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Citation: Chin. Phys. B, 2023, 32 (**12**): 120205. DOI: 10.1088/1674-1056/ad01a5
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Rational solutions of Painlevé-II equation as Gram determinant

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(Received 30 August 2023; revised manuscript received 8 September 2023; accepted manuscript online 10 October 2023)

Under the Flaschka–Newell Lax pair, the Darboux transformation for the Painlevé-II equation is constructed by the limiting technique. With the aid of the Darboux transformation, the rational solutions are represented by the Gram determinant, and then we give the large y asymptotics of the determinant and the rational solutions. Finally, the solution of the corresponding Riemann–Hilbert problem is obtained from the Darboux matrices.

Keywords: Painlevé-II equation, Darboux transformation, rational solutions

PACS: 02.30.Ik, 04.20.Jb, 02.60.Cb **DOI:** 10.1088/1674-1056/ad01a5

1. Introduction

The six Painlevé equations ($P_{\rm I}$–$P_{\rm IV}$) are a class of nonlinear ordinary differential equations which have long been studied. It originated from when the Painlevé school tried to answer Picard's question in Ref. [1] in the late 19th century. Because the general solutions cannot be reduced to elementary or known functions, today it is often regarded as a nonlinear type of special function. Due to transcendency, the solutions to the Painlevé equations are often called the Painlevé transcendents. In the late 1970s, Painlevé equations garnered new attention when they were found to be "integrable", thus not only extending the tools to analyze their solutions, but also expanding the theory of integrability. Although these Painlevé equations were first derived from strictly mathematical considerations, these Painlevé equations have appeared in many physical applications. Perhaps Ref. [2] is the first example of these applications. Subsequently, Painlevé equations were widely used in many aspects of physics. For example, in Ref. [3], the authors showed that the density matrix of a one-dimensional system of impenetrable Bose gases at zero temperature can be precisely represented by the solution of a nonlinear differential equation: the Painlevé transcendence and its generalization. In Ref. [4], the authors gave a detailed derivation by using the Painlevé equation to describe the dynamics of the electrons and holes in a semiconductor. In Ref. [5], by using an auxiliary nonlinear Schrödinger equation, the authors found that the second Painlevé transcendent solution can be used to study the propagation of the optical pulses in single mode dispersive fibers. In addition, the Painlevé equation also has a good application in the light–matter interaction in nematic liquid crystals,[6] the high-energy physics[7] and the planar fluid flow.[8]

We are particularly interested in certain pattern formation related to rogue waves in integrable nonlinear waves and their connection to the Painlevé transcendents. Very recently, the locations of the rogue wave patterns in nonlinear Schrödinger equation and the lump patterns in Kadomtsev–Petviashvili equation are found to be determined by the root structure of the Yablonskii–Vorob'ev polynomials hierarchy.[9–11] It has long been observed in Ref. [12] that the Painlevé-II rational solutions can be represented by the Yablonskii–Vorob'ev polynomials, so the root structure is exactly the pole structure for Painlevé-II rational solutions. This directly inspired us to add to the ample existing studies of Painlevé-II rational solutions from a different perspective.

The general formula of Painlevé-II equation is given by

$$p_{yy} - 2p^3 - \frac{2}{3}py + \frac{2}{3}m = 0. \tag{1}$$

As mentioned, the general solutions of Painlevé equations are transcendental. However, Painlevé-II equation is known to have rational solutions for some special parameters. Indeed, in Ref. [12] Yablonskii first found an iteration formula for the rational solutions with $m \in \mathbb{Z}$ in Eq. (1), and Airault[13] showed that the parameter choice is both necessary and sufficient. Like soliton solutions in integrable systems, the rational solutions can be given by Bäcklund transformations. The Bäcklund transformation for Pailevé-II equation is given by Lukashevich in 1971.[14] In 1985,[15] Murata derived the rational solutions of the second and the fourth Painlevé equation with different integer parameters, and in particular showed the necessary and sufficient conditions for rational solutions to exist. There are numerous studies of rational solutions[13,16–21] through the Bäcklund transformation.

Painlevé equations are considered integrable in the sense that they admit Lax representation. Indeed, Painlevé-II equa-

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tion admits the following Flaschka–Newell Lax representation

$$\begin{aligned}&\frac{\partial \boldsymbol{v}}{\partial \lambda}=\boldsymbol{A}^{\mathrm{FN}}(\lambda;y)\boldsymbol{v},\\&\boldsymbol{A}^{\mathrm{FN}}(\lambda;y):=\begin{bmatrix}-6\mathrm{i}\lambda^2-3\mathrm{i}p^2-\mathrm{i}y & 6p\lambda+3\mathrm{i}p'+m\lambda^{-1}\\ 6p\lambda-3\mathrm{i}p'+m\lambda^{-1} & 6\mathrm{i}\lambda^2+3\mathrm{i}p^2+\mathrm{i}y\end{bmatrix},\\&\frac{\partial \boldsymbol{v}}{\partial y}=\boldsymbol{B}^{\mathrm{FN}}(\lambda;y)\boldsymbol{v},\ \ \boldsymbol{B}^{\mathrm{FN}}(\lambda;y):=\begin{bmatrix}-\mathrm{i}\lambda & p\\ p & \mathrm{i}\lambda\end{bmatrix}.\end{aligned}\tag{2}$$

The compatibility condition

$$\frac{\partial \boldsymbol{A}^{\mathrm{FN}}}{\partial y}-\frac{\partial \boldsymbol{B}^{\mathrm{FN}}}{\partial \lambda}+\left[\boldsymbol{A}^{\mathrm{FN}},\boldsymbol{B}^{\mathrm{FN}}\right]=0$$

yields the Painlevé-II equation (1). Due to integrability, one powerful tool for analysis of the Painlevé equations is the Riemann–Hilbert problems. The book[22] is a particularly comprehensive and systematic documentation of this modern subject. The Riemann–Hilbert problem is also used to study the rogue wave patterns that we are interested in. Specifically in literature,[23] the authors found that in semiclassical focusing nonlinear Schrödinger equation, near a gradient catastrophe point, universal "rogue wave"-like patterns form can be mapped to the poles of the Painlevé-I tritronquée solution. In Ref. [24], the authors analyzed the asymptotics of the infinite order rogue waves, and the asymptotics in the transitional region was related to the Tritronquée solutions of Painlevé-II equation.

There are many ways to derive Painlevé-II rational solutions. One standard method is the Bäcklund transformation, given nearly 50 years ago. However, to the best of our knowledge, few has studied the rational solution of Painlevé equations with the Darboux transformation[25,26] directly. The Darboux transformation for Painlevé-II rational solutions is still an open problem up to now. Observe that the ∂y-equation in the Lax pair (2) is similar to the spectral problem in the AKNS system. It is also shown that the Painlevé-II equation is related the modified Korteweg–de Vries equation via a self-similar transformation.[27–30] Thus it is natural to borrow the idea from the Darboux transformation of AKNS system to construct the solution of the Painlevé-II equation.

In the present work, we modify the Darboux transformation to construct the rational Painlevé-II solutions. Under the same framework, we also derive the rational solution with a Gram determinant representation. With our determinantal formula, we get the asymptotics of large y by expansion. The rational solutions of Painlevé-II equation are often compared to the soliton solutions in integrable nonlinear wave equations, since they can be represented by determinantal formula, and correspond to fully discrete spectral data etc.[29] However, the Flaschka–Newell determinantal formula is different from our Gram determinant formula. Indeed, in our Darboux transformation, all of the spectral data are at the same point so we have to employ limiting technique. The process is much more like deriving the rogue wave solutions in integrable nonlinear waves. The Gram determinant representation is also comparable to rogue waves. This also in a way attests to the connection between the roots of Yablonskii–Vorob'ev polynomial (poles of Painlevé-II rational solutions) and rogue wave patterns found in Refs. [10,11]. However, the transformation for Painlevé-II has even one more difference, the generalized Darboux–Bäcklund transformation we used is not the so called auto-transformation in comparison, which we believe also makes our work even more interesting.

Ultimately we believe to study the rogue wave pattern, Riemann–Hilbert method can give us a better analytical handle, therefore we always have the Riemann–Hilbert problem in mind. In Ref. [24], the authors studied the asymptotics of infinite order rogue waves via the Riemann-Hilbert method, which was first constructed by using the Darboux transformation. Based on this idea, we are able to solve the Painlevé-II Riemann–Hilbert problem given in Ref. [31].

The innovation of this paper consists of the following two points: (i) We derive the rational solutions of Painlevé-II equation via two types of generalized Darboux transformation. In the first case, from the "seed solution" $p=0$, $m=0$, we give the iterated Darboux transformation at the same spectrum $\lambda_1=0$ and derive the corresponding Bäcklund transformation. Compared to some other rational solutions in integrable systems, the derivation of Painlevé-II rational solution is more difficult. To achieve it, we would need to use special limiting techniques: the Darboux transformation is iterated at the same spectrum and the spectrum λ_1 must be chosen at a special point. In the second case, we rewrite the generalized Darboux transformation by another equivalent formula and derive the rational solutions as a Gram determinant. In particular, we can analyze the asymptotics of large y by this formula. (ii) Our generalized Darboux transformation can be used to construct the solution of Riemann–Hilbert problem given in Ref. [31], which provides a new way for solving the corresponding Riemann–Hilbert problem for the Painlevé-II equation.

The outline of this paper is as follows. In Section 2, we give detailed derivation of rational solutions via the generalized Darboux transformation. Then, we also give the corresponding Bäcklund transformation. In Section 3, the generalized Darboux transformation is rewritten as another equivalent formula, which gives the solution in Gram determinant form, and can be used to analyze the asymptotics for large y. In Section 4, we give a brief introduction to the Riemann–Hilbert problem given by Ref. [31] and solve it with the aid of the Darboux transformation. The final part is the conclusion.

2. The Darboux transformation of Painelvé-II equation

In this section, we would like to construct the generalized Darboux transformation of Painlevé-II equation and derive the rational solutions. The rational solutions of Painlevé-II equation play an analogous role as the solitons of the integrable partial differential equations. However, to the best of our knowledge, the rational solutions of Painlevé-II equation have never been constructed by the well known Darboux transformation. From the Lax pair of Eq. (2), we can see that the ∂y-part of Lax pair is the same as the ∂x-equation, what we usually call the Lax equation, in the Lax pair for the defocusing NLS equation,[22] while the $\partial\lambda$-part is different. It has two types of singularity. One is at $\lambda=\infty$ and the other one is at $\lambda=0$. What is more, the $\partial\lambda$-equation contains a constant m, which leads to a new difficulty in the Darboux transformation. To overcome these difficulties, we use two kinds of limiting skills following the original idea of the so-called generalized Darboux transformation.[33,34] We want to emphasize here that there is a difference between the rational solutions of Painlevé-II equation and the higher order rogue wave of integrable equations. The former ones are meromorphic functions with simple poles in the complex plane but the latter ones are global solutions in the (x,t)-plane. The detailed derivation is shown as follows.

