noise measurement strategy, which can result in the better persistency.

Note also that we consider only two different noise factors. From experimental perspectives, it would be more interesting to analyze errors that occur during physical implementation, such as the exponential decrease in coherence of quantum states. Apart from loss in coherence, nonlocality sharing becomes challenging due to photon loss, noise in photon detection, and many other factors.

Appendix A: Proof of Theorem 1

Proof of Eq. (35) Denote by $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$ the two orthogonal eigenvectors corresponding to the two largest eigenvalues of $T(\bar{\rho}_{AB^{(1)}})T^{\mathrm{T}}(\bar{\rho}_{AB^{(1)}})$, respectively. We define $\boldsymbol{b}_i = \frac{T^{\mathrm{T}}(\bar{\rho}_{AB^{(1)}})\boldsymbol{c}_i}{\|T^{\mathrm{T}}(\bar{\rho}_{AB^{(1)}})\boldsymbol{c}_i\|}$, $i=0,1$, and $\sigma_{\boldsymbol{r}} = r_1\sigma_1+r_2\sigma_2+r_3\sigma_3$ for $\boldsymbol{r}=(r_1,r_2,r_3)$, where $\{\sigma_1,\sigma_2,\sigma_3\}$ are Pauli operators. It is worth noting that for two vectors $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$ we have $\mathrm{tr}[\rho(\sigma_{\boldsymbol{a}}\otimes\sigma_{\boldsymbol{b}})] = (\boldsymbol{a},T(\rho)\boldsymbol{b})$, where $(\cdot,\cdot)$ is the Euclidean inner product for the real vector space $\mathbb{R}^3$.

When the noisy measurement strategy Eqs. (27)–(32) are performed, then the CHSH value of Alice and Bob$^{(k)}$ can be computed as follows:

$$\begin{aligned}
S_{\mathrm{CHSH}}^{(k)\mathrm{noisy}} &= \mathrm{tr}[\rho_{AB^{(k)}}((\bar{A}_0+\bar{A}_1)\otimes\bar{B}_0^{(k)})] + \mathrm{tr}[\rho_{AB^{(k)}}((\bar{A}_0-\bar{A}_1)\otimes\bar{B}_1^{(k)})]\\
&= \mathrm{tr}[\rho_{AB^{(k)}}((\beta_1A_0+\beta_2A_1)\otimes\gamma_kB_0^{(k)})] + \mathrm{tr}[\rho_{AB^{(k)}}((\beta_1A_0-\beta_2A_1)\otimes\mu_kB_1^{(k)})]\\
&= \beta_1\gamma_k\mathrm{tr}\{\rho_{AB^{(k)}}[(\cos(\theta)\sigma_{\boldsymbol{c}_0}+\sin(\theta)\sigma_{\boldsymbol{c}_1})\otimes\sigma_{\boldsymbol{b}_0}]\} + \beta_2\gamma_k\mathrm{tr}\{\rho_{AB^{(k)}}[(\cos(\theta)\sigma_{\boldsymbol{c}_0}-\sin(\theta)\sigma_{\boldsymbol{c}_1})\otimes\sigma_{\boldsymbol{b}_0}]\}\\
&\quad+\beta_1\mu_k\eta_k\mathrm{tr}\{\rho_{AB^{(k)}}[(\cos(\theta)\sigma_{\boldsymbol{c}_0}+\sin(\theta)\sigma_{\boldsymbol{c}_1})\otimes\sigma_{\boldsymbol{b}_1}]\} - \beta_2\mu_k\eta_k\mathrm{tr}\{\rho_{AB^{(k)}}[(\cos(\theta)\sigma_{\boldsymbol{c}_0}-\sin(\theta)\sigma_{\boldsymbol{c}_1})\otimes\sigma_{\boldsymbol{b}_1}]\}. \quad (A1)
\end{aligned}$$

