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Quintessence anisotropic stellar models in quadratic and Born–Infeld modified teleparallel Rastall gravity

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This study aims to discuss anisotropic solutions that are spherically symmetric in the quintessence field, which describe compact stellar objects in the modified Rastall teleparallel theory of gravity. To achieve this goal, the Krori and Barua arrangement for spherically symmetric components of the line element is incorporated. We explore the field equations by selecting appropriate off-diagonal tetrad fields. Born–Infeld function of torsion $f(T)=\beta\sqrt{\lambda T+1}-1$ and power law form $h(T)=\delta T^n$ are used. The Born–Infeld gravity was the first modified teleparallel gravity to discuss inflation. We use the linear equation of state $p_r=\xi\rho$ to separate the quintessence density. After obtaining the field equations, we investigate different physical parameters that demonstrate the stability and physical acceptability of the stellar models. We use observational data, such as the mass and radius of the compact star candidates PSRJ 1416-2230, Cen X-3, & 4U 1820-30, to ensure the physical plausibility of our findings.

Keywords: anisotropic spheres, quintessence field, modified Rastall teleparallel gravity, equation of state (EoS), $f(T)$ gravity

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1. Introduction

In 1998, a ground-breaking discovery was made about the universe: it was expanding at an accelerating rate, defying the expectations of observers.[1] This led to a deep curiosity among scientists about the cause of this expansion. Eventually, they identified the existence of dark energy, a negative pressure fluid known as cosmological constant Λ, as the driving force behind this expansion. However, it opposed the energy conditions, leaving many scientists to wonder if general relativity (GR) theory could handle it effectively. GR theory, based on a symmetric and torsion-free Levi–Civita connection,[2] is renowned for explaining gravity at the local level. However, it falls short when discussing gravity at a wider, global level. To address this shortcoming, scientists started considering modifications to GR. Many of these modifications focused on extending GR's geometry, with $f(R)$ theories being one of the most promising. In these theories, the Lagrangian function is expressed in terms of the Ricci scalar R.[3,4]

Teleparallel gravity, also known as TEGR, has gained popularity due to its close resemblance to GR. TEGR is a torsion-based theory that is free of curvature. Its field equations are nearly identical to GR's, making it a popular substitute for GR in certain situations. However, despite their similarities, these two theories are mathematically interpreted differently. A more generalized version of TEGR is the $f(T)$ gravity theory, which has a close correlation with $f(R)$ theory. Unlike GR, this theory is based on the Weitzenbock connection, which means it is a curvature-free theory with non-zero torsion.[2] Einstein initially defined the concept of space–time based on torsion.[5] In TEGR, the tetrad plays a crucial role in setting up the field equations, rather than the metric function. Different tetrads can lead to different field equations, making the choice of the correct tetrad crucial to free the function "f" from constraints. This structure provides a foundation for modifying TEGR. However, using a diagonal tetrad can produce a theory that is only well-established with Birkhoff's Theorem, and the Schwarzschield metric is not a solution to this particular theory. On the other hand, an off-diagonal tetrad does not impose any constraints on the value of "f" or "T". In this paper, we use an off-diagonal tetrad for this reason.

The TEGR and GR theories are similar, and every solution of TEGR is also a solution of GR. Therefore, the problems that remain unaddressed in GR also carry over to TEGR. One of the problems faced by researchers is the cosmological constant Λ, which serves as a negative pressure fluid ($p_\Lambda=-\rho_\Lambda$). While exploring the present behavior of the accelerating universe, GR can only deal with it by adding a scalar field as an extra term. However, the observed value of Λ drastically varies from the expected value, and this issue is known as the "problem of cosmological constant" among researchers. According to the collective view of researchers, this problem can only be addressed by modifying GR, either by including an extra scalar field or by altering the standard model of physics. Some theories suggest that modifying GR can de-

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scribe the late-time cosmic expansion behavior without involving Λ. In contrast, the early universe experienced a rapid expansion known as the radiation era, which cannot be handled by cosmological constant. This situation can be best explained by including a scalar field known as inflation. Moreover, GR does not provide any knowledge about the beginning and nature of inflation. Modified gravity theories[6–8] collectively explain the inflationary era, called early-time expansion, and the present dark energy era. The problem of coincidence, which suggests that today's matter energy density and end dark energy are the same, cannot be explained by GR. Some theorists argue that it is not an issue, just a coincidence. However, some GR modifications may address this issue.[8] Additionally, the literature[9,10] has pointed out many more issues that cannot be addressed in detail by GR. However, when one generalizes TEGR to the $f(T)$ gravity theory, this similarity is scattered as we take the Lagrangian as a function of torsion scalar.[11,12] Due to this reason, $f(T)$ gravity is a strong candidate to explain the acceleration aspect of the universe's expansion by eliminating the existence of dark energy.[13,14]

Einstein's theory of general relativity proposes a fundamental connection between matter field and spacetime geometry, which results in the energy–momentum tensor (EMT) conservation law. However, this conservation law holds only true for Minkowski's flat spacetime or weak gravitational fields, making the theory susceptible to modifications. The established consensus acknowledges that Einstein's gravitational theory requires the condition $\Theta^{\nu}\mu;\nu=0$ to be satisfied. However, Peter Rastall has challenged the basis of this assumption, asserting that it relies on uncertain premises. In response, Rastall proposed an alternative statement, $\Theta^{\nu}\mu;\nu=a_{\mu}$, where the function a_{μ} should produce a null outcome within a flat spacetime context.[15] It is widely accepted that the curvature of spacetime and the effects of gravity are inherently interconnected. The presence of matter gives rise to a gravitational field, which reciprocally results in curvature. As a result, the quantity $T_{\mu\nu}$ has a close association with curvature. This idea can be demonstrated using the analogy of an elastic sphere symbolizing a fundamental particle. In scenarios where curvature is not zero, tidal gravitational forces become evident. These forces distort the sphere and impact its rest mass and energy.[15] Tidal gravitational forces are a consequence of the curved nature of spacetime, as gravity arises from the existence of matter, leading to both curvature and vice versa. In the case of a flat spacetime, $\Theta^{\nu}{}_{\mu;\nu}=0$. Rastall[15,16] introduced a modification to GR that accounts for the conservation law's violation in curved spacetime. This modification involves the Rastall coupling parameter, λ, which measures the extent to which spacetime geometry couples with matter field in a non-minimal way. When λ has a limiting value of zero coupling, the modified form reverts to GR. Various modifications have been proposed for Rastall gravity since its inception. For example, in a study,[17] the authors generalized Rastall's theory by proposing $\nabla_{\mu}T^{\mu\nu}=\nabla_{\nu}(\lambda' R)$, where λ' is a function varying with space–time coordinates. In another reference,[18] the authors assumed $\nabla_{\mu}T^{\mu\nu}=\lambda\nabla^{\nu}f(R)$, which led to the presentation of a solution for an electrically and magnetically neutral regular black hole. Additionally, a modified version of teleparallel gravity based on Rastall's assumption is introduced in Refs. [19,20].