To construct the rational solutions, we start with the "seed solution" $p=0$, $m=0$. Since the ∂y-part of Lax pair is the spectral problem of AKNS system, we can set the Darboux transformation matrix as

$$\boldsymbol{T}_1(\lambda;y)=\mathbb{I}_2-\frac{\lambda_1-\lambda_1^*}{\lambda-\lambda_1^*}\frac{\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\dagger}\sigma_3}{\boldsymbol{\phi}_1^{\dagger}\sigma_3\boldsymbol{\phi}_1}, \tag{3}$$

where $\boldsymbol{\phi}_1$ is a special solution of Lax pair Eq. (2) with $\lambda=\lambda_1\in\mathrm{i}\mathbb{R}$, that is $\boldsymbol{\phi}_1=\mathrm{e}^{\left(-\mathrm{i}\lambda_1 y-2\mathrm{i}\lambda_1^3\right)\sigma_3}\boldsymbol{c}$, and $\boldsymbol{c}=\left[c_1,c_2\right]^{\mathrm{T}}$ is a constant vector to be determined. We use σ_3 to denote one of the three Pauli matrices

$$\sigma_1=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \sigma_2=\begin{bmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{bmatrix},\quad \sigma_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}. \tag{4}$$

For convenience, denote the entries of $\boldsymbol{\phi}_j$ $(j=1,2,\ldots)$ and $\boldsymbol{\phi}_j^{[k]}$ $(j=1,2,\ldots,k=1,2,\ldots)$ as follows:

$$\boldsymbol{\phi}_j:=\begin{bmatrix}\phi_{j,1}\\\phi_{j,2}\end{bmatrix},\quad \boldsymbol{\phi}_j^{[k]}:=\begin{bmatrix}\phi_{j,1}^{[k]}\\\phi_{j,2}^{[k]}\end{bmatrix}. \tag{5}$$

The Darboux transformation is given as follows. Let $\boldsymbol{v}_1=\boldsymbol{T}_1(\lambda;y)\boldsymbol{v}$, then $\boldsymbol{v}_1$ satisfies a new Lax pair equation

$$\begin{aligned}
&\frac{\partial\boldsymbol{v}_1}{\partial\lambda}=\boldsymbol{A}_1^{\mathrm{FN}}(\lambda;y)\boldsymbol{v}_1,\\
&\boldsymbol{A}_1^{\mathrm{FN}}(\lambda;y)=\boldsymbol{T}_{1,\lambda}(\lambda;y)\boldsymbol{T}_1^{-1}(\lambda;y)\\
&\qquad+\boldsymbol{T}_1(\lambda;y)\boldsymbol{A}^{\mathrm{FN}}(\lambda;y)\boldsymbol{T}_1^{-1}(\lambda;y),\\
&\frac{\partial\boldsymbol{v}_1}{\partial y}=\boldsymbol{B}_1^{\mathrm{FN}}(\lambda;y)\boldsymbol{v}_1,\\
&\boldsymbol{B}_1^{\mathrm{FN}}(\lambda;y)=\boldsymbol{T}_{1,y}(\lambda;y)\boldsymbol{T}_1^{-1}(\lambda;y)\\
&\qquad+\boldsymbol{T}_1(\lambda;y)\boldsymbol{B}^{\mathrm{FN}}(\lambda;y)\boldsymbol{T}_1^{-1}(\lambda;y).
\end{aligned} \tag{6}$$

Equipped with the Darboux transformation defined above, we know that the original Lax pair (2) is converted to a new one by replacing p,m with $p^{[1]},m_1$. In the transformed Lax pair we use the superscript $[1]$ in $p^{[1]}$ to mean the first order rational solution, and the subscript in m_1 is the constant in correspondence with $p^{[1]}$.

Expanding $\boldsymbol{B}_1^{\mathrm{FN}}(\lambda;y)$ at $\lambda\to\infty$, the first order rational solution $p^{[1]}$ is given by

$$\begin{aligned}
p^{[1]}&=p+\frac{2\mathrm{i}\left(\lambda_1-\lambda_1^*\right)\phi_{1,1}\phi_{1,2}^*}{|\phi_{1,1}|^2-|\phi_{1,2}|^2}\\
&=p+\frac{2\mathrm{i}\left(\lambda_1-\lambda_1^*\right)\phi_{1,1}^*\phi_{1,2}}{|\phi_{1,1}|^2-|\phi_{1,2}|^2},
\end{aligned} \tag{7}$$

which imposes one symmetric constraint for $\boldsymbol{\phi}_1$: $\phi_{1,1}\phi_{1,2}^*=\phi_{1,1}^*\phi_{1,2}$.

In Ref. [34], the authors showed that the ∂y-equation in the new Lax pair is valid, i.e., it has the same form except with new p under this transformation. Therefore we only need to check whether the Darboux transformation also satisfies the $\partial\lambda$-part of Lax pair. Plugging the Darboux matrix Eq. (3) and the original seed solution $p=0,m=0$ into the new Lax pair, we have

$$\begin{aligned}
\boldsymbol{A}_1^{\mathrm{FN}}(\lambda;y)&=\frac{\lambda_1-\lambda_1^*}{\left(\lambda-\lambda_1^*\right)^2}\frac{\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\dagger}\sigma_3}{\boldsymbol{\phi}_1^{\dagger}\sigma_3\boldsymbol{\phi}_1}\left(\mathbb{I}_2-\frac{\lambda_1^*-\lambda_1}{\lambda-\lambda_1}\frac{\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\dagger}\sigma_3}{\boldsymbol{\phi}_1^{\dagger}\sigma_3\boldsymbol{\phi}_1}\right)\\
&\quad-\left(\mathbb{I}_2-\frac{\lambda_1-\lambda_1^*}{\lambda-\lambda_1^*}\frac{\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\dagger}\sigma_3}{\boldsymbol{\phi}_1^{\dagger}\sigma_3\boldsymbol{\phi}_1}\right)\\
&\quad\times\left(6\mathrm{i}\lambda^2\sigma_3+\mathrm{i}y\sigma_3+m\lambda^{-1}\sigma_1\right)\\
&\quad\times\left(\mathbb{I}_2-\frac{\lambda_1^*-\lambda_1}{\lambda-\lambda_1}\frac{\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\dagger}\sigma_3}{\boldsymbol{\phi}_1^{\dagger}\sigma_3\boldsymbol{\phi}_1}\right).
\end{aligned} \tag{8}$$

Expanding Eq. (8) in the neighborhood of $\lambda=\infty$ and comparing the coefficient of the off-diagonal part of Eq. (8), we know that the coefficient of λ^{-1} is

$$m_1:=\frac{4\mathrm{i}\lambda_1\phi_{1,1}\phi_{1,2}^*}{|\phi_{1,1}|^2-|\phi_{1,2}|^2}y+\frac{24\mathrm{i}\lambda_1^3\phi_{1,1}\phi_{1,2}^*}{|\phi_{1,1}|^2-|\phi_{1,2}|^2}, \tag{9}$$

which should be a constant independent of y. Therefore, the spectral parameter λ_1 is chosen to be zero and the entries of $\boldsymbol{\phi}_1$ should satisfy $|\phi_{1,1}|=|\phi_{1,2}|$. To achieve this, we can set the constant vector $\boldsymbol{c}$ as $\boldsymbol{c}=(1,-1)^{\mathrm{T}}$, then $\phi_{1,1}\phi_{1,2}^*=\phi_{1,1}^*\phi_{1,2}$ and $|\phi_{1,1}|=|\phi_{1,2}|$. Note that now both the numerator and

the denominator are zero. Expand for infinitesimal λ_1, we get $m_1 = 1$, and the first order rational solution $p^{[1]}$ is

$$p^{[1]} = \frac{1}{y}. \tag{10}$$

In order to generate the 2nd order rational solution, the second fold Darboux transformation matrix can be set as

$$\boldsymbol{T}_2(\lambda;y) = \mathbb{I} - \frac{\lambda_2 - \lambda_2^*}{\lambda - \lambda_2^*} \frac{\boldsymbol{\phi}_2^{[1]} \boldsymbol{\phi}_2^{[1],\dagger} \sigma_3}{\boldsymbol{\phi}_2^{[1],\dagger} \sigma_3 \boldsymbol{\phi}_2^{[1]}}, \tag{11}$$

where $\boldsymbol{\phi}_2^{[1]}$ is defined as

$$\boldsymbol{\phi}_2^{[1]} = \boldsymbol{T}_1(\lambda_2, y)\boldsymbol{\phi}_2, \tag{12}$$

and $\boldsymbol{\phi}_2 = \mathrm{e}^{\left(-\mathrm{i}\lambda_2 y - 2\mathrm{i}\lambda_2^3\right)\sigma_3}\boldsymbol{c}$. Recall $\boldsymbol{c} = (1,-1)^{\mathrm{T}}$, then $\boldsymbol{v}_2 = \boldsymbol{T}_2(\lambda;y)\boldsymbol{v}_1$ satisfies the new Lax pair

$$\begin{aligned}
&\frac{\partial \boldsymbol{v}_2}{\partial \lambda} = \boldsymbol{A}_2^{\mathrm{FN}}(\lambda;y)\boldsymbol{v}_2,\\
&\boldsymbol{A}_2^{\mathrm{FN}}(\lambda;y) = \boldsymbol{T}_{2,\lambda}(\lambda;y)\boldsymbol{T}_2^{-1}(\lambda;y)\\
&\qquad\qquad + \boldsymbol{T}_2(\lambda;y)\boldsymbol{A}_1^{\mathrm{FN}}(\lambda;y)\boldsymbol{T}_2^{-1}(\lambda;y),\\
&\frac{\partial \boldsymbol{v}_2}{\partial y} = \boldsymbol{B}_2^{\mathrm{FN}}(\lambda;y)\boldsymbol{v}_2,\\
&\boldsymbol{B}_2^{\mathrm{FN}}(\lambda;y) = \boldsymbol{T}_{2,y}(\lambda;y)\boldsymbol{T}_2^{-1}(\lambda;y)\\
&\qquad\qquad + \boldsymbol{T}_2(\lambda;y)\boldsymbol{B}_1^{\mathrm{FN}}(\lambda;y)\boldsymbol{T}_2^{-1}(\lambda;y),
\end{aligned} \tag{13}$$

with new rational solution $p^{[2]}$ and new constant m_2. The off-diagonal term of $\boldsymbol{B}_2^{\mathrm{FN}}$ is exactly $p^{[2]}$, therefore the second order rational solution $p^{[2]}$ is given by

$$p^{[2]} = p^{[1]} + \frac{2\mathrm{i}\left(\lambda_2 - \lambda_2^*\right)\phi_{2,1}^{[1]}\phi_{2,2}^{[1],*}}{\left|\phi_{2,1}^{[1]}\right|^2 - \left|\phi_{2,2}^{[1]}\right|^2}. \tag{14}$$