By using the Lüders rule, we have

$$\begin{aligned}
\rho_{AB^{(k)}} &= \frac{1}{2}\sum_{b,y}\left(I_2\otimes\sqrt{\bar{B}_{b|y}^{(k-1)}}\right)\rho_{AB^{(k-1)}}\left(I_2\otimes\sqrt{\bar{B}_{b|y}^{(k-1)}}\right)\\
&= \frac{1}{2}\left[\left(I_2\otimes\sqrt{\bar{B}_{0|0}^{(k-1)}}\right)\rho_{AB^{(k-1)}}\left(I_2\otimes\sqrt{\bar{B}_{0|0}^{(k-1)}}\right)+\left(I_2\otimes\sqrt{\bar{B}_{1|0}^{(k-1)}}\right)\rho_{AB^{(k-1)}}\left(I_2\otimes\sqrt{\bar{B}_{1|0}^{(k-1)}}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(I_2\otimes\sqrt{\bar{B}_{0|1}^{(k-1)}}\right)\rho_{AB^{(k-1)}}\left(I_2\otimes\sqrt{\bar{B}_{0|1}^{(k-1)}}\right)+\left(I_2\otimes\sqrt{\bar{B}_{1|1}^{(k-1)}}\right)\rho_{AB^{(k-1)}}\left(I_2\otimes\sqrt{\bar{B}_{1|1}^{(k-1)}}\right)\Big]\\
&=\frac{1}{8}\left(2+2\sqrt{1-\gamma_{k-1}^2}\right)\rho_{AB^{(k-1)}}+\frac{1}{8}(2-2\sqrt{1-\gamma_{k-1}^2})(I_2\otimes\sigma_{\boldsymbol{b}_0})\rho_{AB^{(k-1)}}(I_2\otimes\sigma_{\boldsymbol{b}_0})\\
&+\frac{1}{8}(2+2\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2})\rho_{AB^{(k-1)}}+\frac{1}{8}\left(2-2\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}\right)(I_2\otimes\sigma_{\boldsymbol{b}_1})\rho_{AB^{(k-1)}}\left(I_2\otimes\sigma_{\boldsymbol{b}_1}\right)\\
&=\left[\frac{1}{2}+\frac{1}{4}\left(\sqrt{1-\gamma_{k-1}^2}+\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}\right)\right]\rho_{AB^{(k-1)}}+\frac{1-\sqrt{1-\gamma_{k-1}^2}}{4}(I_2\otimes\sigma_{\boldsymbol{b}_0})\rho_{AB^{(k-1)}}(I_2\otimes\sigma_{\boldsymbol{b}_0})\\
&+\frac{1-\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{4}(I_2\otimes\sigma_{\boldsymbol{b}_1})\rho_{AB^{(k-1)}}(I_2\otimes\sigma_{\boldsymbol{b}_1}),
\end{aligned}
\tag{A2}
$$

where the third equation comes from

$$
\sqrt{\frac{1}{2}(I_2\pm\eta_k\sigma_{\boldsymbol{b}_i})}=\frac{(\sqrt{1+\eta_k}+\sqrt{1-\eta_k})I_2\pm(\sqrt{1+\eta_k}-\sqrt{1-\eta_k})\sigma_{\boldsymbol{b}_i}}{2\sqrt{2}}
\tag{A3}
$$

for $i=0,1$. Then, using Eq. (A2), we have

$$
\begin{aligned}
(\boldsymbol{c}_0,T(\rho_{AB^{(k)}})\boldsymbol{b}_0)&=\left(\frac{1}{2}+\frac{\sqrt{1-\gamma_{k-1}^2}+\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{4}\right)(\boldsymbol{c}_0,T(\rho_{AB^{(k-1)}})\boldsymbol{b}_0)\\
&+\frac{1-\sqrt{1-\gamma_{k-1}^2}}{4}\mathrm{tr}[(I_2\otimes\sigma_{\boldsymbol{b}_0})\rho_{AB^{(k-1)}}(I_2\otimes\sigma_{\boldsymbol{b}_0})(\sigma_{\boldsymbol{c}_0}\otimes\sigma_{\boldsymbol{b}_0})]\\
&+\frac{1-\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{4}\mathrm{tr}[(I_2\otimes\sigma_{\boldsymbol{b}_1})\rho_{AB^{(k-1)}}(I_2\otimes\sigma_{\boldsymbol{b}_1})(\sigma_{\boldsymbol{c}_0}\otimes\sigma_{\boldsymbol{b}_0})]\\
&=\left(\frac{1}{2}+\frac{\sqrt{1-\gamma_{k-1}^2}+\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{4}\right)(\boldsymbol{c}_0,T(\rho_{AB^{(k-1)}})\boldsymbol{b}_0)\\
&+\frac{1-\sqrt{1-\gamma_{k-1}^2}}{4}\mathrm{tr}[\rho_{AB^{(k-1)}}(\sigma_{\boldsymbol{c}_0}\otimes\sigma_{\boldsymbol{b}_0}\sigma_{\boldsymbol{b}_0}\sigma_{\boldsymbol{b}_0})]+\frac{1-\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{4}\mathrm{tr}[\rho_{AB^{(k-1)}}(\sigma_{\boldsymbol{c}_0}\otimes\sigma_{\boldsymbol{b}_1}\sigma_{\boldsymbol{b}_0}\sigma_{\boldsymbol{b}_1})]\\
&=\left(\frac{1}{2}+\frac{\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{2})(\boldsymbol{c}_0,T(\rho_{AB^{(k-1)}})\boldsymbol{b}_0\right).
\end{aligned}
\tag{A4}
$$