It is important to highlight that the adjusted Rastall theory (RT) shares all of the vacuum solutions solutions with GR. Nevertheless, when we take into account the Rastall parameter λ, the solutions involving non-vacuum case exhibit significant deviations from those found in GR. This has aroused a growing interest among scholars in the Rastall theory, as evidenced by their diverse remarks on the originality and accuracy of RT. While researchers have displayed an escalating curiosity in delving into the captivating and enigmatic aspects of RT, a recent assertion by Visser[21] posits that RT is essentially identical to GR. However, this viewpoint has been contested by Darabi *et al.*,[22,23] who contend that these theories stand apart, rendering Visser's assertion inaccurate. Visser contends that Rastall's proposed EMT is flawed, reducing Rastall's proposition to a mere reorganization of the material component of GR. In contrast, Darabi *et al.* argue that Rastall's definition of EMT aligns with the traditional interpretation of EMT. In support of their argument, they present a compelling illustration from the $f(R)$ theory of gravity, following a similar approach as Visser,[21] yet effectively demonstrating that $f(R)$ theory is not equivalent to GR. Fundamentally, the RT can be understood as a modified version of GR, given our previous observation that GR can be reinstated for a specific value of λ the Rastall parameter. Notably, the thorough analysis conducted by Darabi *et al.*[22,23] gains additional support from the recent investigation carried out by Hansraj *et al.*[24]

Compact objects represent the final phase in the development process of ordinary stars, making them invaluable for studying highly dense matter. Pulsars and other spinning stars with strong magnetic fields are among the most significant discoveries in astrophysics. The composition of highly dense matter in these objects is believed to be composed of subatomic particles such as baryons, leptons, mesons, and strange quark matter. However, observational data on the exact composition of these objects is not available. Hence, the native problem faced by astrophysicists in constructing their configurations is determining the geometry and distribution of interior surface matter. Recently, some compact body configurations have been discussed in different modified theories of gravity like; thermodynamics of a new perturbed black hole solution have been discussed in $f(Q)$ gravity,[25] generic wormhole models stability via thin-shell approach have also been

discussed.[26–28] Compact stars formation have been discussed in Refs. [29–31].

Initial studies suggest that the matter distribution in spherically symmetric systems is based on isotropic (perfect) fluid, which leads to the application of the isotropic condition ($p_r = p_t$) on EFEs as the tangential and radial components of pressure coincide with each other. However, Jeans[32] predicted in 1922 that unusual conditions dominate inside the interior of stellar objects, which suggested the involvement of an anisotropic factor for a better understanding of the distribution of matter inside heavenly bodies. Anisotropy, measured as $\Delta(p_t - p_r)$, is simply a measure of deviation from isotropy. A considerable amount of detailed material is available in literature[33,34] to study the effects of anisotropy in stellar structures under spherical symmetry. In relativistic stellar systems, anisotropy arises due to the existence of various fluids such as superfluids, magnetic or external fields, phase transitions, rotational motions, and other fluids.

This research work deals with the static and spherically symmetric system of stellar objects, so anisotropy may arise due to the existence of a superfluid, elastic nature of the superfluid, or an anisotropic fluid. Numerous studies[35–37] have been done in this regard. Ruderman[38] predicted in his pioneer study of anisotropy in astrophysics that it is an inherent property in high-density nuclear matter distribution. Bower and Liang[37] predicted that anisotropy is the result of strong interactions between superconductivity and superfluidity inside heavily denser matter. It is noteworthy that in diverse situations when the radial component p_r is not equal to the tangential component p_t of pressure ($p_t \neq p_r$), anisotropy arises, which is known as anisotropic pressure. When the spatial gradient of the scalar field is non-zero, the physical system generates anisotropic pressure. Herrera and Santos[33] deeply elaborated the generation of local-level anisotropy and its effects in self-gravitating systems for the first time. After that, the effects of local anisotropy came into discussion among several authors (one can consult the current literature[39–44]) on static bodies having spherical symmetry.

We will now proceed with the next phase of our study, which follows the outlined scheme: In Section 2, we will introduce the fundamentals of modified teleparallel Rastall gravity and assess the field equations using an off-diagonal tetrad. Additionally, in this section, we will obtain the generalized solutions using the Karori Barua space–time and quadratic and squared-root form of torsion functions. In Section 3, we will match the interior and exterior geometries to determine the constant parameters utilized in our stellar modeling. Following this, we will present a discussion of our results in Section 4, followed by a conclusion in Section 5.

2. MTRG: basic formulation

To fully comprehend the structure of stars, it is crucial to have a fundamental understanding of their inner workings. This is where spherically symmetric spacetime comes in as a valuable model. With uniform properties in all directions, this spacetime can be represented mathematically using various models such as the Schwarzschild metric. Through the analysis of spherically symmetric spacetimes, scientists are able to gain valuable insights into the behavior and evolution of stars, which have far-reaching implications across different areas of astrophysics.

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu(r)}\mathrm{d}t^2 + \mathrm{e}^{\lambda(r)}\mathrm{d}t^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 + \mathrm{d}\theta^2. \quad (1)$$

It is possible to represent the metric tensor $g_{\mu\nu}$ defined on a manifold using the tetrad fields e^i_μ and the Minkowski metric $\eta_{ij} = \mathrm{diag}(-1,1,1,1)$.

$$g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu, \quad (2)$$

where Greek alphabet $(\mu,\nu,\ldots = 0,1,2,3)$ represent space–time indices and the Latin alphabet $(i,j,\ldots = 0,1,2,3)$ represent tangent space indices. The Weitzenbck connection in a mathematical concept is defined as

$$\Gamma^\mu_{\mu\nu} = e^\alpha_i \partial_\nu e^i_\mu = -e^i_\mu \partial_\nu e^\alpha_i. \quad (3)$$

The specific type of connection that possesses non-zero torsion but zero curvature is employed in the teleparallel theory. These connections play a crucial role in defining the torsion tensor, which can be represented as follows:

$$T^\sigma_{\mu\nu} \equiv \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\mu\nu} = e^\sigma_i \left(\partial_\mu e^i_\nu - \partial_\nu e^i_\mu\right). \quad (4)$$

The Weitzenbock connection is linked to the Levi–Civita connection, $\bar{\Gamma}^\sigma_{\mu\nu}$, by the following relation:

$$\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} - K^\sigma_{\mu\nu}, \quad (5)$$

where $K_{\mu\nu}$ is notions the contorsion tensor, which is given by

$$K^\sigma_{\mu\nu} = \frac{1}{2}\left(T^\sigma_\mu \sigma_\nu + T^\sigma_\nu \sigma_\mu - T^\sigma_{\mu\nu}\right). \quad (6)$$

The expression for torsion scalar reads

$$T = S^{\sigma\mu\nu} T_{\sigma\mu\nu}, \quad (7)$$

where expression for super-potential $S^{\sigma\mu\nu}$ is given by

$$S^{\sigma\mu\nu} = -S^{\sigma\nu\mu} = \frac{1}{2}\left(K^{\mu\nu\sigma} - g^{\sigma\nu} T^{\alpha\mu\mu} + g^{\sigma\mu} T^{a\nu}\right). \quad (8)$$