To find m_2, expand $\boldsymbol{A}_2^{\mathrm{FN}}(\lambda;y)$ at $\lambda \to \infty$. The coefficient of λ^{-1} is

$$\begin{aligned}
m_2 \;=\; & \frac{4\mathrm{i}\lambda_2\phi_{2,1}^{[1]}\phi_{2,2}^{[1],*}}{\left|\phi_{2,1}^{[1]}\right|^2 - \left|\phi_{2,2}^{[1]}\right|^2}\left(3p^{[1],2} + y + 6\lambda_2^2\right)\\
& -6\mathrm{i}\lambda_2 \frac{\left|\phi_{2,1}^{[1]}\right|^2 + \left|\phi_{2,2}^{[1]}\right|^2}{\left|\phi_{2,1}^{[1]}\right|^2 - \left|\phi_{2,2}^{[1]}\right|^2} p_y^{[1]}\\
& +12\lambda_2^2 \frac{\left|\phi_{2,1}^{[1]}\right|^4 - 2\left(\phi_{2,1}^{[1]}\phi_{2,2}^{[1],*}\right)^2 + \left|\phi_{2,2}^{[1]}\right|^4}{\left(\left|\phi_{2,1}^{[1]}\right|^2 - \left|\phi_{2,2}^{[1]}\right|^2\right)^2} p^{[1]} + 1.
\end{aligned} \tag{15}$$

We first look at the first term

$$\frac{4\mathrm{i}\lambda_2\phi_{2,1}^{[1]}\phi_{2,2}^{[1],*}}{\left|\phi_{2,1}^{[1]}\right|^2 - \left|\phi_{2,2}^{[1]}\right|^2}\left(3p^{[1],2} + y + 6\lambda_2^2\right).$$

Substituting Eq. (10) into Eq. (15), $3p^{[1],2} + y + 6\lambda_2^2$ is a function of y. If Eq. (15) is a constant, then the spectral parameter λ_2 must also be zero, i.e., $\lambda_2 = \lambda_1 = 0$. When $\lambda_2 = \lambda_1$, the function $\boldsymbol{\phi}_2^{[1]}$ is degenerate. Thus the Darboux transformation is no longer valid. Following the idea in Ref. [33], we can introduce the generalized Darboux transformation and modify $\boldsymbol{\phi}_2^{[1]}$ as $\boldsymbol{\phi}_1^{[1]}$ with the following relation:

$$\boldsymbol{\phi}_1^{[1]} = \frac{\partial \boldsymbol{\phi}_2^{[1]}}{\partial \lambda_2}\Big|_{\lambda_2 = \lambda_1}. \tag{16}$$

Applying the generalized Darboux transformation, the second order rational solution $p^{[2]}$ can just be given by replacing $\boldsymbol{\phi}_2^{[1]}$ with $\boldsymbol{\phi}_1^{[1]}$.

In fact, if we want to derive the kth order rational solutions for the Painlevé-II equation, the spectral parameters λ_k in the kth iteration will always be zero. We can construct the kth iteration of the Darboux transformation by continuing the above procedures. Begin by setting the kth step Darboux transformation matrix as

$$\boldsymbol{T}_k(\lambda;y) = \mathbb{I} - \frac{\lambda_1 - \lambda_1^*}{\lambda - \lambda_1^*} \frac{\boldsymbol{\phi}_1^{[k-1]} \boldsymbol{\phi}_1^{[k-1],\dagger} \sigma_3}{\boldsymbol{\phi}_1^{[k-1]\dagger} \sigma_3 \boldsymbol{\phi}_1^{[k-1]}}, \tag{17}$$

where $\boldsymbol{\phi}_1^{[k-1]}$ is defined as

$$\boldsymbol{\phi}_1^{[k-1]} = \frac{\partial^{k-1}\left(\boldsymbol{T}_{k-1}(\lambda_1,\lambda_k;y)\cdots\boldsymbol{T}_2(\lambda_1,\lambda_k;y)\boldsymbol{T}_1(\lambda_1,\lambda_k;y)\boldsymbol{\phi}_k\right)}{\partial(\lambda_k)^{k-1}}\Big|_{\lambda_k=\lambda_1}, \tag{18}$$

with $\boldsymbol{\phi}_k = \mathrm{e}^{\left(-\mathrm{i}\lambda_k y - 2\mathrm{i}\lambda_k^3\right)\sigma_3}\boldsymbol{c}$. Then the kth order rational solution $p^{[k]}$ is given by

$$p^{[k]} = p^{[k-1]} + \frac{2\mathrm{i}\left(\lambda_1 - \lambda_1^*\right)\phi_{1,1}^{[k-1]}\phi_{1,2}^{[k-1],*}}{\left|\phi_{1,1}^{[k-1]}\right|^2 - \left|\phi_{1,2}^{[k-1]}\right|^2}. \tag{19}$$

Correspondingly, the kth transformed Lax pair becomes

$$\begin{aligned}
&\frac{\partial \boldsymbol{v}_k}{\partial \lambda} = \boldsymbol{A}_k^{\mathrm{FN}}(\lambda;y)\boldsymbol{v}_k,\\
&\boldsymbol{A}_k^{\mathrm{FN}}(\lambda;y) = \boldsymbol{T}_{k,\lambda}(\lambda;y)\boldsymbol{T}_k^{-1}(\lambda;y)\\
&\qquad\qquad + \boldsymbol{T}_k(\lambda;y)\boldsymbol{A}_{k-1}^{\mathrm{FN}}(\lambda;y)\boldsymbol{T}_k^{-1}(\lambda;y),\\
&\frac{\partial \boldsymbol{v}_k}{\partial y} = \boldsymbol{B}_k^{\mathrm{FN}}(\lambda;y)\boldsymbol{v}_k,\\
&\boldsymbol{B}_k^{\mathrm{FN}}(\lambda;y) = \boldsymbol{T}_{k,y}(\lambda;y)\boldsymbol{T}_k^{-1}(\lambda;y)\\
&\qquad\qquad + \boldsymbol{T}_k(\lambda;y)\boldsymbol{B}_{k-1}^{\mathrm{FN}}(\lambda;y)\boldsymbol{T}_k^{-1}(\lambda;y).
\end{aligned} \tag{20}$$

Similar as above, the coefficient of λ^{-1} in the off-diagonal elements in $\boldsymbol{A}_k^{\mathrm{FN}}(\lambda;y)$ is

$$m_k \;=\; 2\mathrm{i}\left(\lambda_1 - \lambda_1^*\right)\frac{\phi_{1,1}^{[k-1]}\phi_{1,2}^{[k-1],*}}{\left|\phi_{1,1}^{[k-1]}\right|^2 - \left|\phi_{1,2}^{[k-1]}\right|^2}\left[3\left(p^{[k-1]}\right)^2 + y\right]$$

$$-3\mathrm{i}\,(\lambda_1-\lambda_1^*)\frac{\left|\phi_{1,1}^{[k-1]}\right|^2+\left|\phi_{1,2}^{[k-1]}\right|^2}{\left|\phi_{1,1}^{[k-1]}\right|^2-\left|\phi_{1,2}^{[k-1]}\right|^2}p_y^{[k-1]}+m_{k-1}. \quad (21)$$

Substituting Eq. (19) into Eq. (21), we have

$$\begin{aligned} m_k =& -3\mathrm{i}\,(\lambda_1-\lambda_1^*)\frac{\left|\phi_{1,1}^{[k-1]}\right|^2+\left|\phi_{1,2}^{[k-1]}\right|^2}{\left|\phi_{1,1}^{[k-1]}\right|^2-\left|\phi_{1,2}^{[k-1]}\right|^2}p_y^{[k-1]} \\ &+\left(p^{[k]}-p^{[k-1]}\right)\left[3\left(p^{[k-1]}\right)^2+y\right]+m_{k-1}. \end{aligned} \quad (22)$$

A transformation from $p^{[k-1]}$ to $p^{[k]}$ is usually called a Bäcklund tranfsformation, therefore Eq. (22) can be regarded as an incomplete Bäcklund transformation, with the coefficient of $p_y^{[k-1]}$ to be determined. If in Eq. (22), the coefficient of $p_y^{[k-1]}$ can also be written as a relation between $p^{[k]}$ and $p^{[k-1]}$, then Eq. (22) will complete our transformation. Next, we give a lemma about the symmetry of $\phi_1^{[k-1]}$, which will help us complete the Bäcklund transformation.