Similar to Eq. (A4), we get

$$
\begin{aligned}
&(\boldsymbol{c}_1,T(\rho_{AB^{(k)}})\boldsymbol{b}_1)\\
&=\left(\frac{1}{2}+\frac{\sqrt{1-\gamma_{k-1}^2}+\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{4}-\frac{1-\sqrt{1-\gamma_{k-1}^2}}{4}+\frac{1-\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{4}\right)(\boldsymbol{c}_1,T(\rho_{AB^{(k-1)}})b_1)\\
&=\left(\frac{1}{2}+\frac{\sqrt{1-\gamma_{k-1}^2}}{2}\right)(\boldsymbol{c}_1,T(\rho_{AB^{(k-1)}})\boldsymbol{b}_1),
\end{aligned}
\tag{A5}
$$

$$
\begin{aligned}
&(\boldsymbol{c}_0,T(\rho_{AB^{(k)}})\boldsymbol{b}_1)\\
&=\left(\frac{1}{2}+\frac{\sqrt{1-\gamma_{k-1}^2}+\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{4}-\frac{1-\sqrt{1-\gamma_{k-1}^2}}{4}+\frac{1-\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{4}\right)(\boldsymbol{c}_0,T(\rho_{AB^{(k-1)}})\boldsymbol{b}_1)\\
&=\left(\frac{1}{2}+\frac{\sqrt{1-\gamma_{k-1}^2}}{2}\right)(\boldsymbol{c}_0,T(\rho_{AB^{(k-1)}})\boldsymbol{b}_1),
\end{aligned}
\tag{A6}
$$

$$
\begin{aligned}
&(\boldsymbol{c}_1,T(\rho_{AB^{(k)}})\boldsymbol{b}_0)\\
&=\left(\frac{1}{2}+\frac{\sqrt{1-\gamma_{k-1}^2}+\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{4}+\frac{1-\sqrt{1-\gamma_{k-1}^2}}{4}-\frac{1-\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{4}\right)(\boldsymbol{c}_1,T(\rho_{AB^{(k-1)}})\boldsymbol{b}_0)\\
&=\left(\frac{1}{2}+\frac{\sqrt{1-\mu_{k-1}^2\eta_{k-1}^2}}{2}\right)(\boldsymbol{c}_1,T(\rho_{AB^{(k-1)}})\boldsymbol{b}_0).
\end{aligned}
\tag{A7}
$$

By recursion, we get

$$(c_0, T(\rho_{AB^{(k)}})b_0) = 2^{1-k}(c_0, T(\bar{\rho}_{AB^{(1)}})b_0)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2}\right), \tag{A8}$$

$$(c_1, T(\rho_{AB^{(k)}})b_1) = 2^{1-k}(c_1, T(\bar{\rho}_{AB^{(1)}})b_1)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right), \tag{A9}$$

$$(c_0, T(\rho_{AB^{(k)}})b_1) = 2^{1-k}(c_0, T(\bar{\rho}_{AB^{(1)}})b_1)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right), \tag{A10}$$

$$(c_1, T(\rho_{AB^{(k)}})b_0) = 2^{1-k}(c_1, T(\bar{\rho}_{AB^{(1)}})b_0)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2}\right). \tag{A11}$$

It follows from Eqs. (A8)–(A11) and (A1) that

$$\begin{aligned} S_{\mathrm{CHSH}}^{(k)\mathrm{noisy}} &= \gamma_k 2^{1-k}\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2}\right) \\ &\quad\times[(\beta_1+\beta_2)\cos(\theta)(c_0, T(\bar{\rho}_{AB^{(1)}})b_0) \\ &\quad+(\beta_1-\beta_2)\sin(\theta)(c_1, T(\bar{\rho}_{AB^{(1)}})b_0)] \\ &\quad+\mu_k\eta_k 2^{1-k}\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right) \\ &\quad\times[(\beta_1+\beta_2)\cos(\theta)(c_0, T(\bar{\rho}_{AB^{(1)}})b_1) \\ &\quad+(\beta_1+\beta_2)\sin(\theta)(c_1, T(\bar{\rho}_{AB^{(1)}})b_1)]. \end{aligned} \tag{A12}$$

Proof of Theorem 1 For the case of $T(\bar{\rho}_{AB^{(1)}}) = \mathrm{diag}(-\alpha\delta, \alpha\delta, \delta)$, from Eq. (A12) we have

$$(c_1, T(\bar{\rho}_{AB^{(1)}})b_0) = \frac{1}{\|T^{\mathrm{T}}(\bar{\rho}_{AB^{(1)}})c_0\|}(c_1, T(\bar{\rho}_{AB^{(1)}})T^{\mathrm{T}}(\bar{\rho}_{AB^{(1)}})c_0) = 0, \tag{A13}$$

$$(c_0, T(\bar{\rho}_{AB^{(1)}})b_1) = \frac{1}{\|T^{\mathrm{T}}(\bar{\rho}_{AB^{(1)}})c_1\|}(c_0, T(\bar{\rho}_{AB^{(1)}})T^{\mathrm{T}}(\bar{\rho}_{AB^{(1)}})c_1) = 0, \tag{A14}$$

$$(c_0, T(\bar{\rho}_{AB^{(1)}})b_0) = \delta, \tag{A15}$$

$$(c_1, T(\bar{\rho}_{AB^{(1)}})b_1) = \alpha\delta. \tag{A16}$$

Using Eqs. (A13)–(A16) in Eq. (A12), we get

$$\begin{aligned} S_{\mathrm{CHSH}}^{(k)\mathrm{noisy}} &= 2^{1-k}(\beta_1+\beta_2)\delta\left[\gamma_k\cos(\theta)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2}\right)\right. \\ &\quad\left.+\alpha\mu_k\eta_k\sin(\theta)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right)\right]. \end{aligned} \tag{A17}$$