The action of the modified teleparallel gravity is presented by[20]

$$S = S_G + S_m = \frac{1}{4\kappa}\int e f(T)\,\mathrm{d}^4x + \int e L_m\,\mathrm{d}^4x, \quad (9)$$

where the function $f(T)$ is torsion dependent, and the tetrad field $e^a{}_F$ determinant is represented as e. Furthermore, the matter Lagrangian is denoted by L_m, and by computing the

variation of the action in regards to the tetrad field, we can derive the corresponding field equation

$$S_i^{\mu\nu} f_{TT} \partial_\mu T + \mathrm{e}^{-1} \partial_\mu \left(e S_i^{\mu\nu} \right) f_T - T^\alpha{}_{\mu i} S_\sigma^{\nu\mu} f_T - \frac{1}{4} e_i{}^\nu f = -\kappa \Theta_\mu^\nu, \tag{10}$$

where Θ_μ^ν is addressed to the usual energy–momentum tensor of the perfect fluid. It is shown that

$$\left(S_i^{\mu\nu} f_{TT} \partial_\mu T + \mathrm{e}^{-1} \partial_\mu (e S_i{}^{\mu\nu}) f_T - T^\sigma{}_{\mu i} S_\sigma^{\nu\mu} f_T - \frac{1}{4} e_i^\nu f \right)_{;\nu} = 0, \tag{11}$$

where semicolon is used to denote the covariant derivative in teleparallel structure

$$V^\mu_{;\nu} = \partial_\nu V^\mu + \left(\Gamma^\mu_{\lambda\nu} - K^\mu_{\lambda\nu} \right) V^\lambda, \tag{12}$$

for any space–time vector V^μ. The above-given expression ensures that the covariant derivative of the energy–momentum tensor also vanishes

$$\Theta^\nu{}_{\mu;\nu} = 0. \tag{13}$$

Both Einstein's theory and our modified teleparallel gravity theory share the same energy–momentum conservation equation. However, Rastall's proposal of a new equation, $T^\mu \nu;;\mu = \lambda R,\nu$, challenged the conservation equation in Einstein's theory. This equation suggests an intriguing interaction between matter and geometry, indicating a connection between the two and leading to a modified field equation. In our modified teleparallel gravity theory, we adopt a similar assumption inspired by Rastall's concept. By linking matter and geometry through the scalar torsion of geometry, we establish a connection where the divergence of the energy–momentum tensor, $\Theta^\nu \mu$, is proportional to the divergence of the torsion scalar.

$$\Theta^\nu{}_{\mu;\nu} = \lambda h(T)_{,\mu}, \tag{14}$$

where λ is a real constant and $h(T)$ is an analytical function of torsion. Then, the field Eq. (11) can be restated as follows:

$$S_u^{\mu\nu} f \gamma\tau \partial_\mu T + \mathrm{e}^{-1} \partial_\mu \left(S_i^{\mu\nu} \right) f_T - T_\mu^\sigma S_\sigma^{\nu\mu} f_T - \frac{1}{4} e_i^\nu f - \delta_\mu^\nu \kappa\lambda h(T) = -\kappa \overset{e\text{-}m}{\mathcal{T}}{}_{\mu\nu}. \tag{15}$$

Here, κ is the gravitational constant in Rastall theory, which is a comparison with Newtonian gravity is expressed as

$$\kappa = \frac{4\gamma - 1}{6\gamma - 1} \kappa_G, \tag{16}$$

where $\gamma = \lambda\kappa$ and κ_G notions the Einstein coupling constant $\kappa_G = 4\pi G$.

Energy momentum tensor for anisotropic fluid characterizing the core of compact star is written by

$$\overset{e\text{-}m}{\mathcal{T}}{}_{\mu\nu} = (\rho + p_t) u_\mu u_\nu - p_t g_{\mu\nu} + (p_r - p_t) v_\mu v_\nu, \tag{17}$$

where $u_\mu = \mathrm{e}^{\frac{\nu}{2}} \delta_\mu^0$ and $v_\mu = \mathrm{e}^{\frac{\lambda}{2}} \delta_\mu^1$, $\overset{e\text{-}m}{\mathcal{T}}{}_\mu^\nu = \overset{matter}{\mathcal{T}}{}_\mu^\nu + \overset{q}{\mathcal{T}}{}_\mu^\nu$, $\overset{q}{\mathcal{T}}{}_\mu^\nu$ is the energy momentum for quintessence field equation with energy density ρ_q and equation of state parameter w_q ($-1 < w_q < -1/3$). Here components of $\overset{q}{\mathcal{T}}{}_\mu^\nu$ are defined as

$$\overset{q}{\mathcal{T}}{}_t^t = \overset{q}{\mathcal{T}}{}_r^r = -\rho_q, \tag{18}$$

$$\overset{q}{\mathcal{T}}{}_\theta^\theta = \overset{q}{\mathcal{T}}{}_\phi^\phi = \frac{(3w_q + 1)\rho_q}{2}, \tag{19}$$

where vector u_ν expresses the four velocities in the time-like direction, while v_μ represents the unit space-like vector in the radial direction. The relationship between these vectors is given by $u_0 u^0 = -v_1 v^1 = 1$. Also w_q represents the quintessential state parameter.

The teleparallel technique used in general relativity involves tetrad fields denoted by e_μ^i. These fields represent the coordinates of the manifold through holonomic Greek indices and the frame through anholonomic Latin indices. By combining the two types of indices, the tetrad matrix e_μ^i and its inverse can be described. It is known that $e_i^\mu e_i^\nu = \delta_\nu^\mu$ and $e_i^\mu e_j^\mu = \delta_{ij}$. The teleparallel technique aims to create a more generalized manifold that includes torsion in addition to curvature. The Riemannian curvature tensor is expected to be zero, which allows either the torsion-free part (geometry) or the torsion part (tetrad) to be used to explain the gravitational field. Therefore, tetrad fields and torsion provide an alternative to the geometric definition of gravity. Tamanini and Bhmer introduced the concept of a "good tetrad" in their work,[45] which refers to a tetrad that does not impose any additional constraints on the functional form of $f(T)$. This allows for the study of a broader class of $f(T)$ cosmologies. Bhmer *et al.*[46] investigated the existence of relativistic stars in $f(T)$ modified gravity and constructed various classes of static perfect fluid solutions for both diagonal and off-diagonal tetrad. However, the diagonal tetrad is unsuitable for spherical symmetry as the exact solutions correspond to a constant torsion scalar. Therefore, off-diagonal tetrad (good tetrad) is preferred, and many aspects of spherically symmetric spacetime have been presented for review in the literature (interested readers can consult some of the available references[13,47–51] and some others).

In this study, we will develop field equations using the off-diagonal tetrad matrix given in Refs. [45,46,52],

$$e^a{}_\mu = \begin{pmatrix} \mathrm{e}^{\nu/2} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{\lambda/2} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \mathrm{e}^{\lambda/2} \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \mathrm{e}^{\lambda/2}\cos\theta & -r\sin\theta & 0 \end{pmatrix}, \tag{20}$$

where $g_{tt} = \mathrm{e}^{\nu(r)}$ and $g_{rr} = \mathrm{e}^{\lambda(r)}$. The determimant of the above tetrad field is given by $e = \det\left(e^{a}{}_{\mu}\right) = r^2 \sin\theta\, \mathrm{e}^{(\nu+\lambda)/2}$. The torsion scalar obtained from Eqs. (4), (7), and (8) is given as

$$T(r) = \frac{2\mathrm{e}^{-\lambda}}{r^2}\left(\mathrm{e}^{\frac{\lambda}{2}} - 1\right)\left(\mathrm{e}^{\frac{\lambda}{2}} - 1 - r\nu'\right). \tag{21}$$