Lemma 1 Let $\phi_k=\mathrm{e}^{\left(-\mathrm{i}\lambda_k y-2\mathrm{i}\lambda_k^3\right)\sigma_3}\boldsymbol{c}$, and the constant vector $\boldsymbol{c}=(1,-1)^{\mathrm{T}}$. Define the vector $\phi_k^{[k-1]}$ as

$$\begin{aligned} &\phi_k^{[k-1]}=\boldsymbol{T}_{k-1}(\lambda_1,\lambda_k;y)\cdots\boldsymbol{T}_1(\lambda_1,\lambda_k;y)\boldsymbol{T}_0(\lambda_1,\lambda_k;y)\phi_k, \\ &(k=1,2,\ldots),\ \ \boldsymbol{T}_0(\lambda_1,\lambda_k;y)=\mathbb{I}_2. \end{aligned} \quad (23)$$

Then for $\lambda_i,\ \lambda_k\in\mathrm{i}\mathbb{R}$, $\phi_k^{[k-1]}$ satisfies the following symmetry relation:

$$\begin{aligned} \phi_{k,1}^{[k-1]}(\lambda_1,\lambda_k;y) &= \left[\phi_{k,1}^{[k-1]}(\lambda_1,\lambda_k;y)\right]^* \\ &= -\left[\phi_{k,2}^{[k-1]}(\lambda_1^*,\lambda_k^*;y)\right]^*, \\ \phi_{k,2}^{[k-1]}(\lambda_1,\lambda_k;y) &= \left[\phi_{k,2}^{[k-1]}(\lambda_1,\lambda_k);y\right]^*. \end{aligned} \quad (24)$$

In comparison, the vector $\phi_1^{[k-1]}$ in generalized Darboux transformation defined in Eq. (18) satisfies a different kind of symmetry relation

$$\begin{aligned} \phi_{1,1}^{[k-1]}(\lambda_1;y) &= (-1)^{k-1}\phi_{1,1}^{[k-1],*}(\lambda_1;y) \\ &= (-1)^k\phi_{1,2}^{[k-1]}(\lambda_1^*;y)=-\phi_{1,2}^{[k-1],*}(\lambda_1^*;y), \end{aligned} \quad (25)$$

with $\lambda_1\in\mathrm{i}\mathbb{R}$.

Proof Using induction. When $k=1,\lambda_1=0$, $\phi_1=\left(\mathrm{e}^{-\mathrm{i}\lambda_1 y-2\mathrm{i}\lambda_1^3},-\mathrm{e}^{\mathrm{i}\lambda_1 y+2\mathrm{i}\lambda_1^3}\right)^{\mathrm{T}}$ immediately satisfies the symmetry relation Eqs. (24) and (25). Supposing that these symmetries are valid under the $(k-1)$th step Darboux transformation, we only need to check that in the kth step, these symmetries are still valid. To prove it, we rewrite Eq. (23) as

$$\begin{aligned} &\phi_k^{[k-1]}=\boldsymbol{T}_{k-1}(\lambda_1,\lambda_k;y)\widehat{\phi}_k^{[k-2]}, \\ &\widehat{\phi}_k^{[k-2]}:=\boldsymbol{T}_{k-2}(\lambda_1,\lambda_k;y)\cdots\boldsymbol{T}_1(\lambda_1,\lambda_k;y)\phi_k, \end{aligned} \quad (26)$$

where $\boldsymbol{T}_{k-1}(\lambda_1,\lambda_k;y)=\mathbb{I}_2-\frac{\lambda_1-\lambda_1^*}{\lambda_k-\lambda_1^*}\frac{\phi_1^{[k-2]}\phi_1^{[k-2],\dagger}\sigma_3}{\phi_1^{[k-2],\dagger}\sigma_3\phi_1^{[k-2]}}$, $\phi_1^{[k-2]}=\left.\frac{\partial^{k-2}(\boldsymbol{T}_{k-2}(\lambda_1,\lambda_{k-1};y)\cdots\boldsymbol{T}_1(\lambda_1,\lambda_{k-1};y)\phi_{k-1})}{\partial(\lambda_{k-1})^{k-2}}\right|_{\lambda_{k-1}=\lambda_1}$. By a simple calculation, $\phi_k^{[k-1]}$ is equal to

$$\phi_k^{[k-1]}=\widehat{\phi}_k^{[k-2]}-\left(\frac{\lambda_1-\lambda_1^*}{\lambda_k-\lambda_1^*}\frac{\phi_1^{[k-2],\dagger}\sigma_3\widehat{\phi}_k^{[k-2]}}{\phi_1^{[k-2],\dagger}\sigma_3\phi_1^{[k-2]}}\right)\phi_1^{[k-2]}. \quad (27)$$

It is clear that if $\widehat{\phi}_k^{[k-2]}$ and $\phi_1^{[k-2]}$ satisfy the symmetry relation Eqs. (24) and (25) respectively under this assumption, then $\phi_k^{[k-1]}$ will also satisfy the symmetry Eq. (24) when $\lambda_k\in\mathrm{i}\mathbb{R}$.

Next, we give the proof of Eq. (25). From the definition of $\phi_1^{[k-1]}$ in Eq. (18), we know that the entries of $\phi_1^{[k-1]}$ can be written as

$$\begin{aligned} \phi_{1,1}^{[k-1]}(\lambda_1;y) &= \left.\frac{\partial^{k-1}\phi_{k,1}^{[k-1]}(\lambda_1,\lambda_k;y)}{\partial(\lambda_k)^{k-1}}\right|_{\lambda_k=\lambda_1}, \\ \phi_{1,1}^{[k-1],*}(\lambda_1;y) &= \left.\frac{\partial^{k-1}\phi_{k,1}^{[k-1],*}(\lambda_1,\lambda_k;y)}{\partial(\lambda_k^*)^{k-1}}\right|_{\lambda_k^*=\lambda_1^*}, \\ \phi_{1,2}^{[k-1]}(\lambda_1^*;y) &= \left.\frac{\partial^{k-1}\phi_{k,2}^{[k-1]}(\lambda_1^*,\lambda_k^*;y)}{\partial(\lambda_k)^{k-1}}\right|_{\lambda_k=\lambda_1}, \\ \phi_{1,2}^{[k-1],*}(\lambda_1^*;y) &= \left.\frac{\partial^{k-1}\phi_{k,2}^{[k-1],*}(\lambda_1^*,\lambda_k^*;y)}{\partial(\lambda_k^*)^{k-1}}\right|_{\lambda_k^*=\lambda_1^*}, \end{aligned} \quad (28)$$

which indicates $\phi_{1,1}^{[k-1]}(\lambda_1;y)=(-1)^{k-1}\phi_{1,1}^{[k-1],*}(\lambda_1;y)=(-1)^k\phi_{1,2}^{[k-1]}(\lambda_1^*;y)=-\phi_{1,2}^{[k-1],*}(\lambda_1^*;y)$, it completes the proof.

Based on the symmetry relation in Lemma 1, when $\lambda_1=0$, we have

$$\begin{aligned} \phi_{1,1}^{[k-1]}(0;y) &= (-1)^{k-1}\phi_{1,1}^{[k-1],*}(0;y) \\ &= (-1)^k\phi_{1,2}^{[k-1]}(0;y)=-\phi_{1,2}^{[k-1],*}(0;y), \end{aligned} \quad (29)$$

substituting Eq. (29) into Eq. (22), we can get an explicit relation between $p^{[k]}$ and $p^{[k-1]}$,

$$\begin{aligned} &\left(p^{[k]}-p^{[k-1]}\right)\left((-1)^{k-1}3p_y^{[k-1]}+3\left(p^{[k-1]}\right)^2+y\right) \\ &+m_{k-1}=m_k. \end{aligned} \quad (30)$$

Then the Bäcklund transformation between $p^{[k]}$ and $p^{[k-1]}$ is

$$p^{[k]}=p^{[k-1]}+\frac{m_k-m_{k-1}}{(-1)^{k-1}3p_y^{[k-1]}+3\left(p^{[k-1]}\right)^2+y}. \quad (31)$$

Summarizing the above calculation, we obtain rational solutions of all orders. In Ref. [13], Airault pointed out that $m\in\mathbb{Z}$ is the necessary and sufficient condition for Painlevé-II solution to be rational. By uniqueness, we conclude all general rational solutions are given by our generalized Darboux transform in Theorem 1.

Theorem 1 Setting $\lambda_1=-\lambda_1^*=0$, the generalized Darboux transformation matrices for the Painlevé-II equation (1) are given by

$$\boldsymbol{T}^{[k]}(\lambda;y)=\boldsymbol{T}_k(\lambda;y)\cdots\boldsymbol{T}_2(\lambda;y)\boldsymbol{T}_1(\lambda;y), \quad (32)$$

where

$$\boldsymbol{T}_k(\lambda;y)=\mathbb{I}-\frac{\lambda_1-\lambda_1^*}{\lambda-\lambda_1^*}\frac{\boldsymbol{\phi}_1^{[k-1]}\boldsymbol{\phi}_1^{[k-1],\dagger}\sigma_3}{\boldsymbol{\phi}_1^{[k-1],\dagger}\sigma_3\boldsymbol{\phi}_1^{[k-1]}}, \tag{33}$$

and $\boldsymbol{\phi}_1^{[k-1]}$ is defined in Eq. (18). By expanding $\boldsymbol{T}^{[k]}(\lambda;y)$ in neighborhood of $\lambda=\infty$, the rational solutions of Painlevé-II Eq. (1) can be given as

$$\begin{aligned}p^{[k]}(y) &= 2\mathrm{i}\lim_{\lambda\to\infty}\lambda\boldsymbol{T}^{[k]}(\lambda;y)_{12}\\ &= p+\sum_{l=0}^{k-1}\frac{2\mathrm{i}\,(\lambda_1-\lambda_1^*)\,\phi_{1,1}^{[l]}\phi_{1,2}^{[l],*}}{\left|\phi_{1,1}^{[l]}\right|^2-\left|\phi_{1,2}^{[l]}\right|^2},\end{aligned} \tag{34}$$

where p is the seed solution $p=0$ in Eq. (3).

We remark on the specificity of Painlevé-II rational solutions in the transformation.

Remark 1 It is clear that each Painlevé-II equation (1) involves one constant m, and that from our transformation in each iteration, we take a different value of m_k. Therefore, the Bäcklund transformation between two rational solutions is not the auto-Bäcklund transformation. For each order of the rational solutions, the constants m_k differ. This makes our Bäcklund transformation different than what usually appears in integrable partial differential equations.

We derived the Bäcklund transformation (31) through generalized Darboux transformation. In fact, apart from this method, we can also give a Bäcklund transformation for $p^{[k]}$ from Eq. (1) itself. Before discussing it, we first give a proposition about the Schwarzian derivative.

Proposition 1 The Schwarzian derivative $S(\sigma)=\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\sigma_{yy}}{\sigma_y}\right)-\frac{1}{2}\left(\frac{\sigma_{yy}}{\sigma_y}\right)^2$ [35] is invariant under the Möbius group $\sigma=\frac{a\eta+b}{c\eta+d}$, where $ad-bc\neq 0$.