Appendix B: Proof of Theorem 2

From $S_{\mathrm{CHSH}}^{(k)\mathrm{noisy}} > 2$, $k = 1, 2, \ldots, n$, it is equivalent to

$$\eta_k > \frac{\frac{2^k}{(\beta_1+\beta_2)\delta} - \gamma_k\cos(\theta)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2}\right)}{\sin(\theta)\alpha\mu_k\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right)}. \tag{B1}$$

Define a function

$$\begin{aligned} &f(\theta, \alpha, \delta, \beta_1, \beta_2, \mu_1, \ldots, \mu_k, \gamma_1, \ldots, \gamma_k, \eta_1, \ldots, \eta_{k-1}) \\ &= \frac{\frac{2^k}{(\beta_1+\beta_2)\delta} - \gamma_k\cos(\theta)\prod_{j=1}^{k-1}(1+\sqrt{1-\mu_j^2\eta_j^2})}{\sin(\theta)\alpha\mu_k\prod_{j=1}^{k-1}(1+\sqrt{1-\gamma_j^2})}. \end{aligned}$$

Obviously, $f(\theta, \alpha, \delta, \beta_1, \beta_2, \mu_1, \ldots, \mu_k, \gamma_1, \ldots, \gamma_k, \eta_1, \ldots, \eta_{k-1}) > 0$, then the inequality (B1) can be written as

$$\eta_k > f(\theta, \alpha, \delta, \beta_1, \beta_2, \mu_1, \ldots, \mu_k, \gamma_1, \ldots, \gamma_k, \eta_1, \ldots, \eta_{k-1}),$$

i.e.,

$$\frac{\eta_k}{f(\theta, \alpha, \delta, \beta_1, \beta_2, \mu_1, \ldots, \mu_k, \gamma_1, \ldots, \gamma_k, \eta_1, \ldots, \eta_{k-1})} > 1,$$

we have

$$\inf\left(\frac{\eta_k}{f(\theta, \alpha, \delta, \beta_1, \beta_2, \mu_1, \ldots, \mu_k, \gamma_1, \ldots, \gamma_k, \eta_1, \ldots, \eta_{k-1})}\right) > 1,$$

this implies that there exists $\varepsilon > 0$ such that $\eta_k = (1+\varepsilon)f(\theta, \alpha, \delta, \beta_1, \beta_2, \mu_1, \ldots, \mu_k, \gamma_1, \ldots, \gamma_k, \eta_1, \ldots, \eta_{k-1})$.

Denote by $\mathcal{X}_k$ the set $\{\varepsilon, \alpha, \delta, \beta_1, \beta_2, \mu_i, \gamma_i\}_{i=1}^k$. Furthermore, we define a sequence $\eta_k(\theta, \mathcal{X}_k)$ as follows:

$$\eta_k(\theta, \mathcal{X}_k) = \begin{cases} (1+\varepsilon)\dfrac{\frac{2^k}{(\beta_1+\beta_2)\delta} - \gamma_k\cos(\theta)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2(\theta, \mathcal{X}_j)}\right)}{\sin(\theta)\alpha\mu_k\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right)} & \text{if } 0 < \eta_{k-1}(\theta, \mathcal{X}_{k-1}) < 1, \\ \infty & \text{otherwise,} \end{cases} \tag{B2}$$

with $\eta_1(\theta, \mathcal{X}_1) = (1+\varepsilon)\frac{\frac{2}{(\beta_1+\beta_2)\delta} - \gamma_1\cos(\theta)}{\sin(\theta)\alpha\mu_1}$ for some $\varepsilon > 0$ and $\theta \in (0, \pi/4]$. We note that $\eta_i(\theta, \mathcal{X}_i) > 0$, $i = 1, 2, \ldots$, for $\theta \in (0, \frac{\pi}{4}]$ and $\varepsilon > 0$. In order to $\eta_k(\theta, \mathcal{X}_k) < 1$, i.e.,

$$(1+\varepsilon)\left[\frac{2^k}{(\beta_1+\beta_2)\delta} - \cos(\theta)\gamma_k\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2(\theta)}\right)\right] < \sin(\theta)\alpha\mu_k\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right). \tag{B3}$$

If $\eta_i(\theta,\mathcal{X}_i)<1$ for $i=1,2,\ldots,k-1$, then

$$\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2(\theta,\mathcal{X}_j)}\right)>\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2}\right).$$