Replacing Eqs. (15), (17) and (20) in the field Eq. (1), the obtained nonzero components of the energy–momentum tensor read

$$\begin{aligned}
&\kappa(\rho(r)+\rho_q(r)) \\
&= \frac{\mathrm{e}^{-\lambda/2}}{r}\left(1-\mathrm{e}^{-\lambda/2}\right)f_T' - \left(\frac{T}{4}-\frac{1}{2r^2}\right)f_T \\
&\quad + \frac{\mathrm{e}^{-\lambda}}{2r^2}\left(r\lambda'-1\right)f_T + \frac{f}{4} + \gamma h(T), \qquad (22)\\
&\kappa(p(r)-\rho_q(r)) \\
&= \left[\frac{T}{4} - \frac{1}{2r^2} + \frac{\mathrm{e}^{-\lambda}}{2r^2}\left(1+r\nu'\right)\right]f_T - \frac{f}{4} - \gamma h(T), \qquad (23)\\
&\kappa\left(p_t(r) + \frac{1}{2}\rho_q(3w_q+1)\right) \\
&= \frac{\mathrm{e}^{-\lambda}}{2}\left(\frac{\nu'}{2} + \frac{1}{r} - \frac{\mathrm{e}^{\lambda/2}}{r}\right)f_T' - \frac{f}{4} - \gamma h(T) \\
&\quad + f_T\left[\frac{T}{4} + \frac{\mathrm{e}^{-\lambda}}{2r}\left[\left(\frac{1}{2}+\frac{r\nu'}{4}\right)\left(\nu'-\lambda'\right) + \frac{r\nu''}{2}\right]\right]. \qquad (24)
\end{aligned}$$

The presence of the Rastall term $\gamma h(T)$ and the coefficient κ in equations has a significant impact on the behavior and magnitude of the components, potentially altering the energy conditions.

To obtain solutions for compact objects, one must consider a range of assumptions for the $f(T)$ and $h(T)$ functions available. In our study, we are using the off-diagonal tetrad which makes the analysis more physical without any additional limitation[45,46] on functions depending upon torsion, i.e., $f(T)$ and $h(T)$. If we talk about the Pioneer modifications of teleparallel gravity, Born–Infeld gravity was the first modified $f(T)$ gravity to discuss the inflationary phenomenon by providing the exact solution.[11] Serving the motive of more generalized solutions, we choose non-linear models, like the Born–Infeld for $f(T)$ function[11,53] and power law form of $h(T)$ function[4,54] given below:

$$f(T) = \beta\left(\sqrt{\lambda T + 1} - 1\right), \tag{25}$$

$$h(T) = \delta T^n, \tag{26}$$

where β, n, δ are arbitrary real constants and λ is Born–Infeld parameter. In order to maintain the nonlinear form here we choose $n = 2$ to ensure the quadratic form of the model. As an interior solution, we use the Krori and Barua[55] space–time in the form of potential components given as

$$\nu(r) = Br^2 + c, \quad \lambda(r) = Ar^2, \tag{27}$$

where values of constants A, B, and c are chosen based on different physical considerations to define this spacetime geometry. This geometry has proven to be effective in simulating self-gravitating stellar models in both general relativity and modified gravity, as documented in references.[29,56–59]

Equation of state (EoS), a tool necessary for studying compact objects in the quintessence field, is used to calculate the quintessence density by developing a relationship between the energy density ρ and radial pressure p_r. In the study of compact stars, EoS proves to be an invaluable asset, particularly when the system has more variables than equations. Numerous EoS are available in literature, such as the asymptotic EoS $p = (1+\frac{2}{n})\rho$ used by Nojiri and Odintsov[60] while discussing the singularity of spherically-symmetric spacetime in quintessence/phantom dark energy universe, and the generalized Chaplygin gas EoS $p = -C/\sigma^{\gamma}$, $0 < \gamma \leq 1$ used by Sharif and Faisal[61] while studying the Stability of Einstein–Power–Maxwell $(2+1)$-dimensional wormholes. For the sake of simplicity, our study employs the EoS provided below:[62,63]

$$p_r = \xi\rho, \quad 0 < \varepsilon < 1. \tag{28}$$

After all, solving Eqs. (22)–(28), we get the final versions of field equations, as given below:

$$\rho = \frac{\beta(6\gamma-1)\lambda\,\mathrm{e}^{-Ar^2}\left(-8B^2\lambda r^2(\mathrm{e}^{\frac{Ar^2}{2}}-1) + 2Af_4 r - \lambda r\mathrm{e}^{Ar^2}(\mathrm{e}^{\frac{Ar^2}{2}}-1) + f_5(r)\right)}{16\pi(4\gamma-1)f_1(r)f_4(r)(\xi+1)}, \tag{29}$$

$$\begin{aligned}
\rho_q &= \frac{\mathrm{e}^{-Ar^2}}{16\pi(4\gamma-1)f_1(r)f_4(r)(\xi+1)r^2}\Big[-4\beta(6\gamma-1)\lambda^2\xi r^2(A+B)\left(2Br^2+1\right) + 4\beta(6\gamma-1)\lambda^2\mathrm{e}^{\frac{Ar^2}{2}} \\
&\quad \times \left(Br^2+1\right)\left(2A\xi r^2 + 2B(2\xi+1)r^2 + \xi + 1\right) + (\xi+1)\mathrm{e}^{2Ar^2}\left(2\lambda + r^2\right) \\
&\quad \times \Big[2\beta(1-6\gamma)\lambda - (6\gamma-1)r^2\left(\beta + \gamma\delta f_2(r)f_3(r)2^{n+2} + \beta\left(-f_2(r)\right)r\right)\Big] + f_{10}(r)\mathrm{e}^{\frac{3Ar^2}{2}} + f_{11}(r)\Big], \qquad (30)
\end{aligned}$$

$$p_r = \xi\rho, \tag{31}$$

$$\begin{aligned}
p_t &= \frac{\mathrm{e}^{-Ar^2}}{32\pi(4\gamma-1)f_1(r)f_4(r)(\xi+1)r^2}\Big[-4\beta(6\gamma-1)\lambda^2\left(2Br^2+1\right) \\
&\quad \times \Big[Ar^2\left(B(\xi+1)r^2 + 2\xi + 3\xi w_q + 1\right) - B^2(\xi+1)r^4 + B\xi r^2(3w_q+1) + \xi + 1\Big]
\end{aligned}$$

$$+4\beta(6\gamma-1)f_7(r)\lambda^2 e^{\frac{Ar^2}{2}}\left(Br^2+1\right)+f_8(r)(\xi+1)e^{2Ar^2}\left(2\lambda+r^2\right)+f_6(r)e^{\frac{3Ar^2}{2}}+f_9(r)e^{Ar^2}\Big], \tag{32}$$

where $f_i(r)$, $i=1, 2, \ldots, 11$ are given in Appendix A.

3. Matching conditions

The metric of the inner border remains constant, irrespective of the star's geometric structure, whether observed from inside or outside. In the case of an emergent scenario, the metric components must be continuous at the boundary, regardless of the reference frame. When analyzing stellar remnants in general relativity, the priority is given to Schwarzschild's solution among all available matching alternatives. It is advisable to consider quasi-pressure and energy density when working with modified gravity theories. Several researchers have made significant contributions to boundary conditions.[64,65] Goswami *et al.*[66] identified matching boundaries by combining unique constraints related to stellar compact structures and thermodynamically relevant properties. In the $f(T)$-gravity theory with generalized functional form, the most suitable exterior spacetime can be the exterior Schwarzschild de-Sitter solution. But, in this study, we are discussing the quintessence stars models. So the most suitable exterior spacetime can be the vacuum case in the quintessence field, as reported in Kiselev's and Toledo's works:[67,68]

$$\begin{aligned} ds^2 =& -\left(1-\frac{2M}{R}-\frac{q}{R^{3w_q+1}}\right)dt^2 \\ &+\left(1-\frac{2M}{R}-\frac{q}{R^{3w_q+1}}\right)^{-1}dt^2 \\ &+r^2\sin^2\theta\, d\phi^2+d\theta^2, \end{aligned} \tag{33}$$

where M is mass of the star, w_q is the quintessential state parameter, and q is the quintessential parameter associated with the density of quintessence defined below. The pressure and density of quintessence are related by the equation of state $p=w_q\rho_q$. To achieve the scenario of accelerated expansion, it is necessary to require that $-1<w_q<-1/3$. As for the quintessential parameter q, it is a positive quantity, and note that when $q=0$, the Schwarzschild solution is recovered.

