

Darboux transformation, infinite conservation laws, and exact solutions for the nonlocal Hirota equation with variable coefficients

Jinzhou Liu(刘锦洲), Xinying Yan(闫鑫颖), Meng Jin(金梦), and Xiangpeng Xin(辛祥鹏)
Citation: Chin. Phys. B, 2023, 32 (**12**): 120401. DOI: 10.1088/1674-1056/acf703
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Nonlinear dynamical wave structures of Zoomeron equation for population models

Ahmet Bekir and Emad H M Zahran
Chin. Phys. B, 2022, 31 (**6**): 060401. DOI: 10.1088/1674-1056/ac48ff

Darboux transformation and soliton solutions of a nonlocal Hirota equation

Yarong Xia(夏亚荣), Ruoxia Yao(姚若侠), and Xiangpeng Xin(辛祥鹏)
Chin. Phys. B, 2022, 31 (**2**): 020401. DOI: 10.1088/1674-1056/ac11e9

The global monopole spacetime and its topological charge

Hongwei Tan(谭鸿威), Jinbo Yang(杨锦波), Jingyi Zhang(张靖仪), Tangmei He(何唐梅)
Chin. Phys. B, 2018, 27 (**3**): 030401. DOI: 10.1088/1674-1056/27/3/030401

Unstable and exact periodic solutions of three-particles time-dependent FPU chains

Liu Qi-Huai (刘期怀), Xing Ming-Yan (邢明燕), Li Xin-Xiang (李新祥), Wang Chao (王超)
Chin. Phys. B, 2015, 24 (**12**): 120401. DOI: 10.1088/1674-1056/24/12/120401

Spherically symmetric solution in higher-dimensional teleparallel equivalent of general relativity

Gamal G. L. Nashed
Chin. Phys. B, 2013, 22 (**2**): 020401. DOI: 10.1088/1674-1056/22/2/020401

Darboux transformation, infinite conservation laws, and exact solutions for the nonlocal Hirota equation with variable coefficients

Jinzhou Liu(刘锦洲), Xinying Yan(闫鑫颖), Meng Jin(金梦), and Xiangpeng Xin(辛祥鹏)[†]

School of Mathematical Sciences, Liaocheng University, Liaocheng 252059, *China*

(Received 22 June 2023; revised manuscript received 5 September 2023; accepted manuscript online 6 September 2023)

This article presents the construction of a nonlocal Hirota equation with variable coefficients and its Darboux transformation. Using zero-seed solutions, 1-soliton and 2-soliton solutions of the equation are constructed through the Darboux transformation, along with the expression for N-soliton solutions. Influence of coefficients that are taken as a function of time instead of a constant, i.e., coefficient function $\delta(t)$, on the solutions is investigated by choosing the coefficient function $\delta(t)$, and the dynamics of the solutions are analyzed. This article utilizes the Lax pair to construct infinite conservation laws and extends it to nonlocal equations. The study of infinite conservation laws for nonlocal equations holds significant implications for the integrability of nonlocal equations.

Keywords: infinite conservation laws, nonlocal Hirota equation with variable coefficient, soliton solutions, Darboux transformation

PACS: 04.20.Jb, 05.45.Yv, 02.30.Ik, 02.30.Jr **DOI:** 10.1088/1674-1056/acf703

1. Introduction

Soliton theory is an important direction in the field of nonlinearity, which can reflect an essential class of natural phenomena. The investigation of nonlinear evolution equations (NLEEs) has been a topic of significant interest.[1–3] NLEEs serve as a valuable modeling tool for a wide range of complex physical phenomena, including mathematical physics, isoparticle physics, fluid dynamics, atmospheric and oceanic processes, etc. Therefore, research of the exact solutions of NLEEs with a variety of physical models is of great significance.[4–8] Through the concerted efforts of many researchers, many effective methods have been developed, such as the Hirota bilinear method,[9–11] the Lie symmetry analysis method,[12] the Riemann–Hilbert method,[13,14] neural network method,[15] and the Darboux transformation.[16–18]

The Darboux transformation is a canonical transformation of the spectral problem for NLEEs. The idea is to construct exact solutions of the integrable equations by means of solutions of the eigenfunctions of the NLEEs associated with the Lax pair. The Darboux transform is a very useful method for obtaining the solutions of soliton equations. In soliton theory, conservation laws also play an important role in integrability of soliton equations.[19,20] Conservation laws are a reaction to the phenomenon that certain physical amounts are not changed with time. The infinite conservation laws are intimately linked with the existence of solitary particles.[21,22] The infinite conservation laws are closely related to the integrability of the equation. If the infinite conservation laws of an equation can be constructed, it shows that the equation is conservatively integrable. In addition, the solutions to the equation must exist and be completely integrable. Therefore, using infinite conservation laws of the equation is also an effective way to study whether the equation is integrable.[23,24]