Thus, $\eta_k(\theta,\mathcal{X}_k)<1$ when

$$(1+\varepsilon)\left[\frac{2^k}{(\beta_1+\beta_2)\delta}-\cos(\theta)\gamma_k\prod_{j=1}^{k-1}(1+\sqrt{1-\mu_j^2})\right]<\sin(\theta)\alpha\mu_k\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right). \tag{B4}$$

Set

$$a_k=(\beta_1+\beta_2)\delta\gamma_k\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2}\right), \tag{B5}$$

$$b_k=(\beta_1+\beta_2)\delta\alpha\mu_k\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right). \tag{B6}$$

We can rewrite inequality (B4) as $(1+\varepsilon)(2^k-\cos(\theta)a_k)<b_k\sin(\theta)$ and get

$$\left[2^{2k}-\left(\frac{b_k}{1+\varepsilon}\right)^2\right]\tan^2(\theta)-\frac{2a_kb_k}{1+\varepsilon}\tan(\theta)+2^{2k}-a_k^2<0. \tag{B7}$$

The solution of inequality (B7) satisfies

$$\tan(\theta_k)\in(x_{k1}(\varepsilon),x_{k2}(\varepsilon)), \tag{B8}$$

where

$$x_{k1}(\varepsilon)=\frac{\frac{a_kb_k}{1+\varepsilon}-2^k\sqrt{a_k^2+(\frac{b_k}{1+\varepsilon})^2-2^{2k}}}{2^{2k}-(\frac{b_k}{1+\varepsilon})^2},$$

$$x_{k2}(\varepsilon)=\frac{\frac{a_kb_k}{1+\varepsilon}+2^k\sqrt{a_k^2+(\frac{b_k}{1+\varepsilon})^2-2^{2k}}}{2^{2k}-(\frac{b_k}{1+\varepsilon})^2}$$

for any $k=1,2,\ldots,n$, i.e., $\theta_k\in(\arctan[x_{k2}(\varepsilon)],\arctan[x_{k1}(\varepsilon)])$. Therefore, if

$$\max_{k\in\{1,2,\ldots,n\}}\arctan[x_{k2}(\varepsilon)]\le\min_{k\in\{1,2,\ldots,n\}}\arctan[x_{k1}(\varepsilon)], \tag{B9}$$

for some $\varepsilon>0$, then there exists $\theta\in(\max_{k\in\{1,2,\ldots,n\}}\arctan[x_{k2}(\varepsilon)],\min_{k\in\{1,2,\ldots,n\}}\arctan[x_{k1}(\varepsilon)])$ such that $\eta_k(\theta,\mathcal{X}_k)<1$ for any $k=1,2,\ldots,n$.

Appendix C: Proof of Theorem 3

From assumption, $\gamma_n(\beta_1+\beta_2)\delta=2$, denote by $\mathcal{Y}_k$ the set of given nonzero parameters $\{\varepsilon,\alpha,\mu_s,\gamma_t\}_{s=1,t=1}^{k,k-1}$, one has

$$\eta_k(\theta):=\begin{cases}(1+\varepsilon)\dfrac{2^{k-1}-\cos(\theta)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2(\theta)}\right)}{\sin(\theta)\alpha\mu_k\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right)} & \text{if } 0<\eta_{k-1}(\theta)<1,\\ \infty & \text{otherwise,}\end{cases} \tag{C1}$$

with $\eta_1(\theta)=(1+\varepsilon)\frac{1-\cos(\theta)}{\sin(\theta)\alpha\mu_1}$, $\theta\in(0,\frac{\pi}{4}]$ and $\varepsilon>0$, which corresponds to inequality (B2).

To obtain the bound of the terms of the sequence $\{\eta_k(\theta)|k=1,2,\ldots,n\}$, we use the following inequalities: $\sqrt{1-x^2}\ge1-x^2$ for $x\in[0,1]$; $\cos(\theta)\ge1-\theta^2/2$ and $\sin(\theta)\ge\theta/2$ for $\theta\in(0,\pi/4]$. Then we have

$$\eta_k(\theta)\le(1+\varepsilon)\frac{2^{k-1}-(1-\frac{\theta^2}{2})\prod_{j=1}^{k-1}(2-\mu_j^2\eta_j^2(\theta))}{\frac{\theta}{2}\alpha\mu_k\prod_{j=1}^{k-1}(2-\gamma_j^2)}=(1+\varepsilon)2^k\frac{1-(1-\frac{\theta^2}{2})\prod_{j=1}^{k-1}(1-\frac{\mu_j^2\eta_j^2(\theta)}{2})}{\theta\alpha\mu_k\prod_{j=1}^{k-1}(2-\gamma_j^2)}. \tag{C2}$$