After comparing the inner spacetime Eq. (1) and outer space–time Eq. (33) at the boundary $r=R$, we arrive at the given below system of equations:

$$e^{BR^2+c}=-\frac{2M}{R}-\frac{q}{R^{3w_q+1}}+1, \tag{34}$$

$$e^{AR^2}=\frac{1}{-\frac{2M}{R}-\frac{q}{R^{3w_q+1}}+1}, \tag{35}$$

$$2BRe^{BR^2+c}=\frac{2M}{R^2}-q(-3w_q-1)R^{-3w_q-2}. \tag{36}$$

The solution of these equations results in the following form of expressions for constants A, B, x:

$$A=-\frac{\log\left(-\frac{2M}{R}-\frac{q}{R^{3w_q+1}}+1\right)}{R^2}, \tag{37}$$

$$B=\frac{\frac{2M}{R^2}-q(-3w_q-1)R^{-3w_q-2}}{2R\left(-\frac{2M}{R}-\frac{q}{R^{3w_q+1}}+1\right)}, \tag{38}$$

$$c=\frac{2\left((2M-R)R^{3w_q}+q\right)\log\left(\frac{-2M-qR^{-3w_q}+R}{R}\right)+2MR^{3w_q}+3qw_q+q}{2\left((2M-R)R^{3w_q}+q\right)}. \tag{39}$$

Specific values of constant parameters are given in Table 1.

Table 1. Values of constants, by fixing $n=0$, $\beta=2$, $\xi=0.23$, $q=7\times10^{-7}$, $\delta=-4$, $\lambda=4\times10^{-4}$, $w_q=0.99$, and $\gamma=0.050$ (solid lines), 0.055 (long dashes), 0.060 (dashes), 0.065 (short dashes), 0.070 (dots).

Name of star	Mass (M_0)	Radius R (km)	A	B	c	$\frac{p_{rc}}{\rho_c}$ $(r=0)$
PSRJ 1614-2230	1.97 ± 0.04	10.249	0.0045003	0.00287503	–0.77472	<1
Cen X-3	1.49 ± 0.08	10.136	0.00360974	0.00218354	–0.595192	<1
4U 1820-30	1.54 ± 0.06	9.71	0.00404785	0.0024628	–0.613851	<1

4. Discussion of calculated results

In this section of our case study, we explore the stellar models by making the choice of squared-root and power law form of gravity for the functions $f(T)$ and $h(T)$ respectively. We also make a choice of three compact star models like PSRJ 1614-2230, Cen X-3, 4U 1820-30. The summary of our obtained results is as follows.

To gain a deeper understanding, let's take a closer look at the metric components $e^{\nu(r)}$ and $e^{\lambda(r)}$. To achieve this, we adopt the ansatz of Krori and Barua spacetime, like e^{Br^2+c} and e^{Ar^2}. One can examine that, as $r\to0$ $e^{Ar^2}=1$ and $e^{Br^2+c}=e^c>0$, both of which demonstrate a smooth evolutionary behavior, as depicted in Fig. 1(a).

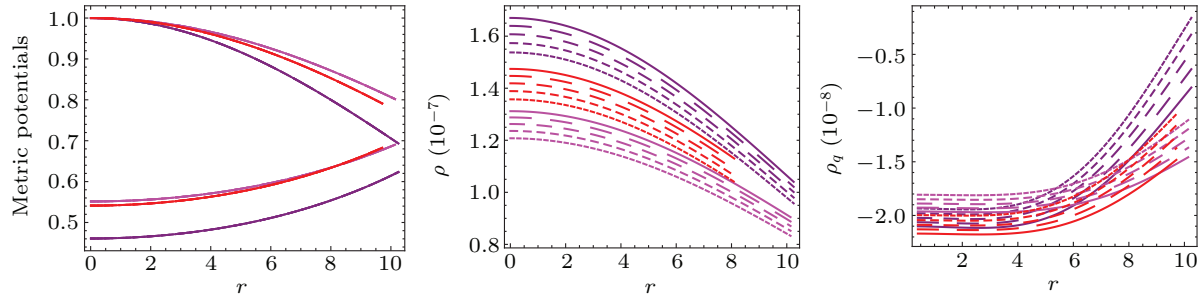


Fig. 1. Metric functions, energy density, and quintessence density versus radial coordinate r for compact star candidates PSRJ 1614-2230, Cen X-3, and 4U 1820-30. Here we fix $n=0$, $\beta=2$, $\xi=0.23$, $q=7\times10^{-7}$, $\delta=-4$, $\lambda=4\times10^{-4}$, $w_q=0.99$, and $\gamma=0.050$ (solid lines), 0.055 (long dashes), 0.060 (dashes), 0.065 (short dashes), 0.070 (dots). Other constant parameters are given in Table 1.

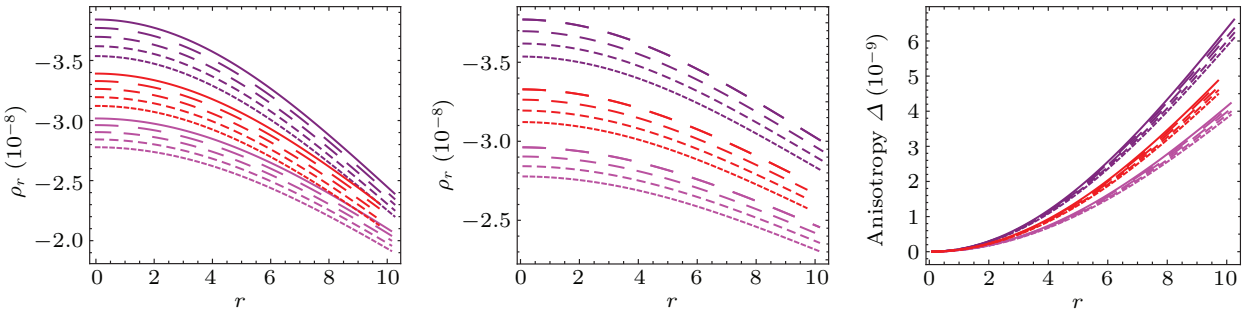


Fig. 2. Radial pressure, tangential pressure, and anisotropy versus radial coordinate r for compact star candidates PSRJ 1614-2230, Cen X-3, and 4U1820-30. Here we fix $n=0$, $\beta=2$, $\xi=0.23$, $q=7\times10^{-7}$, $\delta=-4$, $\lambda=4\times10^{-4}$, $w_q=0.99$, and $\gamma=0.050$ (solid lines), 0.055 (long dashes), 0.060 (dashes), 0.065 (short dashes), 0.070 (dots). Other constant parameters are given in Table 1.

Moreover, to effectively study compact stars, it is important to ensure physical validity, as a study lacking in physical admissibility would not be worthwhile. The density parameter, ρ, can be used as a tool to ensure the study's physical affirmation. Figure 1 provides useful information regarding the propagation of energy parameters. Specifically, Fig. 1(b) illustrates that the energy density behaves in accordance with the physical criteria, with maximum values at the center and smooth, positive declines everywhere within the star's distribution ($0<r\leq R$). This confirms the physical plausibility of the celestial body.