Proof From the Möbius transformation $\sigma=\frac{a\eta+b}{c\eta+d}$, we have

$$c\sigma\eta+d\sigma-a\eta-b=0. \tag{35}$$

Taking the derivative of Eq. (35) with respect to y for once, twice and thrice respectively, then we have

$$\begin{bmatrix}(\sigma\eta)_y & \sigma_y & -\eta_y\\ (\sigma\eta)_{yy} & \sigma_{yy} & -\eta_{yy}\\ (\sigma\eta)_{yyy} & \sigma_{yyy} & -\eta_{yyy}\end{bmatrix}\begin{bmatrix}c\\ d\\ a\end{bmatrix}=0, \tag{36}$$

which indicates

$$\begin{vmatrix}(\sigma\eta)_y & \sigma_y & -\eta_y\\ (\sigma\eta)_{yy} & \sigma_{yy} & -\eta_{yy}\\ (\sigma\eta)_{yyy} & \sigma_{yyy} & -\eta_{yyy}\end{vmatrix}=0. \tag{37}$$

Through a simple calculation, we can get the relation between σ and η,

$$2\sigma_y\eta_y^2\sigma_{yyy}-3\eta_y^2\sigma_{yy}^2-2\eta_y\sigma_y^2\eta_{yyy}+3\sigma_y^2\eta_{yy}^2=0,$$

$$\Rightarrow\ \frac{2\sigma_{yyy}}{\sigma_y}-\frac{3}{2}\left(\frac{\sigma_{yy}}{\sigma_y}\right)^2=\frac{2\eta_{yyy}}{\eta_y}-\frac{3}{2}\left(\frac{\eta_{yy}}{\eta_y}\right)^2, \tag{38}$$

which is exactly the Schwarzian derivative. This completes the proof.

Next, we give the Bäcklund transformation in another way following the idea in Ref. [36]. For convenience, we rewrite Eq. (1) in equivalent form

$$\left(\frac{\mathrm{d}}{\mathrm{d}y}+2p\right)\left(\frac{\mathrm{d}p}{\mathrm{d}y}-p^2\right)-\frac{2}{3}py+\frac{2}{3}m=0. \tag{39}$$

Setting $p=\frac{1}{2}\frac{\sigma_{yy}}{\sigma_y}$ and substituting into Eq. (39), we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}y}+\frac{\sigma_{yy}}{\sigma_y}\right)\left[\frac{1}{2}\frac{\sigma_{yyy}}{\sigma_y}-\frac{3}{4}\left(\frac{\sigma_{yy}}{\sigma_y}\right)^2\right]-\frac{1}{3}\frac{\sigma_{yy}}{\sigma_y}y+\frac{2}{3}m=0. \tag{40}$$

Notice $\frac{1}{2}\frac{\sigma_{yyy}}{\sigma_y}-\frac{3}{4}\left(\frac{\sigma_{yy}}{\sigma_y}\right)^2$ is the Schwarzian derivative of σ, which is invariant under the Möbius transformation (Proposition 1). Thus, it is natural to introduce the Möbius transformation group $\sigma=\frac{a\eta+b}{c\eta+d}$, $ad-bc\neq 0$, and substitute it into Eq. (40),

$$\begin{aligned}&\left(\frac{\mathrm{d}}{\mathrm{d}y}+\frac{\eta_{yy}}{\eta_y}-\frac{2c\eta_y}{c\eta+d}\right)\left[\frac{1}{2}\frac{\eta_{yyy}}{\eta_y}-\frac{3}{4}\left(\frac{\eta_{yy}}{\eta_y}\right)^2\right]\\ &-\frac{1}{3}\left(\frac{\eta_{yy}}{\eta_y}-\frac{2c\eta_y}{c\eta+d}\right)y+\frac{2}{3}m=0.\end{aligned} \tag{41}$$

Moreover, imposing that Eq. (41) satisfies the transformed Painlevé-II equation, i.e., it has the same form as Eq. (40), except with a different constant $\tilde{m}$,

$$\begin{aligned}&\left(\frac{\mathrm{d}}{\mathrm{d}y}+\frac{\eta_{yy}}{\eta_y}\right)\left[\frac{1}{2}\frac{\eta_{yyy}}{\eta_y}-\frac{3}{4}\left(\frac{\eta_{yy}}{\eta_y}\right)^2\right]\\ &-\frac{1}{3}\left(\frac{\eta_{yy}}{\eta_y}\right)y+\frac{2}{3}\tilde{m}=0,\end{aligned} \tag{42}$$

then the identity between the difference of two Painlevé-II solutions, $\frac{1}{2}\frac{\sigma_{yy}}{\sigma_y}-\frac{1}{2}\frac{\eta_{yy}}{\eta_y}=-\frac{c\eta_y}{c\eta+d}$, can be regarded as the Bäcklund transformation for Painlevé-II solutions with respect to different constants m and $\tilde{m}$. Subtracting Eq. (42) from Eq. (41), we have

$$\begin{aligned}&\left(\frac{2c\eta_y}{c\eta+d}\right)\left[\frac{1}{2}\frac{\eta_{yyy}}{\eta_y}-\frac{3}{4}\left(\frac{\eta_{yy}}{\eta_y}\right)^2\right]\\ &-\frac{2}{3}\frac{c\eta_y}{c\eta+d}y+\frac{2}{3}(\tilde{m}-m)=0.\end{aligned} \tag{43}$$

Equivalently,

$$\frac{1}{2}\frac{\eta_{yyy}}{\eta_y}-\frac{3}{4}\left(\frac{\eta_{yy}}{\eta_y}\right)^2=\frac{(m-\tilde{m})(c\eta+d)}{3c\eta_y}+\frac{1}{3}y. \tag{44}$$

Next, substitute Eq. (44) into Eq. (42),

$$\left(\frac{\mathrm{d}}{\mathrm{d}y}+\frac{\eta_{yy}}{\eta_y}\right)\left(\frac{(m-\tilde{m})(c\eta+d)}{3c\eta_y}+\frac{1}{3}y\right)$$

$$-\frac{1}{3}\frac{\eta_{yy}}{\eta_y}y+\frac{2}{3}\tilde{m}=0. \tag{45}$$

Direct computation shows the relation between m and $\tilde{m}$, that is $m=-1-\tilde{m}$. Plugging $\tilde{p}=\frac{1}{2}\frac{\eta_{yy}}{\eta_y}$ into Eq. (43), the Bäcklund transformation factor $\frac{-c\eta_y}{c\eta+d}$ is thus given by

$$\frac{-c\eta_y}{c\eta+d}=p-\tilde{p}=-\frac{(1+2\tilde{m})}{-3\tilde{p}_y+3\tilde{p}^2+y}. \tag{46}$$

Furthermore, from this Bäcklund transformation (46) and the discrete symmetry $(p(y),m)\to(-p(y),-m)$, the Bäcklund transformation about the Painlevé-II equation can be given by

$$p=\tilde{p}-\frac{2\tilde{m}\pm1}{\mp3\tilde{p}_y+3\tilde{p}^2+y},\quad m=\mp1-\tilde{m}. \tag{47}$$

In the following section, we will prove in Theorem 2 that the constant m_k corresponding to the kth order rational solution $p^{[k]}$ is $m_k=(-1)^{k-1}k$. If we set $(-1)^{k-2}p^{[k-1]}=\tilde{p}$, $(-1)^{k-2}p^{[k]}=p$, then the two Bäcklund transformations Eqs. (31) and (46) are equivalent.

3. The asymptotics for large y

In the last section, through generalized Darboux transformations (32), we obtain the rational solutions in Eq. (34). The Bäcklund transformation (31) is also directly given from iteration. In this section, we will continue to discuss some properties of the rational solutions of Painlevé-II equation. So far, the rational solutions in Eq. (34) are in abstract form, which does not lend itself to easy analysis of their properties. In this section, following the idea in Ref. [37], we convert the generalized Darboux transformation (32) into another equivalent formulation and rewrite the rational solutions as a Gram determinant. A great benefit of this type of formula is that it can be easily used to analyze the asymptotics for large y. The detailed calculation is shown as follows in Corollary 1.

Corollary 1 The generalized Darboux transformation matrices (32) can be rewritten as the following equivalent formula:

$$\boldsymbol{T}^{[k]}(\lambda;y)=\mathbb{I}-\boldsymbol{Y}_k\boldsymbol{M}^{-1}\boldsymbol{D}\boldsymbol{Y}_k^{\dagger}\sigma_3, \tag{48}$$

where

$$\boldsymbol{Y}_k=\left[\phi^{[0]},\phi^{[1]},\ldots,\phi^{[k-1]}\right],\quad \boldsymbol{D}=\begin{bmatrix}\frac{1}{\lambda} & 0 & \cdots & 0\\ \frac{1}{\lambda^2} & \frac{1}{\lambda} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ \frac{1}{\lambda^k} & \frac{1}{\lambda^{k-1}} & \cdots & \frac{1}{\lambda}\end{bmatrix}, \tag{49}$$

with $\phi^{[l]}=\frac{1}{l!}\frac{\partial^l\phi_1}{\partial\lambda_1^l}\Big|_{\lambda_1=0}$, and the elements M_{ij} in $\boldsymbol{M}$ are defined by expanding $\frac{\phi_1^{\dagger}\sigma_3\phi_1}{\lambda_1-\lambda_1^*}$ at $\lambda_1=-\lambda_1^*=0$, that is

$$\begin{aligned}\frac{\phi_1^{\dagger}\sigma_3\phi_1}{\lambda_1-\lambda_1^*} &= \frac{-2\mathrm{i}\sin\left[(\lambda_1-\lambda_1^*)\left(2\lambda_1^2+2\lambda_1^{*,2}+2\lambda_1\lambda_1^*+y\right)\right]}{\lambda_1-\lambda_1^*}\\ &= \sum_{l=1}^{[\frac{k+1}{2}]}\frac{(-2\mathrm{i})(-1)^{l-1}(\lambda_1-\lambda_1^*)^{2l-2}\left(2\lambda_1^2+2\lambda_1^{*,2}+2\lambda_1\lambda_1^*+y\right)^{2l-1}}{(2l-1)!}+\mathscr{O}(\lambda_1-\lambda_1^*)^{2\left[\frac{k+1}{2}\right]}\\ &:= \sum_{i=1}^{k}\sum_{j=1}^{k}M_{ij}\lambda_1^{i-1}\lambda_1^{*,j-1}+\mathscr{O}(\lambda_1^{k+1},\lambda_1^{*,k+1}).\end{aligned} \tag{50}$$

Then the kth order rational solutions of Painlevé-II equation can be given by

$$p^{[k]}=2\mathrm{i}\frac{\det(\boldsymbol{G})}{\det(\boldsymbol{M})}, \tag{51}$$

where $\boldsymbol{G}=\begin{bmatrix}\boldsymbol{M} & Y_{k,2}^{+}\\ -Y_{k,1} & 0\end{bmatrix}$, and $Y_{k,j}$ $(j=1,2)$ represents the jth row of $\boldsymbol{Y}_k$.