We can define a new sequence $(p_i(\theta))_i$ of the following form:

$$p_k(\theta):=\begin{cases}(1+\varepsilon)2^k\dfrac{1-(1-\frac{\theta^2}{2})\prod_{j=1}^{k-1}(1-\frac{\mu_j^2p_j^2(\theta)}{2})}{\theta\alpha\mu_k\prod_{j=1}^{k-1}(2-\gamma_j^2)} & \text{if } 0<p_{k-1}(\theta)<1,\\ \infty & \text{otherwise,}\end{cases} \tag{C3}$$

with $p_1(\theta)=(1+\varepsilon)\frac{\theta}{\alpha\mu_1}$ and $\eta_k(\theta)\le p_k(\theta)$. We have $p_1(0)=0$, and $p_1(\theta)$ is a continuous positive function and $0<p_1(\theta)<1$ for all $\theta\in(0,\theta_1)$ with $\theta_1=\frac{\alpha\mu_1}{1+\varepsilon}$. As $p_2(\theta)=(1+\varepsilon)\theta\frac{(1+\varepsilon)^2(4-\theta^2)+2\alpha^2}{\alpha^3\mu_2(2-\gamma_1^2)}$, therefore, $p_2(\theta)$ is also a continuous positive function and $p_2(0)=0$, this implies that there exists $0<p_2(\theta)<1$ for $\theta\in(0,\theta_2)$.

Next, we proceed by induction to show that there exists $\theta_k > 0$ such that $0 < p_k(\theta) < 1$ when $\theta \in (0, \theta_k)$. Suppose that $p_j(\theta)$ is a continuous function and $p_j(0) = 0$, then we can find interval $(0, \theta_j)$ such that $0 < p_j(\theta) < 1$ for $j = 1, 2, \ldots, k-1$. It implies that $0 < p_j(\theta) < 1$ for any $j = 1, 2, \ldots, k-1$ when $\theta \in (0, \min\limits_{j\in\{1,2,\ldots,k-1\}} \theta_j)$. Then, we can calculate to get $p_k(0) = 0$ and $p_k(\theta)$ is also a continuous function. It follows from $0 < p_j(\theta) < 1$ for any $j = 1, 2, \ldots, k-1$ when $\theta \in (0, \min\limits_{j\in\{1,2,\ldots,k-1\}} \theta_j)$ that $p_k(\theta)$ is a positive function. Therefore, we can get that $0 < p_k(\theta) < 1$ on the interval $(0, \theta_k)$. It follows by induction that for any $n \in \mathbb{N}$ we can find a $\theta \in (0, \min\limits_{j\in\{1,2,\ldots,n\}} \theta_j)$ such that $0 < p_n(\theta) < 1$.

Finally, as $\eta_k(\theta) \leq p_k(\theta)$ we have $0 < \eta_i(\theta) < 1$ when $\theta \in (0, \min\limits_{j\in\{1,2,\ldots,n\}} \theta_j)$ for all $i = 1, 2 \ldots, n$.

Appendix D: Proof of Theorem 4

In the following, we give a brief proof, which is similar to the proof of Theorem 1. When both s and t are even, we use Eqs. (43), (44) and Eqs. (45), (46) as noisy measurements, we get

$$\begin{aligned}\tilde{S}^{(k)\text{noisy}}_{\text{CHSH}} &= \text{tr}[\rho_{AB^{(k)}}((\bar{A}_0 + \bar{A}_1) \otimes \bar{B}^{(k)}_0)] \\ &\quad + \text{tr}[\rho_{AB^{(k)}}((\bar{A}_0 - \bar{A}_1) \otimes \bar{B}^{(k)}_1)] \\ &= \text{tr}[\rho_{AB^{(k)}}((\beta_1 A_0 + \beta_2 A_1) \otimes \gamma_k B^{(k)}_0)] \\ &\quad + \text{tr}[\rho_{AB^{(k)}}((\beta_1 A_0 - \beta_2 A_1) \otimes \mu_k B^{(k)}_1)] \\ &= \beta_1\gamma_k \text{tr}[\rho_{AB^{(k)}}(A_0 \otimes B^{(k)}_0)] \\ &\quad + \beta_2\gamma_k \text{tr}[\rho_{AB^{(k)}}(A_1 \otimes B^{(k)}_0)] \\ &\quad + \beta_1\mu_k \text{tr}[\rho_{AB^{(k)}}(A_0 \otimes B^{(k)}_1)] \\ &\quad - \beta_2\mu_k \text{tr}[\rho_{AB^{(k)}}(A_1 \otimes B^{(k)}_1)], \end{aligned} \tag{D1}$$

where

$$\begin{aligned} A_0 &= I_{\frac{s}{2}} \otimes (\cos(\theta)\sigma_3 + \sin(\theta)\sigma_1), \\ A_1 &= I_{\frac{s}{2}} \otimes (\cos(\theta)\sigma_3 - \sin(\theta)\sigma_1), \end{aligned} \tag{D2}$$

$$B^{(k)}_0 = I_{\frac{t}{2}} \otimes \sigma_3, \; B^{(k)}_1 = \eta_k I_{\frac{t}{2}} \otimes \sigma_1. \tag{D3}$$