Figure 1(c) visually represents the propagation of quintessence density, ρ_q, which is a crucial characteristic that must remain negative throughout the distribution of stars. Our case also holds true for this property.

The physical characteristics of a celestial object are determined by several critical factors, including the pressure components, such as energy density (ρ). The pressure profiles, namely, p_r (as shown in Fig. 2(a)) and p_t (as displayed in Fig. 2(b)), play a crucial role in this determination. The peak value of the pressure is observed at the central point, where r approaches 0, and subsequently decreases smoothly with an increase in r up to the radius R. It is noteworthy that the pressure components, p_r and p_t, at the surface ($r=R$) are positive. It is important to note that EoS $p_r=\xi\rho$ constrains p_r from approaching zero at the boundary; hence, it remains positive throughout the object, which is reasonable and conforms to the expected behavior of a celestial body.

Anisotropy provides a repulsive force that counterbalances the effects of gradient components, leading to a substantial improvement in the equilibrium and stability of stellar models. The positive anisotropy observed confirms the enduring benefits of these repulsive forces. The anisotropy is determined by the condition that $\Delta|_{(0<r\leq R)}>0$ when $p_t>p_r$, where $\Delta=p_t-p_r$. However, as $r\to0$, Δ tends towards zero. The anisotropy Δ depicted in Fig. 2(c) conforms to the required behavior.

Gradients typically exhibit a negative and decreasing trend, starting from zero at the center ($\mathrm{d}\rho/\mathrm{d}r=\mathrm{d}p_r/\mathrm{d}r=\mathrm{d}p_t/\mathrm{d}r)|_{r\to0}=0$, except for ($\mathrm{d}\rho/\mathrm{d}r$, $\mathrm{d}p_r/\mathrm{d}r$, $\mathrm{d}p_t/\mathrm{d}r)|_{0<r\leq R}<0$ in their graphical representation. The results in Fig. 3(b) confirm that the computed gradients fall within this range.

The composition of compact stellar systems, whether they consist of normal matter or dark matter, is of significant importance. To ensure that the system is made up of normal matter, the values of w_r and w_t for realistic or byronic matter, the EoS must be within the range of $0\leq w_r<1$ and $0<w_t<1$. The EoS expressions are given by

$$w_r=\frac{p_r}{\rho}=\xi,\qquad w_t=\frac{p_t}{\rho}. \tag{40}$$

Figure 3(a) shows that these EoS parameters satisfy the required limiting criteria, ensuring that matter is generally distributed throughout the system.

In GR, the energy–momentum tensor defines momentum, mass, and stress, representing the distribution of matter fields

and gravitation-free fields (GFF) in spacetime. However, the Einstein field equations (EFEs) do not directly relate to the state of matter or allowable GFF in the spacetime manifold. To ensure physically valid solutions of the field equations, energy conditions are employed, which sanction all forms of matter, contradict GFF in GR, and ensure a realistic and physically acceptable distribution of matter. The anisotropic conduct of energy must remain positive and obey specific limiting constraints throughout the stellar body to achieve this distribution. These constraints, referred to as the strong energy condition (SEC), weak energy condition (WEC), null energy condition (NEC), and dominant energy condition (DEC), are expressed as

$$\text{SEC}: \rho + p_\gamma \geq 0,\ \rho + p_r + 2p_t \geq 0, \tag{41}$$

$$\text{WEC}: \rho \geq 0,\ \rho + p_\gamma \geq 0, \tag{42}$$

$$\text{NEC}: \rho + p_\gamma \geq 0, \tag{43}$$

$$\text{DEC}: \rho > |p_\gamma|, \tag{44}$$

where $\gamma = r,t$; r and t denote the radial and tangential coordinates. The findings of our study, depicted in Fig. 4, align with the conventional criteria utilized in researching compact stars.

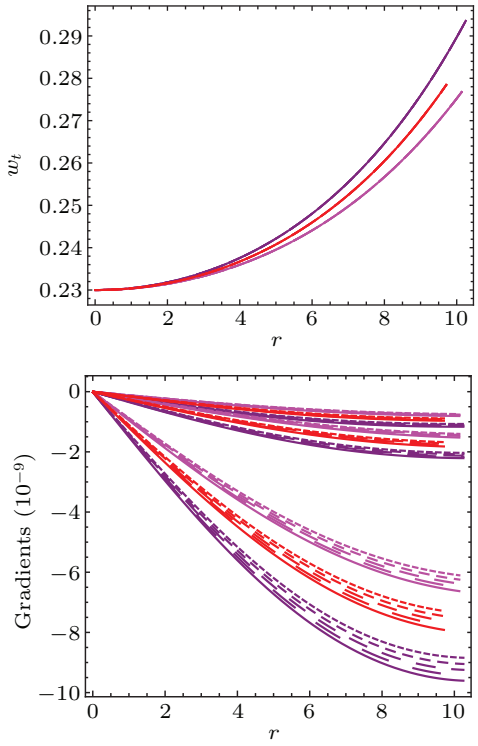


Fig. 3. Tangential component of EoS and gradient versus radial coordinate, r for compact star candidates PSRJ 1614-2230, Cen X-3, and 4U 1820-30. Here we fix $n = 0$, $\beta = 2$, $\xi = 0.23$, $q = 7 \times 10^{-7}$, $\delta = -4$, $\lambda = 4 \times 10^{-4}$, $w_q = 0.99$, and $\gamma = 0.050$ (solid lines), 0.055 (long dashes), 0.060 (dashes), 0.065 (short dashes), 0.070 (dots). Other constant parameters are given in Table 1.

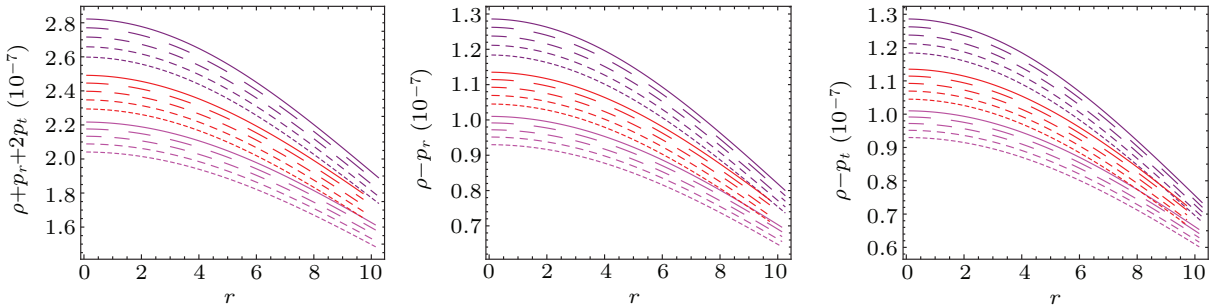


Fig. 4. Energy conditions versus radial coordinate r for compact star candidates PSRJ 1614-2230, Cen X-3, and 4U 1820-30. Here we fix $n = 0$, $\beta = 2$, $\xi = 0.23$, $q = 7 \times 10^{-7}$, $\delta = -4$, $\lambda = 4 \times 10^{-4}$, $w_q = 0.99$, and $\gamma = 0.050$ (solid lines), 0.055 (long dashes), 0.060 (dashes), 0.065 (short dashes), 0.070 (dots). Other constant parameters are given in Table 1.