Proof 3 A simple way to obtain the generalized Darboux transformation is by using the limiting technique. We use an alternative form of the k-fold Darboux transformation matrix, see[37]

$$\widehat{\boldsymbol{T}}^{[k]}(\lambda;y)=\mathbb{I}+\widehat{\boldsymbol{Y}}\widehat{\boldsymbol{M}}^{-1}\widehat{\boldsymbol{D}}^{-1}\widehat{\boldsymbol{Y}}^{\dagger}\sigma_3, \tag{52}$$

where $\widehat{\boldsymbol{Y}}=[\phi_1,\phi_2,\ldots,\phi_k]$, ϕ_k is the special solution of Lax pair Eq. (2), $\phi_k=\mathrm{e}^{(-\mathrm{i}\lambda_k y-2\mathrm{i}\lambda_k^3)\sigma_3}(1,-1)^{\mathrm{T}}$, with $\lambda=\lambda_k\in\mathrm{i}\mathbb{R}$, and $\widehat{\boldsymbol{M}}=\left(\frac{\phi_i^{\dagger}\sigma_3\phi_j}{\lambda_i^*-\lambda_j}\right)_{1\le i,j\le k}$, $\widehat{\boldsymbol{D}}=\mathrm{diag}\left(\lambda-\lambda_1^*,\lambda-\lambda_2^*,\ldots,\lambda-\lambda_k^*\right)$.

Recall the rational solutions of Painlevé-II equation is recovered by Eq. (34). We have proven that all the spectral parameters λ_i $(i=1,2,\ldots,k)$ are equal to 0, so we set the spectral parameters $\lambda_2,\lambda_3,\ldots,\lambda_k$ as

$$\begin{aligned}&\lambda_2=\varepsilon_2,\ \ \lambda_3=\varepsilon_3,\ \ \ldots,\ \lambda_k=\varepsilon_k,\ \ \phi_2=\phi_1(\varepsilon_2),\\ &\phi_3=\phi_1(\varepsilon_3),\ \ \phi_k=\phi_1(\varepsilon_k).\end{aligned} \tag{53}$$

Directly substitute the above to the Darboux matrix (52) and take the limit $\varepsilon_i\to0$, $i=2,3,\ldots,k$, then we can get the formula (48). Correspondingly, the rational solution of Painlevé-

II equation can be given with the following formula:

$$p^{[k]} = 2\mathrm{i}\lim_{\lambda\to\infty}\lambda \boldsymbol{T}_{12}^{[k]}(\lambda;y), \tag{54}$$

which equals to Eq. (51), it completes the proof.

Apparently the rational solutions in Eqs. (34) and (51) have different forms, each of which has its own advantage. The former can be used to derive the Bäcklund transformation, to which the latter does not have an obvious connection. However, the latter writes all rational Painlevé-II solution in a compact formula. Moreover, the formula is made up of determinants, which renders analytical properties for large y asymptotics very accessible. Using the new representation, we find the constant m_k in Eq. (1) for the kth order rational solution $p^{[k]}$, thus also completing the Bäcklund transformation Eq. (31).

Theorem 2 The constant m_k in the Painlevé-II equation Eq. (1), corresponding to the kth order rational solutions $p^{[k]}$ in Eq. (51) is equal to $(-1)^{k-1}k$.

Proof Following the idea in Ref. [38], we first expand $p^{[k]}$ Eq. (51) at the neighborhood of $y=\infty$ and substitute this series into the Painlevé-II equation (1). Then the constant m_k can be calculated by comparing the coefficients of y polynomials. Clearly, the rational solution Eq. (51) is the quotient of two Gram determinants, thus we can look at the denominator and the numerator separately. Firstly, we discuss the asymptotics of the denominator $\det(\boldsymbol{M})$. From the definition of M_{ij} in Eq. (50), we know that while M_{ij} does not generally admit simple formula, the leading behavior for large y is quite easy to compute:

$$M_{ij} = \begin{cases} (-2\mathrm{i}y)(-1)^{\frac{i-j}{2}}\dfrac{y^{i+j-2}}{(i-1)!(j-1)!(i+j-1)} + \mathscr{O}(y^{i+j-5}), & i+j \text{ is even}, \\ 0, & i+j \text{ is odd}. \end{cases} \tag{55}$$

Based on this asymptotical expression in Eq. (55), when y is large and k is even, the denominator $\det(\boldsymbol{M})$ can be expanded in the following formula:

$$\det(\boldsymbol{M}) = \begin{vmatrix} -2\mathrm{i}y & 0 & \cdots & \frac{-2\mathrm{i}y(-\mathrm{i}y)^{k-2}}{(k-2)!(k-1)} & 0 \\ 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^2}{3} & \cdots & 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^k}{(k-1)!(k+1)} \\ \frac{-2\mathrm{i}y(-\mathrm{i}y)^2}{2!3} & 0 & \cdots & \frac{-2\mathrm{i}y(-\mathrm{i}y)^k}{2!(k-2)!(k+1)} & 0 \\ 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^4}{3!5} & \cdots & 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^{k+2}}{3!(k-1)!(k+3)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{-2\mathrm{i}y(-\mathrm{i}y)^{k-2}}{(k-2)!(k-1)} & 0 & \cdots & \frac{-2\mathrm{i}y(-\mathrm{i}y)^{2k-4}}{(k-2)!(k-2)!(2k-3)} & 0 \\ 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^k}{(k-1)!(k+1)} & \cdots & 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^{2k-2}}{(k-1)!(k-1)!(2k-1)} \end{vmatrix} + \mathscr{O}(y^{k^2-1})$$

$$= \begin{vmatrix} -2\mathrm{i}y & \cdots & \frac{-2\mathrm{i}y(-\mathrm{i}y)^{k-2}}{(k-2)!(k-1)} & 0 & \cdots & 0 \\ \frac{-2\mathrm{i}y(-\mathrm{i}y)^2}{2!3} & \cdots & \frac{-2\mathrm{i}y(-\mathrm{i}y)^k}{2!(k-2)!(k+1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{-2\mathrm{i}y(-\mathrm{i}y)^{k-2}}{(k-2)!(k-1)} & \cdots & \frac{-2\mathrm{i}y(-\mathrm{i}y)^{2k-4}}{((k-2)!)^2(2k-3)} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^2}{3} & \cdots & \frac{2\mathrm{i}y(-\mathrm{i}y)^k}{(k-1)!(k+1)} \\ 0 & \cdots & 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^4}{3!5} & \cdots & \frac{2\mathrm{i}y(-\mathrm{i}y)^{k+2}}{3!(k-1)!(k+3)} \\ 0 & \cdots & 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^k}{(k-1)!(k+1)} & \cdots & \frac{2\mathrm{i}y(-\mathrm{i}y)^{2k-2}}{((k-1)!)^2(2k-1)} \end{vmatrix} + \mathscr{O}(y^{k^2-1})$$

$$= \frac{2^k(-\mathrm{i}y)^{k^2}}{(2!3!\cdots(k-1)!)^2}\begin{vmatrix} 1 & \frac{1}{3} & \cdots & \frac{1}{k-1} & 0 & \cdots & \cdots & 0 \\ \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{k+1} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{k-1} & \frac{1}{k+1} & \cdots & \frac{1}{2k-3} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & -\frac{1}{3} & -\frac{1}{5} & \cdots & -\frac{1}{k+1} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{5} & -\frac{1}{7} & \cdots & -\frac{1}{k+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & -\frac{1}{k+1} & -\frac{1}{k+3} & \cdots & -\frac{1}{2k-1} \end{vmatrix} + \mathscr{O}(y^{k^2-1}). \tag{56}$$

Similarly, when y is large, the numerator $\det(\boldsymbol{G})$ can be expanded as

$$\det(\boldsymbol{G}) = \begin{vmatrix} -2\mathrm{i}y & \cdots & \frac{-2\mathrm{i}y(-\mathrm{i}y)^{k-2}}{(k-2)!(k-1)} & 0 & \cdots & 0 & 1 \\ \frac{-2\mathrm{i}y(-\mathrm{i}y)^{2}}{2!3} & \cdots & \frac{-2\mathrm{i}y(-\mathrm{i}y)^{k}}{2!(k-2)!(k+1)} & 0 & \cdots & 0 & \frac{(-\mathrm{i}y)^{2}}{2!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{-2\mathrm{i}y(-\mathrm{i}y)^{k-2}}{(k-2)!(k-1)} & \cdots & \frac{-2\mathrm{i}y(-\mathrm{i}y)^{2k-4}}{((k-2)!)^2(2k-3)} & 0 & \cdots & 0 & \frac{(-\mathrm{i}y)^{k-2}}{(k-2)!} \\ 0 & \cdots & 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^{2}}{3} & \cdots & \frac{2\mathrm{i}y(-\mathrm{i}y)^{k}}{(k-1)!(k+1)} & -\mathrm{i}y \\ 0 & \cdots & 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^{4}}{3!5} & \cdots & \frac{2\mathrm{i}y(-\mathrm{i}y)^{k+2}}{3!(k-1)!(k+3)} & \frac{(-\mathrm{i}y)^{3}}{3!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{2\mathrm{i}y(-\mathrm{i}y)^{k}}{(k-1)!(k+1)} & \cdots & \frac{2\mathrm{i}y(-\mathrm{i}y)^{2k-2}}{((k-1)!)^2(2k-1)} & \frac{(-\mathrm{i}y)^{k-1}}{(k-1)!} \\ 1 & \cdots & \frac{(-\mathrm{i}y)^{k-2}}{(k-2)!} & -\mathrm{i}y & \cdots & \frac{(-\mathrm{i}y)^{k-1}}{(k-1)!} & 0 \end{vmatrix} + \mathscr{O}(y^{k^2-2})$$

$$= \frac{2^k(-\mathrm{i}y)^{k^2-1}}{(2!3!\cdots(k-1)!)^2} \begin{vmatrix} 1 & \frac{1}{3} & \cdots & \frac{1}{k-1} & 0 & \cdots & \cdots & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{k+1} & 0 & \cdots & \cdots & 0 & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{k-1} & \frac{1}{k+1} & \cdots & \frac{1}{2k-3} & 0 & \cdots & \cdots & 0 & \frac{1}{2} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{3} & -\frac{1}{5} & \cdots & -\frac{1}{k+1} & \frac{1}{2} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{5} & -\frac{1}{7} & \cdots & -\frac{1}{k+3} & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & -\frac{1}{k+1} & -\frac{1}{k+3} & \cdots & -\frac{1}{2k-1} & \frac{1}{2} \\ 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 \end{vmatrix} + \mathscr{O}(y^{k^2-2}). \tag{57}$$