In Ref. [25], Chen *et al.* investigated the integrable SWW equation of the Ablowitz–Kaup–Newell–Segur (AKNS) type. They obtained soliton-cnoidal wave interaction solutions by solving the initial value problem and deriving the Darboux transformation for the extended system. In Ref. [26], Li *et al.* focused on a class of classical nonlinear Schrödinger equation. They proposed an extended generalized Darboux transformation method to construct mixed solutions involving rogue waves and breather waves. In Ref. [27], Yang *et al.* utilized the Darboux transformation to provide localized wave solutions in matrix form for the nonlocal integrable Lakshmanan–Porsezian–Daniel equation. In this work, we employ the Darboux transformation method to study the nonlocal variable-coefficient Hirota equation and construct first-order soliton solutions.

We will study a $(1+1)$-dimensional nonlocal Hirota equation with variable coefficient

$$u_t - 2\delta(t)\mathrm{i}u(-x,-t)u^2\gamma + 6\delta(t)\rho u_x u(-x,-t)u - \delta(t)\gamma\mathrm{i}u_{xx} + \delta(t)\rho u_{xxx} = 0, \tag{1}$$

where γ and ρ are arbitrary constants, and $\delta(t)$ is an arbitrary function of t. In the literature,[28] Zhang *et al.* followed the AKNS procedure to construct Lax pairs with spectral parameters and obtained the coupled Hirota equation. In this paper, we construct the nonlocal Hirota equation with variable coefficients by means of modifying its Lax pair, and the symmetric

[†]Corresponding author. E-mail: xinxiangpeng@lcu.edu.cn

© 2023 Chinese Physical Society and IOP Publishing Ltd

reduction. For the constant coefficient equation, since its coefficients are fixed, the physical model described by the equation is also very restrictive. By transforming the constant coefficient equation into a variable coefficient equation, the equation is able to describe a much wider variety of natural phenomena. Therefore, research of variable coefficient equations is of even more physical significance.[29–31] When $\delta(t)=1$, Eq. (1) is the third flow of the NLS structure, and this nonlinear evolution equation can be used to describe important models in physics, fibre optics and other engineering disciplines. Equation (1) can be simplified to a nonlinear Schrödinger equation when $\delta(t)=1$, $\gamma=1$, and $\rho=0$.

The nonlocal variable coefficient Hirota equations play a critical role in various realms of physics. They unveil the underlying mechanisms of complex nonlinear phenomena in multiple areas, including the propagation of laser pulses in optics, the interaction of waves in plasma physics, the nonlinear characteristics of phonon propagation in solids, shockwaves and water wave phenomena in fluid dynamics, as well as the conduction of neural impulses in biological systems. As a result, the nonlocal variable coefficient Hirota equations have become powerful tools for studying and explaining these intricate nonlinear phenomena. They have a significant impact on research and applications in fields such as optics, plasma physics, solid-state physics, fluid dynamics, and biomedical science.

The main innovations of this paper lie in the introduction of variable coefficients to the nonlocal Hirota equation and the construction of the corresponding Darboux transformation. This approach imparts variable coefficients to the traditional Hirota equation, thereby enriching the equation's behavior. Through the Darboux transformation, new exact solutions can be generated from known zero-seed solutions, thereby expanding the scope and nature of solutions. This paper also explores the impact of coefficients on solutions by selecting the coefficient function, revealing the effects of specific parameter choices on solution characteristics. Furthermore, by utilizing the Lax pair, the infinite conservation laws of the variable coefficient nonlocal Hirota equation are established, and they are extended to the variable coefficient nonlocal equation. Investigating the infinite conservation laws of nonlocal equations aids in determining their integrability, consequently deriving more exact solutions and understanding the structure of solutions.

The outline of this article is as follows: In Section 2, the $(1+1)$-dimensional nonlocal Hirota equation with variable coefficients is constructed by means of the linear spectral problem. In Section 3, the Darboux transformation is constructed with the help of the Lax pair. In Section 4, the application of the constructed Darboux transform to the zero seed solution results in the discovery of several new exact solutions. In Section 5, the infinite conservation laws for the nonlocal Hirota equation with variable coefficients are constructed based on the Lax pair. Finally, this work is summarized in Section 6.