Tolman–Oppenheimer–Volkoff (TOV) equation[69,70] predicts the stability criteria of the stellar system. The generalized format of the TOV equation for MTRG is given as

$$\frac{\mathrm{d}p_r}{\mathrm{d}r} + \frac{\nu'(\rho + p_r)}{2} - \frac{2(p_t - p_r)}{r} - \frac{\gamma}{4\gamma - 1}\left(\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{\mathrm{d}p_r}{\mathrm{d}r} - \frac{\mathrm{d}p_t}{\mathrm{d}r}\right) = 0, \tag{45}$$

$$F_g + F_h + F_a + F_r = 0, \tag{46}$$

where

$$F_g = -\frac{\nu'(\rho + p_r)}{2}, \quad F_h = -\frac{\mathrm{d}p_r}{\mathrm{d}r}, \quad F_a = \frac{2(p_t - p_r)}{r},$$

$$F_e = -\frac{\gamma}{4\gamma - 1}\left(\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{\mathrm{d}p_r}{\mathrm{d}r} - \frac{\mathrm{d}p_t}{\mathrm{d}r}\right).$$

The TOV equation dictates that a stellar system attains equilibrium when the four forces F_a, F_g, F_h, and F_r are balanced such that their net effect is zero, as presented in Eq. (45). This equilibrium state is critical in avoiding the formation of a singular point during the system's gravitational collapse. Figure 5(a) demonstrates that all the forces in this study section are appropriately balanced, thereby guaranteeing the stability of our solutions and preventing any collapse.

In this discussion, we will analyze the stability of the stellar system by examining two stability parameters v_r^2, the speed along the radial direction, and v_t^2, the speed along the tangent direction. Additionally, we must consider the concept of anisotropic matter distribution, known as the Herrera cracking concept. The Herrera cracking concept states that for stability to be maintained, the sound speeds must satisfy the conditions $0 < v_r^2, v_t^2 < 1$, where the speed of light $c = 1$, and both speeds

are less than the speed of light. The formula for sound speeds is as follows:

$$v_r^2 = \frac{\mathrm{d}p_r}{\mathrm{d}\rho} = \xi, \qquad v_t^2 = \frac{\mathrm{d}p_t}{\mathrm{d}\rho}. \tag{47}$$

Abreu *et al.*[71] introduced an alternative stability criterion, stating that the region is considered to be potentially stable if $v_r^2 > v_t^2$, with $v_r^2 - v_t^2$ remaining unchanged. Later on, Andreasson[72] generalized this criterion to $0 < |v_t^2 - v_r^2| < 1$, implying that there is no cracking and a stable region. Our results, shown in Figs. 5(b) and 5(c), are consistent with the Abreu and Andreasson stability criteria, indicating that our solutions for compact star research are stable.

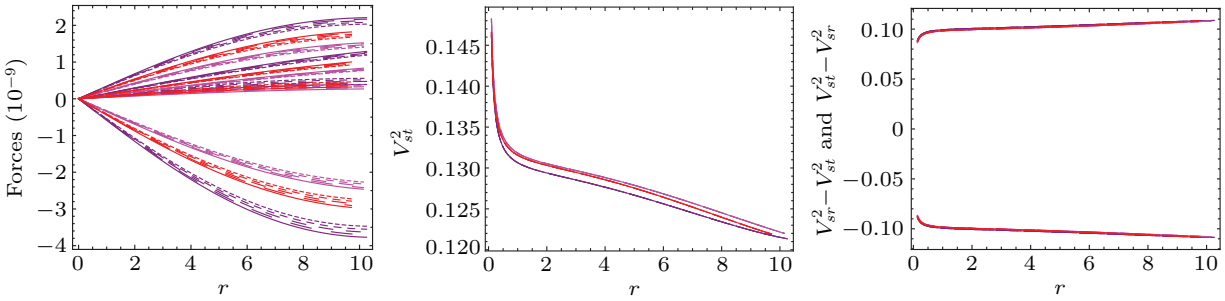


Fig. 5. TOV forces, tangential sound speeds, and Abreu condition versus radial coordinate r for compact star candidates PSRJ 1614-2230, Cen X-3, and 4U 1820-30. Here we fix $n = 0$, $\beta = 2$, $\xi = 0.23$, $q = 7 \times 10^{-7}$, $\delta = -4$, $\lambda = 4 \times 10^{-4}$, $w_q = 0.99$, and $\gamma = 0.050$ (solid lines), 0.055 (long dashes), 0.060 (dashes), 0.065 (short dashes), 0.070 (dots). Other constant parameters are given in Table 1.

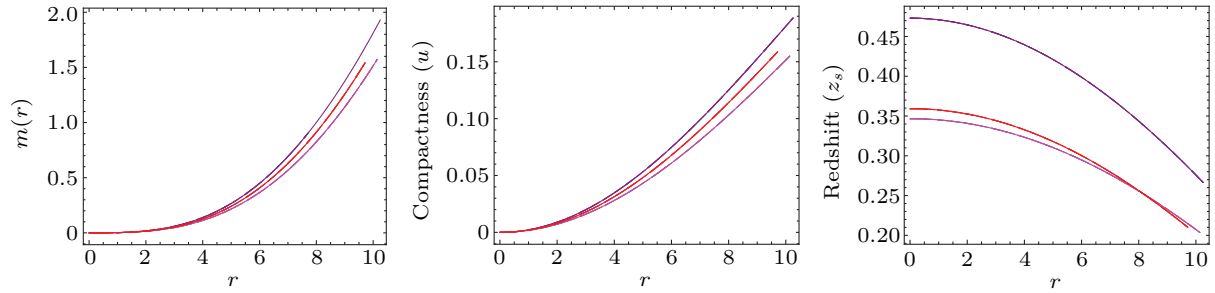


Fig. 6. Mass function, compactness, and gravitational redshift profiles versus radial coordinate r for compact star candidates PSRJ 1614-2230, Cen X-3, and 4U 1820-30. Here we fix $n = 0$, $\beta = 2$, $\xi = 0.23$, $q = 7 \times 10^{-7}$, $\delta = -4$, $\lambda = 4 \times 10^{-4}$, $w_q = 0.99$, and $\gamma = 0.050$ (solid lines), 0.055 (long dashes), 0.060 (dashes), 0.065 (short dashes), 0.070 (dots). Other constant parameters are given in Table 1.

The compactness ratio $m(R)/R$ is a crucial measure of the compactness of a star. The mass can be obtained using the following formula:

$$m(R) = \frac{r}{2}\left(1 - \mathrm{e}^{-\lambda(r)}\right). \tag{48}$$

By incorporating the contribution of Eq. (48), we can derive expressions for the compactness parameter $u(r)$ and the redshift function z_s:

$$u = \frac{m(R)}{R}, \tag{49}$$

$$z_s = \mathrm{e}^{-\frac{a(r)}{2}} - 1. \tag{50}$$

Buchdahl[73] established a maximum value of the compactness parameter $u = m(R)/R < 4/9$. This criterion was generalized for anisotropic matter distributions in Ref. [71]. Buchdahl also set a maximum value criterion for the redshift parameter, $z_s \leq 4.77$.[37] Our study produced a smooth and regular mass function, as shown in Fig. 6(a). Figures 6(b) and 6(c) demonstrate that our results for the compactness and redshift parameters satisfy the physical admissibility criteria for the stellar system.