Therefore,

$$p^{[k]} = -\frac{2}{y}\frac{\begin{vmatrix} \boldsymbol{H}_{\mathrm{e}} & \frac{1}{2} \\ \mathbf{1} & 0 \end{vmatrix}}{\det(\boldsymbol{H}_{\mathrm{e}})} + \mathscr{O}(y^{-2}), \tag{58}$$

where $\boldsymbol{H}_{\mathrm{e}}$ is a block matrix defined by

$$\boldsymbol{H}_{\mathrm{e}} = \begin{bmatrix} 1 & \frac{1}{3} & \cdots & \frac{1}{k-1} & 0 & \cdots & \cdots & 0 \\ \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{k+1} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{k-1} & \frac{1}{k+1} & \cdots & \frac{1}{2k-3} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & -\frac{1}{3} & -\frac{1}{5} & \cdots & -\frac{1}{k+1} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{5} & -\frac{1}{7} & \cdots & -\frac{1}{k+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & -\frac{1}{k+1} & -\frac{1}{k+3} & \cdots & -\frac{1}{2k-1} \end{bmatrix}. \tag{59}$$

Here we use the subscript e to mean the even case. With a simple calculation, we have

$$p^{[k]} = -\frac{2}{y}\left(-(1,1,\cdots,1)\,\boldsymbol{H}_{\mathrm{e}}^{-1}\left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right)^{\mathrm{T}}\right) + \mathscr{O}(y^{-2})$$

$$= \frac{\sum_{i,j=1}^{k}\left(\boldsymbol{H}_{\mathrm{e}}^{-1}\right)_{i,j}}{y} + \mathscr{O}(y^{-2}). \tag{60}$$

The upper left block of $\boldsymbol{H}_{\mathrm{e}}$ is a Hilbert matrix, while the lower right block is the negative of a Hilbert matrix. Employing basic property of Hilbert matrices, we can derive the following identity:

$$\sum_{i,j=1}^{k}\left(\boldsymbol{H}_{\mathrm{e}}^{-1}\right)_{i,j} = -k.$$

Hence $p^{[k]} = -\frac{k}{y} + \mathscr{O}(y^{-2})$. Substituting into Eq. (1) and comparing the coefficient of y^0, we know that the constant m_k must be $-k$. Analogously, when k is odd, we may still expand both the denominator and the numerator. In this case, the denominator and the numerator have the following expansions as $y \to \infty$:

$$\det(\boldsymbol{M}) = \frac{2^k(-\mathrm{i}y)^{k^2}}{(2!3!\cdots(k-1)!)^2} \begin{vmatrix} 1 & \frac{1}{3} & \cdots & \frac{1}{k} & 0 & \cdots & \cdots & 0 \\ \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{k+2} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{k} & \frac{1}{k+2} & \cdots & \frac{1}{2k-1} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & -\frac{1}{3} & -\frac{1}{5} & \cdots & -\frac{1}{k} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{5} & -\frac{1}{7} & \cdots & -\frac{1}{k+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & -\frac{1}{k} & -\frac{1}{k+2} & \cdots & -\frac{1}{2k-3} \end{vmatrix} + \mathscr{O}(y^{k^2-1}), \tag{61}$$

$$\det(\boldsymbol{G})=\frac{2^k(-\mathrm{i}y)^{k^2-1}}{(2!3!\cdots(k-1)!)^2}\begin{vmatrix}1&\frac{1}{3}&\cdots&\frac{1}{k}&0&\cdots&\cdots&0&\frac{1}{2}\\ \frac{1}{3}&\frac{1}{5}&\cdots&\frac{1}{k+2}&0&\cdots&\cdots&0&\frac{1}{2}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \frac{1}{k}&\frac{1}{k+2}&\cdots&\frac{1}{2k-1}&0&\cdots&\cdots&0&\frac{1}{2}\\ 0&\cdots&\cdots&0&-\frac{1}{3}&-\frac{1}{5}&\cdots&-\frac{1}{k}&\frac{1}{2}\\ 0&\cdots&\cdots&0&-\frac{1}{5}&-\frac{1}{7}&\cdots&-\frac{1}{k+2}&\frac{1}{2}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 0&\cdots&\cdots&0&-\frac{1}{k}&-\frac{1}{k+2}&\cdots&-\frac{1}{2k-3}&\frac{1}{2}\\ 1&\cdots&\cdots&1&1&\cdots&\cdots&1&0\end{vmatrix}+\mathscr{O}(y^{k^2-2}). \tag{62}$$

A similar asymptotic formula as Eq. (60) for $p^{[k]}$ with odd k can also be obtained, with $\boldsymbol{H}_{\mathrm{e}}$ replaced by $\boldsymbol{H}_{\mathrm{o}}$. The subscript o indicates the odd case. In the odd case, $\boldsymbol{H}_{\mathrm{o}}$ is instead given by

$$\boldsymbol{H}_{\mathrm{o}}=\begin{bmatrix}1&\frac{1}{3}&\cdots&\frac{1}{k}&0&\cdots&\cdots&0\\ \frac{1}{3}&\frac{1}{5}&\cdots&\frac{1}{k+2}&0&\cdots&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \frac{1}{k}&\frac{1}{k+2}&\cdots&\frac{1}{2k-1}&0&\cdots&\cdots&0\\ 0&\cdots&\cdots&0&-\frac{1}{3}&-\frac{1}{5}&\cdots&-\frac{1}{k}\\ 0&\cdots&\cdots&0&-\frac{1}{5}&-\frac{1}{7}&\cdots&-\frac{1}{k+2}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 0&\cdots&\cdots&0&-\frac{1}{k}&-\frac{1}{k+2}&\cdots&-\frac{1}{2k-3}\end{bmatrix}. \tag{63}$$

For this matrix, we have

$$\sum_{i,j=1}^{k}\left(\boldsymbol{H}_{\mathrm{o}}^{-1}\right)_{i,j}=k.$$

Consequently, the asymptotics of $p^{[k]}$ becomes $p^{[k]}=k/y+\mathscr{O}(y^{-2})$. Substituting this expression into Eq. (1), then we obtain that the constant m_k is equal to k.

Example Next, we compute here the first several rational solutions derived from the generalized Darboux transformation. When $k=1$, the Darboux transformation $\boldsymbol{T}_1(\lambda;y)$ and the first order solution $p^{[1]}$ are

$$\boldsymbol{T}_1(\lambda;y)=\mathbb{I}-\frac{\mathrm{i}}{2\lambda y}\begin{bmatrix}-1&-1\\ 1&1\end{bmatrix},\quad p^{[1]}=\frac{1}{y}. \tag{64}$$

When $k=2$, the Darboux transformation $\boldsymbol{T}^{[2]}(\lambda;y)$ and the second order rational solution $p^{[2]}$ are

$$\boldsymbol{T}^{[2]}(\lambda;y)=\mathbb{I}+\begin{bmatrix}\frac{-4\mathrm{i}\lambda y^3-6\mathrm{i}\lambda-3y^2}{2\lambda^2y(y^3+6)}&\frac{2\mathrm{i}\lambda y^3-6\mathrm{i}\lambda-3y^2}{2\lambda^2y(y^3+6)}\\ -\frac{2\mathrm{i}\lambda y^3-6\mathrm{i}\lambda+3y^2}{2\lambda^2y(y^3+6)}&\frac{4\mathrm{i}\lambda y^3+6\mathrm{i}\lambda+3y^2}{2\lambda^2y(y^3+6)}\end{bmatrix},$$
$$p^{[2]}=-\frac{2\left(y^3-3\right)}{y\left(y^3+6\right)}. \tag{65}$$

Obviously, the first order rational solution $p^{[1]}$ satisfies the Painlevé-II equation (1) with $m_1=1$ and the second order rational solution $p^{[2]}$ satisfies it with $m_2=-2$. Similar to the rational solutions of KdV equation,[39] both the rational solutions $p^{[1]}$ and $p^{[2]}$ are meromorphic functions with simple poles. Indeed these solutions have different analytic properties from the rogue waves, even though in Ref. [11] the authors showed the close connection of the two, which is an important motivation for our study. In Fig. 1, we show the location of zeros and poles of higher order rational solutions.

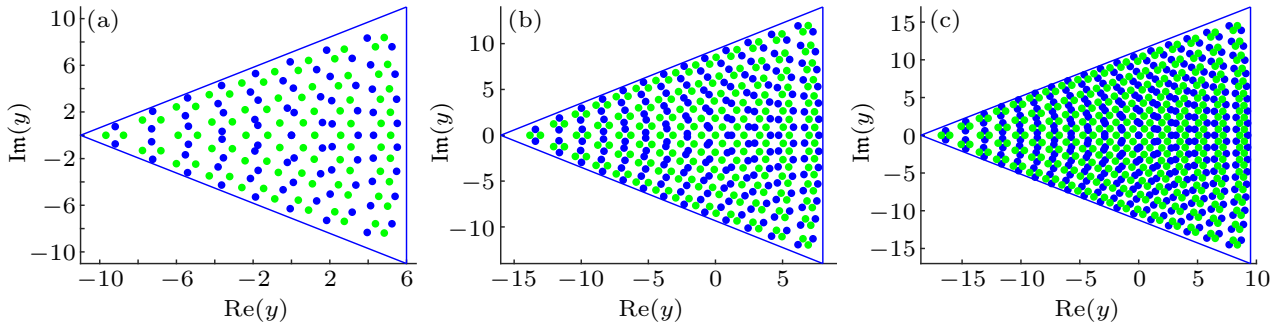


Fig. 1. The poles (green) and the zeros (blue) of the rational solutions of Painlevé-II equation. The corresponding parameters are $m_9=9$ (a), $m_{14}=-14$ (b) and $m_{18}=-18$ (c), respectively.