For the case $k = 1$, we have

$$\begin{aligned} &\text{tr}[\rho_{AB^{(1)}}(I_{\frac{s}{2}} \otimes \sigma_3 \otimes I_{\frac{t}{2}} \otimes \sigma_1)] \\ &= \text{tr}[\rho_{AB^{(1)}}(I_{\frac{s}{2}} \otimes \sigma_1 \otimes I_{\frac{t}{2}} \otimes \sigma_3)] = 0, \end{aligned} \tag{D4}$$

$$\text{tr}[\rho_{AB^{(1)}}(I_{\frac{s}{2}} \otimes \sigma_3 \otimes I_{\frac{t}{2}} \otimes \sigma_3)] = \sum_i c_i^2 = 1, \tag{D5}$$

$$\text{tr}[\rho_{AB^{(1)}}(I_{\frac{s}{2}} \otimes \sigma_1 \otimes I_{\frac{t}{2}} \otimes \sigma_1)] = L, \tag{D6}$$

where $L = 2(c_1c_2 + c_3c_4 + \cdots + c_{2[\frac{s}{2}]-1}c_{2[\frac{s}{2}]})$. Then we get

$$\begin{aligned} &\text{tr}[\tilde{\rho}_{AB^{(1)}}(I_{\frac{s}{2}} \otimes \sigma_3 \otimes I_{\frac{t}{2}} \otimes \sigma_1)] \\ &= \text{tr}[\tilde{\rho}_{AB^{(1)}}(I_{\frac{s}{2}} \otimes \sigma_1 \otimes I_{\frac{t}{2}} \otimes \sigma_3)] = 0, \end{aligned} \tag{D7}$$

$$\text{tr}[\tilde{\rho}_{AB^{(1)}}(I_{\frac{s}{2}} \otimes \sigma_3 \otimes I_{\frac{t}{2}} \otimes \sigma_3)] = \nu \sum_i c_i^2 = \nu, \tag{D8}$$

$$\text{tr}[\tilde{\rho}_{AB^{(1)}}(I_{\frac{s}{2}} \otimes \sigma_1 \otimes I_{\frac{t}{2}} \otimes \sigma_1)] = \nu L. \tag{D9}$$

It follows from the Lüders rule, Eqs. (A3)–(A5) in Ref. [33] and Eqs. (D2), (D3), (D7)–(D9) that Eq. (D1) is transformed into

$$\begin{aligned} &\tilde{S}^{(k)\text{noisy}}_{\text{CHSH}} \\ &= (\beta_1 + \beta_2)[\gamma_k \cos(\theta) \text{tr}[\rho_{AB^{(k)}}(I_{\frac{s}{2}} \otimes \sigma_3 \otimes I_{\frac{t}{2}} \otimes \sigma_3)] \\ &\quad + \mu_k\eta_k \sin(\theta) \text{tr}[\rho_{AB^{(k)}}(I_{\frac{s}{2}} \otimes \sigma_1 \otimes I_{\frac{t}{2}} \otimes \sigma_1)]] \\ &= (\beta_1 + \beta_2)[\gamma_k \cos(\theta)(2^{1-k}\nu \prod_{j=1}^{k-1}(1 + \sqrt{1 - \mu_j^2\eta_j^2})) \\ &\quad + \mu_k\eta_k \sin(\theta) 2^{1-k}\nu L \prod_{j=1}^{k-1}(1 + \sqrt{1 - \gamma_j^2})] \\ &= (\beta_1 + \beta_2)2^{1-k}\nu[\gamma_k \cos(\theta) \prod_{j=1}^{k-1}(1 + \sqrt{1 - \mu_j^2\eta_j^2}) \\ &\quad + \mu_k\eta_k L \sin(\theta) \prod_{j=1}^{k-1}(1 + \sqrt{1 - \gamma_j^2})]. \end{aligned} \tag{D10}$$

Then, we consider the case that both s and t are odd, which is the most complex case. In this case, we use Eqs. (47), (48) and Eqs. (49), (50) as the noisy measurements for Alice and Bob$^{(k)}$, respectively. By calculation, we get

$$\text{tr}[\rho_{AB^{(1)}}(A_0 \otimes B^{(k)}_0)] = \cos(\theta)(1 - c_s^2) + c_s^2, \tag{D11}$$

$$\text{tr}[\rho_{AB^{(1)}}(A_0 \otimes B^{(k)}_1)] = \eta_k(c_s^2 + \sin(\theta)L), \tag{D12}$$

$$\text{tr}[\rho_{AB^{(1)}}(A_1 \otimes B^{(k)}_0)] = \cos(\theta)(1 - c_s^2) + c_s^2, \tag{D13}$$

$$\text{tr}[\rho_{AB^{(1)}}(A_1 \otimes B^{(k)}_1)] = \eta_k(c_s^2 - \sin(\theta)L), \tag{D14}$$