2. The (1+1)-dimensional nonlocal Hirota equation with variable coefficients

Our main work in this section is to construct Eq. (1) by means of Lax pairs. The Lax pair form is

$$\begin{cases}\psi_x = U\psi,\\ \psi_y = V\psi,\end{cases} \tag{2}$$

where $\psi=(\psi_1(x,t),\psi_2(x,t))^{\mathrm{T}}$, and

$$U=\begin{pmatrix}-\mathrm{i}\lambda & u\\ v & \mathrm{i}\lambda\end{pmatrix},\quad V=\begin{pmatrix}H & J\\ K & -H\end{pmatrix}, \tag{3}$$

where

$$\begin{aligned}H&=\delta(t)[-4\mathrm{i}\rho\lambda^3-2\mathrm{i}\gamma\lambda^2-2\mathrm{i}\rho\lambda uv-\mathrm{i}\gamma uv+\rho(vu_x-uv_x)],\\ J&=\delta(t)[4\rho\lambda^2u+(2\mathrm{i}\rho u_x+2\gamma u)\lambda+\mathrm{i}\gamma u_x-\rho(u_{xx}-2u^2v)],\\ K&=\delta(t)[4\rho\lambda^2v-(2\mathrm{i}\rho v_x-2\gamma v)\lambda-\mathrm{i}\gamma v_x-\rho(v_{xx}-2v^2u)],\end{aligned} \tag{4}$$

where u and v are smooth potentials with independent variables x and t; ψ is a vector function; and λ is a constant spectral parameter. In addition, ψ must satisfy the compatibility condition $\psi_{xt}=\psi_{tx}$.

The variable coefficient coupled Hirota equation can be obtained under the compatibility condition as follows:

$$\begin{aligned}&u_t+2\mathrm{i}\delta(t)vu^2\gamma-6\delta(t)u_xvu\rho-\mathrm{i}\delta(t)u_{xx}\gamma+u_{xxx}\delta(t)\rho=0,\\ &v_t-2\mathrm{i}\delta(t)v^2u\gamma-6\delta(t)vuv_x\rho+\mathrm{i}\delta(t)v_{xx}\gamma+v_{xxx}\delta(t)\rho=0.\end{aligned} \tag{5}$$

To obtain the nonlocal form of Eq. (5), we assume that a symmetric reduction is

$$v(x,t)=-u(-x,-t). \tag{6}$$

By means of Eq. (6), Eq. (5) can be transformed to

$$\begin{aligned}&u(x,t)_t+\delta(t)(2\mathrm{i}u(-x,-t)u(x,t)^2\gamma\\ &-6u(x,t)_xu(-x,-t)u(x,t)\rho-\mathrm{i}u(x,t)_{xx}\gamma+u(x,t)_{xxx}\rho)=0,\\ &u(x,t)_t+\delta(-t)(2\mathrm{i}u(-x,-t)u(x,t)^2\gamma\\ &-6u(x,t)_xu(-x,-t)u(x,t)\rho-\mathrm{i}u(x,t)_{xx}\gamma+u(x,t)_{xxx}\rho)=0.\end{aligned} \tag{7}$$

When $\delta(t)=\delta(-t)$, the nonlocal form of Eq. (7) can be obtained.

3. Darboux transformation

In order to study the Darboux transformation of the nonlocal Hirota equation with variable coefficients, we firstly in-

vestigate the coupled Hirota equation with variable coefficients. A canonical transformation is firstly introduced as follows:

$$\psi^{[1]} = T^{[1]}\psi. \tag{8}$$

By means of the canonical transformation mentioned above, we can transform the spectral problem (2) into a new problem with the same matrix form

$$\psi_x^{[1]} = U^{[1]}\psi^{[1]}, \quad \psi_t^{[1]} = V^{[1]}\psi^{[1]}, \tag{9}$$

where

$$\begin{aligned} U^{[1]} &= (T^{[1]}U + T_x^{[1]})(T^{[1]})^{-1}, \\ V^{[1]} &= (T^{[1]}V + T_t^{[1]})(T^{[1]})^{-1}. \end{aligned} \tag{10}$$

In order to make Eq. (1) invariant under transformation (8), one has to find a matrix $T^{[1]}$ so that $U^{[1]}, V^{[1]}$ and U, V have the same forms. A Darboux matrix in Eq. (8) is assumed to be

$$T^{[1]} = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} &A(\lambda_i) = \lambda^N + \sum_{i=0}^{N-1} A_i(x,t)\lambda^i, \ \ B(\lambda_i) = \sum_{i=0}^{N-1} B_i(x,t)\lambda^i, \\ &C(\lambda_i) = \sum_{i=0}^{N-1} C_i(x,t)\lambda^i, \ \ D(\lambda_i) = \lambda^N + \sum_{i=0}^{N-1} D_i(x,t)\lambda^i. \end{aligned} \tag{12}$$