4. Riemann–Hilbert representation about the rational solutions of Painlevé-II equation

In the previous two sections, we constructed two types of Darboux transformations about the rational solutions and derived the corresponding Bäcklund transformation. Both of them are studied under the Flaschka–Newell Lax representation Eq. (2). It is well known that the Painlevé-II equation has three different Lax representations, i.e., the Flaschka–Newell Lax representation, the Jimbo–Miwa Lax representation and Bortola–Bothner representation. In Ref. [31], the au-

thors gave a detailed introduction of the three representations and constructed three different Riemann–Hilbert problems accordingly. In this section, we will give the solution of the first (Flaschka–Newell) Riemann–Hilbert problem for Painlevé-II equation.

First we give a brief review of the Riemann–Hilbert problem corresponding to the Flaschka–Newell Lax representation from Ref. [31]. Firstly, the authors constructed two types of fundamental solution matrices $\boldsymbol{V}_\infty(\lambda;y)$ and $\boldsymbol{V}_0(\lambda;y)$ of the Flaschka–Newell Lax pair Eq. (2). The former solution $\boldsymbol{V}_\infty(\lambda;y)$ has a convergent expansion when $|\lambda|$ is large and the latter solution is given in the neighborhood of $\lambda=0$. Then there exists a so-called monodromy matrix $\boldsymbol{G}_m(\lambda;y)$ such that

$$\boldsymbol{V}_\infty(\lambda;y)=\boldsymbol{V}_0(\lambda;y)\boldsymbol{G}_m,\ \ 0<|\lambda|<\infty. \tag{66}$$

One can show the following symmetry of $\boldsymbol{V}_\infty(\lambda;y)$ and $\boldsymbol{V}_0(\lambda;y)$:

$$\begin{aligned}&\sigma_1\boldsymbol{V}_\infty(-\lambda;y)\sigma_1=\boldsymbol{V}_\infty(\lambda;y),\\&\sigma_1\boldsymbol{V}_0(-\lambda;y)\sigma_1=\boldsymbol{V}_0(\lambda;y)\begin{bmatrix}0&(-1)^m\\(-1)^{m+1}&0\end{bmatrix}.\end{aligned} \tag{67}$$

The general formula of the connection matrix $\boldsymbol{G}_m$ is given by $\boldsymbol{G}_m=\begin{bmatrix}\alpha&(-1)^m\alpha\\(-1)^{m+1}(2\alpha)^{-1}&(2\alpha)^{-1}\end{bmatrix}$, $\alpha\neq0$. Define two sectional matrices $\boldsymbol{M}_m(\lambda;y)$ depending on these two fundamental solutions $\boldsymbol{V}_\infty(\lambda;y)$ and $\boldsymbol{V}_0(\lambda;y)$, that is,

$$\boldsymbol{M}_m(\lambda;y)=\begin{cases}\boldsymbol{V}_\infty(\lambda;y)\mathrm{e}^{\mathrm{i}\theta(\lambda;y)\sigma_3}\lambda^{-m\sigma_3},&|\lambda|>1,\\\boldsymbol{V}_0(\lambda;y)\mathrm{e}^{\mathrm{i}\theta(\lambda;y)\sigma_3}\lambda^{-m\sigma_3},&|\lambda|<1,\end{cases} \tag{68}$$

where $\theta(\lambda;y)=\lambda y+2\lambda^3$. This immediately gives a Riemann–Hilbert problem for $\boldsymbol{M}_m(\lambda;y)$.

Riemann–Hilbert problem 1[31] Let $m\in\mathbb{Z}$ and $y\in\mathbb{C}$, seek $\boldsymbol{M}_m(\lambda;y)$ satisfying the following conditions.

(i) Analyticity $\boldsymbol{M}_m(\lambda;y)$ is analytic for $|\lambda|\neq1$, and $\boldsymbol{M}_{m,+}(\lambda;y)$, $\boldsymbol{M}_{m,-}(\lambda;y)$ are the continuous boundary values from the interior and the exterior.

(ii) Jump condition When $|\lambda|=1$, $\boldsymbol{M}_m(\lambda;y)$ satisfies the following jump condition:

$$\begin{aligned}\boldsymbol{M}_{m,+}(\lambda;y)=&\ \boldsymbol{M}_{m,-}(\lambda;y)\lambda^{m\sigma_3}\mathrm{e}^{-\mathrm{i}\theta(\lambda;y)\sigma_3}\\&\times\boldsymbol{G}_m^{-1}\mathrm{e}^{\mathrm{i}\theta(\lambda;y)\sigma_3}\lambda^{-m\sigma_3},\ \ |\lambda|=1.\end{aligned} \tag{69}$$

(iii) Normalization When $\lambda\to\infty$, $\boldsymbol{M}_m(\lambda;y)$ is normalized with the following formula:

$$\lim_{\lambda\to\infty}\boldsymbol{M}_m(\lambda;y)\lambda^{m\sigma_3}=\mathbb{I}. \tag{70}$$

Then the rational solutions are given by

$$\begin{aligned}p^{[m]}(y)=&\ 2\mathrm{i}\lim_{\lambda\to\infty}\lambda^{1+m}M_{m,12}(\lambda;y)\\=&\ -2\mathrm{i}\lim_{\lambda\to\infty}\lambda^{1-m}M_{m,21}(\lambda;y).\end{aligned} \tag{71}$$

To solve this Riemann–Hilbert problem, in Sections 2 and 3, we constructed two types of generalized Darboux transformation and derived the general rational solutions of Painlevé-II equation. The solution of the Riemann–Hilbert problem 1 can be directly given by the Darboux transformation.

Theorem 3 The solution of Riemann–Hilbert problem 1 can be given by

$$\boldsymbol{M}_m(\lambda;y)=\begin{cases}\boldsymbol{M}_{m,-}(\lambda;y)=(-1)^{\frac{m+1}{2}\sigma_3}\boldsymbol{T}^{[m]}(\lambda;y)(-1)^{\frac{m+1}{2}\sigma_3}\lambda^{-m\sigma_3},\\\boldsymbol{M}_{m,+}(\lambda;y)=(-1)^{\frac{m+1}{2}\sigma_3}\boldsymbol{T}^{[m]}(\lambda;y)(-1)^{\frac{m+1}{2}\sigma_3}\mathrm{e}^{-\mathrm{i}\theta\sigma_3}\boldsymbol{G}_m^{-1}\mathrm{e}^{\mathrm{i}\theta\sigma_3}\lambda^{-m\sigma_3},\end{cases} \tag{72}$$

where $\boldsymbol{T}^{[m]}(\lambda;y)$ is the generalized Darboux transformation defined in Eq. (48).

Proof Based on the basic property of Darboux transformation, we know that $\boldsymbol{T}^{[m]}(\lambda;y)\mathrm{e}^{-\mathrm{i}\theta\sigma_3}$ can solve the Lax pair Eq. (6) with p taken as the mth order rational solution $p^{[m]}$ and the constant as $(-1)^{m-1}m$. From the symmetry in the ∂y-equation of the Lax pair, we have $\sigma_1\boldsymbol{B}^{\mathrm{FN}}(-\lambda;y)\sigma_1=\boldsymbol{B}^{\mathrm{FN}}(\lambda;y)$. Therefore, the Darboux transformation matrix also have symmetry

$$\sigma_1\boldsymbol{T}^{[m]}(-\lambda;y)\sigma_1=\boldsymbol{T}^{[m]}(\lambda;y).$$

In addition, the Darboux transformation matrix has a convergent series expansion for $\lambda\to\infty$. Notice that $\boldsymbol{T}^{[m]}$ almost coincides with the fundamental solution $\boldsymbol{V}_\infty(\lambda;y)$ defined in Ref. [31], except for a sign difference for even m. With the discrete symmetry of the Painlevé-II equation $(p(y),m)\to(-p(y),-m)$, we can take a simple transformation to $\boldsymbol{T}^{[m]}(\lambda;y)$, and now the formula is valid for all $m\in\mathbb{Z}$. Thus, we have derived the connection between the generalized Darboux transformation and the Riemann–Hilbert problem 1 when $|\lambda|>1$. For $|\lambda|<1$, the solution of the Riemann Hilbert problem 1 can also be given by the jump condition. Therefore, the solution of the Riemann–Hilbert problem 1 is given in the whole complex plane $\mathbb{C}$, that is,

$$\begin{aligned}&\boldsymbol{M}_{m,-}(\lambda;y)=(-1)^{\frac{m+1}{2}\sigma_3}\boldsymbol{T}^{[m]}(\lambda;y)(-1)^{\frac{m+1}{2}\sigma_3}\lambda^{-m\sigma_3},\\&\boldsymbol{M}_{m,+}(\lambda;y)=(-1)^{\frac{m+1}{2}\sigma_3}\boldsymbol{T}^{[m]}(\lambda;y)(-1)^{\frac{m+1}{2}\sigma_3}\\&\qquad\times\mathrm{e}^{-\mathrm{i}\theta\sigma_3}\boldsymbol{G}_m^{-1}\mathrm{e}^{\mathrm{i}\theta\sigma_3}\lambda^{-m\sigma_3}.\end{aligned} \tag{73}$$

5. Discussion and conclusion

In this paper, we construct the general rational solutions to Painlevé-II equation by the generalized Darboux transformation. We are able to compactly write the rational solutions as Gram determinant. In the generalized Darboux transformation, the spectral parameter λ_1 is chosen as a special value $\lambda_1 = 0$. Under this condition, the fundamental solution ϕ_1 becomes a polynomial. In particular, the spectrum in the Darboux transformation is a removable singularity, which is similar to the study of rogue wave. In Section 2, we use two methods to derive the Bäcklund transformation of the rational solutions. One is directly from the Painlevé-II equation itself, and the other one is derived from the iteration steps in the Darboux transformation. The two transformations are shown to be equivalent by a simple transformation. In addition, from the exact form of rational solutions simply represented by Gram determinants, we can also analyze the asymptotics of large y. In Section 4, we prove that our Darboux transformation can solve the Riemann–Hilbert problem in Ref. [31], which has never been reported before.

It is known that the generalized Darboux transformation can be used to derive the rogue wave of NLS equation and the rational solutions of Painlevé-II equation, both written as a Gram determinant. In view of the result in Ref. [10], we conjecture that there exist certain connections between these two kinds of solutions. Our goal is to continue the study of this topic via the Riemann–Hilbert problem in the future.

Acknowledgments

The authors sincerely thank Professor Liming Ling for his guidance and help. Project supported by the National Natural Science Foundation of China (Grant No. 12101246).

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