where

$$A_0 = \begin{pmatrix} I_{[\frac{s}{2}]} \otimes (\cos(\theta)\sigma_3 + \sin(\theta)\sigma_1) & 0 \\ 0 & 1 \end{pmatrix}, \tag{D15}$$

$$A_1 = \begin{pmatrix} I_{[\frac{s}{2}]} \otimes (\cos(\theta)\sigma_3 - \sin(\theta)\sigma_1) & 0 \\ 0 & 1 \end{pmatrix}, \tag{D16}$$

$$B^{(k)}_0 = \begin{pmatrix} I_{[\frac{t}{2}]} \otimes \sigma_3 & 0 \\ 0 & 1 \end{pmatrix}, \tag{D17}$$

$$B^{(k)}_1 = \eta_k \begin{pmatrix} I_{[\frac{t}{2}]} \otimes \sigma_1 & 0 \\ 0 & 1 \end{pmatrix} \tag{D18}$$

for some $\theta \in (0, \pi/4]$ and $k = 1, 2, \ldots, n$. Similar to the proof of Theorem 1, we can get

$$\begin{aligned} &\text{tr}[\rho_{AB^{(k)}}(A_0 \otimes B^{(k)}_0)] \\ &= \prod_{j=1}^{k-1}\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \mu_j^2\eta_j^2}\right) \text{tr}[\tilde{\rho}_{AB^{(1)}}(A_0 \otimes B^{(k)}_0)], \end{aligned} \tag{D19}$$

$$\begin{aligned} &\text{tr}[\rho_{AB^{(k)}}(A_0 \otimes B^{(k)}_1)] \\ &= \prod_{j=1}^{k-1}\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \gamma_j^2}\right) \text{tr}[\tilde{\rho}_{AB^{(1)}}(A_0 \otimes B^{(k)}_1)], \end{aligned} \tag{D20}$$

$$\text{tr}[\rho_{AB^{(k)}}(A_1 \otimes B^{(k)}_0)]$$

$$= \prod_{j=1}^{k-1}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\mu_j^2\eta_j^2}\right)\mathrm{tr}[\tilde{\rho}_{AB^{(1)}}(A_1\otimes B_0^{(k)})], \quad \text{(D21)}$$

$$\mathrm{tr}[\rho_{AB^{(k)}}(A_1\otimes B_1^{(k)})]$$

$$= \prod_{j=1}^{k-1}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\gamma_j^2}\right)\mathrm{tr}[\tilde{\rho}_{AB^{(1)}}(A_1\otimes B_1^{(k)})]. \quad \text{(D22)}$$

Combining the form of noisy initial state $\tilde{\rho}_{AB^{(1)}}$ with Eqs. (D11)–(D22) and $\beta_1 \geq \beta_2$, Eq. (D1) can be rewritten as

$$\begin{aligned}
\tilde{S}_{\mathrm{CHSH}}^{(k)\mathrm{noisy}} &= \beta_1\gamma_k\mathrm{tr}[\rho_{AB^{(k)}}(A_0\otimes B_0^{(k)})] \\
&\quad + \beta_2\gamma_k\mathrm{tr}[\rho_{AB^{(k)}}(A_1\otimes B_0^{(k)})] \\
&\quad + \beta_1\mu_k\mathrm{tr}[\rho_{AB^{(k)}}(A_0\otimes B_1^{(k)})] \\
&\quad - \beta_2\mu_k\mathrm{tr}[\rho_{AB^{(k)}}(A_1\otimes B_1^{(k)})] \\
&= \nu\Bigg[(\beta_1+\beta_2)\gamma_k(\cos(\theta)(1-c_s^2)+c_s^2) \\
&\quad \times \prod_{j=1}^{k-1}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\mu_j^2\eta_j^2}\right) \\
&\quad + \eta_k\mu_k[c_s^2(\beta_1-\beta_2)+(\beta_1+\beta_2)L\sin(\theta)] \\
&\quad \times \prod_{j=1}^{k-1}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\gamma_j^2}\right)\Bigg] \\
&\quad + \frac{1-\nu}{st}\Bigg[(\beta_1+\beta_2)\gamma_k\prod_{j=1}^{k-1}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\mu_j^2\eta_j^2}\right) \\
&\quad + (\beta_1-\beta_2)\mu_k\eta_k\prod_{j=1}^{k-1}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\gamma_j^2}\right)\Bigg] \\
&\geq 2^{1-k}(\beta_1+\beta_2)\nu\Bigg[\gamma_k\cos(\theta)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\mu_j^2\eta_j^2}\right) \\
&\quad + L\mu_k\eta_k\sin(\theta)\prod_{j=1}^{k-1}\left(1+\sqrt{1-\gamma_j^2}\right)\Bigg], \quad \text{(D23)}
\end{aligned}$$

where $L = 2(c_1c_2 + c_3c_4 + \cdots + c_{2[\frac{s}{2}]-1}c_{2[\frac{s}{2}]})$.

For the case of even (odd) t and odd (even) s, it is analogous to prove inequality (51).

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