5. Conclusion

This study of compact objects is based on a straightforward modification of $f(T)$, specifically the modified teleparallel Rastall gravity (MTRG) theory which differs from $f(T)$ gravity due to the inclusion of Rastall's term. To obtain admissible results, we incorporate the Krori and Barua ansatz for spherical symmetric spacetime as an interior solution, and the vacuum case of quintessence spacetime[67,68] as the outer solution. We observe that the Rastall parameter significantly affects the results. We explore different forms of MTRG by choosing the squared-root and power law forms of gravity for the functions $f(T)$ and $h(T)$, respectively. To diagnose the anisotropic nature of the quintessence field in detail, we choose three candidates, namely, PSRJ 1614-2230, Cen X-3, and 4U 1820-30, as stellar models. The main findings of our study are summarized as follows:

Our analysis successfully captures the anisotropic behavior of all parameters. The smooth behavior of the metric potentials, with e^{λ} approaching 1 as $r \to 0$ and e^{ν} remaining positive, is indicative of a stable system. Our expressions for ρ, ρ_q, p_r, and p_t conform to the required behavior of stellar configurations. Additionally, the anisotropy parameter Δ

demonstrates a smooth behavior from the center to the boundary, with negative gradients. Throughout the stellar configurations, energy conditions show positive behavior, and the EoS, speed of sound, and causality limits fulfill required criteria. TOV forces ensure the stability of the system, while the mass function, compactification, and redshift functions exhibit the expected behavior.

In essence, our results align with the Krori and Barua ansatz in the MTRG theory of gravity, representing a physically acceptable framework. While Ditta and Xia[44] used the environment of Rastall teleparallel gravity, an extended version of Rastall's gravity, to discuss the stellar structure using an anisotropic fluid distribution with a spherically symmetric metric, our study explores the environment of MTRG, an extended version of $f(T)$ gravity. As a comparison to the research available in Ref. [44], the compact formation of stellar bodies in this manuscript is less dense. Moreover, research in Ref. [44], is based on Tolaman Kuchowz spacetime, while in this study we use the ansatz suggested by Krori and Barua.

Appendix A

$$
\begin{aligned}
f_1(r) &= \sqrt{\frac{2\lambda\, e^{-Ar^2}\left(e^{\frac{Ar^2}{2}}-1\right)\left(e^{\frac{Ar^2}{2}}-2Br^2-1\right)}{r^2}+1}, \quad f_2(r)=\sqrt{\frac{f_4(r)\, e^{-Ar^2}}{r^2}},\\
f_3(r) &= \left(\frac{e^{-Ar^2}(e^{\frac{Ar^2}{2}}-1)(e^{\frac{Ar^2}{2}}-2Br^2-1)}{r^2}\right)^n,\\
f_4(r) &= -4\lambda\, e^{\frac{Ar^2}{2}}\left(Br^2+1\right)+e^{Ar^2}\left(2\lambda+r^2\right)+2\lambda\left(2Br^2+1\right), \quad f_5(r)=2B\left(-4\lambda\, e^{\frac{Ar^2}{2}}+e^{Ar^2}\left(2\lambda+r^2\right)+2\lambda\right),\\
f_6(r) &= 6B(6\gamma-1)\lambda(\xi+1)r^4(w_q+1)\left(\gamma\delta f_2(r)f_3(r)2^{n+3}+\beta\left(3-2f_2(r)\right)\right)+2\lambda(\xi+1)r^2\\
&\quad\times\left[3\gamma(6\gamma-1)\delta f_3(r)f_2(r)2^{n+3}(w_q+1)-\beta(6\gamma-1)\left(w_q\left(-18B\lambda+6f_2(r)-9\right)-14B\lambda+6f_2(r)-9\right)\right]\\
&\quad+\beta(6\gamma-1)\lambda^2r^3(\xi(3w_q+2)+1)+4\beta(6\gamma-1)\lambda^2(\xi+1)(9w_q+7),\\
f_7(r) &= 2Ar^2\left(B(\xi+1)r^2+2\xi+3\xi w_q+1\right)-2B^2(\xi+1)r^4+2Br^2(4\xi+6\xi w_q+3w_q+3)+(\xi+1)(3w_q+5),\\
f_8(r) &= -24\beta\gamma\lambda+4\beta\lambda-3(6\gamma-1)r^2(w_q+1)\left(\beta+\gamma\delta f_2(r)f_3(r)2^{n+2}+\beta\left(-f_2(r)\right)\right)-36\beta\gamma\lambda w_q+6\beta\lambda w_q,\\
f_9(r) &= -2(6\gamma-1)\lambda r^4\left[A\beta(2B\lambda(\xi+1)+2\xi+3\xi w_q+1)+10\beta B^2\lambda\xi+10\beta B^2\lambda+12\beta B^2\lambda\xi w_q+12\beta B^2\lambda w_q\right.\\
&\quad+7\beta B\xi+6\beta B+3B\gamma\delta f_2(r)f_3(r)2^{n+3}\xi+3B\gamma\delta f_2(r)f_3(r)2^{n+3}+3B\gamma\delta f_2(r)f_3(r)2^{n+3}\xi w_q+3B\gamma\delta\\
&\quad\left.\times f_2(r)f_3(r)2^{n+3}w_q-6\beta Bf_2(r)\xi-6\beta Bf_2(r)-6\beta Bf_2(r)\xi w_q-6\beta Bf_2(r)w_q+9\beta B\xi w_q+6\beta Bw_q\right]\\
&\quad+r^2\left[2\beta(6\gamma-1)\lambda\left[w_q\left[-6A\lambda\xi-6B\lambda(7\xi+6)+3f_2(r)\xi+3f_2(r)-3\xi-3\right]-4A\lambda\xi-2A\lambda-34B\lambda\xi\right.\right.\\
&\quad\left.-32B\lambda+3f_2(r)\xi+3f_2(r)-4\xi-4\left[-3\gamma(6\gamma-1)\delta f_3(r)f_2(r)\lambda 2^{n+3}(\xi+1)(w_q+1)\right]-2\beta B(6\gamma-1)\right.\\
&\quad\left.\times\lambda(\xi+1)r^6(A-B)-\beta B(6\gamma-1)\lambda^2(\xi+1)r^5-\beta(6\gamma-1)\lambda^2r^3(\xi(3w_q+2)+1)-36\beta(6\gamma-1)\lambda^2(\xi+1)(w_q+1),\right.\\
f_{10}(r) &= 12\beta(6\gamma-1)\lambda^2(\xi+1)+2B(6\gamma-1)\lambda(\xi+1)r^4\left(\gamma\delta f_2(r)f_3(r)2^{n+3}+\beta\left(3-2f_2(r)\right)\right)\\
&\quad+2\lambda(\xi+1)r^2\left(\gamma(6\gamma-1)\delta f_3(r)f_2(r)2^{n+3}-\beta(6\gamma-1)\left(-6B\lambda+2f_2(r)-3\right)\right)+\beta(6\gamma-1)\lambda^2\xi r^3,\\
f_{11}(r) &= e^{Ar^2}\left[-2(6\gamma-1)\lambda r^4\left[A\beta\xi+4\beta B^2\lambda(\xi+1)+B\gamma\delta f_2(r)f_3(r)2^{n+3}(\xi+1)+\beta B\left[\xi\left(3-2f_2(r)\right)-2f_2(r)+2\right]\right]\right.\\
&\quad+r^2\left[2\beta(6\gamma-1)\lambda\left[-2A\lambda\xi-2B\lambda(7\xi+6)+f_2(r)\xi r+f_2(r)-\xi-1\right]\right.\\
&\quad\left.\left.-\gamma(6\gamma-1)\delta f_3(r)f_2(r)\lambda 2^{n+3}(\xi+1)\right]-12\beta(6\gamma-1)\lambda^2(\xi+1)+\beta(1-6\gamma)\lambda^2\xi r^3\right],
\end{aligned}
$$

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