Based on the symmetry reduction from Eq. (6), we can get the constraint condition of Eq. (12) as follows:

$$A_i(x,t) = C_i(-x,-t). \tag{13}$$

Moreover, A_i, B_i, C_i and D_i satisfy the following linear relationships:

$$\begin{aligned} &\sum_{i=0}^{N-1} (A_i + \omega_j B_i)\lambda^i = -\lambda_j^N, \\ &\sum_{i=0}^{N-1} (C_i + \omega_j D_i)\lambda^i = -\omega_j \lambda_j^N, \end{aligned} \tag{14}$$

where

$$\omega_j = \frac{h_2(\lambda_j) - \mu_j b_2(\lambda_j)}{h_1(\lambda_j) - \mu_j b_1(\lambda_j)}, \ \ 1 \le j \le 2N. \tag{15}$$

Here $h(\lambda_j) = (h_1(\lambda_j),\ h_2(\lambda_j))^{\mathrm{T}}$ and $b(\lambda_j) = (b_1(\lambda_j), b_2(\lambda_j))^{\mathrm{T}}$ are the two fundamental solutions of the spectral problem. By selecting appropriate coefficients for $\lambda_j, \mu_j (\lambda_j \neq \mu_j^*)$, the coefficients in system (14) can be made to be non-zero. As a result, A_i, B_i, C_i and D_i are uniquely determined by system (14).

From Eq. (12), it is obvious that $\det T(\lambda)$ is a polynomial of degree $2N$ in terms of λ. Furthermore,

$$\det T^{[1]}(\lambda_j) = A(\lambda_j)D(\lambda_j) - B(\lambda_j)C(\lambda_j). \tag{16}$$

As derived from Eq. (14), we can obtain

$$A(\lambda_j) = -\omega_j B(\lambda_j), \ \ C(\lambda_j) = -\omega_j D(\lambda_j), \tag{17}$$

hence, $\det T^{[1]}(\lambda_j) = 0$. It can be concluded that λ_j $(1 \le j \le N)$ constitutes $2N$ roots of $\det T^{[1]}(\lambda)$, implying

$$\det T^{[1]}(\lambda) = \prod_{j=1}^{2N} (\lambda - \lambda_j). \tag{18}$$

Based on the aforementioned facts, we can demonstrate the following propositions.

Proposition 1 According to Eq. (10), it can be determined that $U^{[1]}$ and U have the same form

$$U^{[1]} = \begin{pmatrix} -\mathrm{i}\lambda & u^{[1]} \\ v^{[1]} & \mathrm{i}\lambda \end{pmatrix}, \tag{19}$$

where $u^{[1]}$ and $v^{[1]}$ can be obtained by Eq. (12).

The transformation formula between the old and new potential functions is given by

$$u^{[1]} = u + 2\mathrm{i}B_{N-1}, \ \ v^{[1]} = v - 2\mathrm{i}C_{N-1}. \tag{20}$$

Assume $T^{-1} = (\det T)^{-1}T^*$, and

$$(T_x + TU)T^* = \begin{pmatrix} h_{11}(\lambda) & h_{12}(\lambda) \\ h_{21}(\lambda) & h_{22}(\lambda) \end{pmatrix} = (h_{ij})_{2\times 2}, \tag{21}$$

where $*$ represents a conjugate complex number; $h_{11}(\lambda)$ and $h_{22}(\lambda)$ are polynomials of degree $2N+1$ with respect to λ; while $h_{12}(\lambda)$ and $h_{21}(\lambda)$ are polynomials of degree $2N$ with respect to λ. This can be deduced from spectral problem (3) and Eq. (15).

$$\omega_{jx} = v + 2\mathrm{i}\lambda_j\omega_j - u\omega_j^2 \tag{22}$$

From Eqs. (17) and (20), we can infer that λ_j $(1 \le j \le 2N)$ are roots of $h_{ij}(\lambda)$ $(i, j = 1, 2)$.

Equation (21) can be expressed in the following form:

$$(T_x + TU)T^* = (\det T)P(\lambda), \tag{23}$$

where

$$P(\lambda) = \begin{pmatrix} p_{11}^{[0]} + p_{11}^{[1]}\lambda & p_{12}^{[0]} \\ p_{21}^{[0]} & p_{22}^{[0]} + p_{22}^{[1]}\lambda \end{pmatrix}, \tag{24}$$

$P_{ij}^{[k]}$ $(i, j = 1, 2, k = 0, 1)$ is independent of λ. According to $T^{-1} = T^*(\det T)^{-1}$, Eq. (23) can be transformed to

$$T_x + TU = P(\lambda)T. \tag{25}$$

By comparing the coefficients of the same power of λ on both sides of Eq. (25) and then according to Eq. (12) we can get

$$P_{11}^{[0]} = 0, \ P_{12}^{[0]} = u^{[1]}, \ P_{21}^{[0]} = v^{[1]}, \ P_{22}^{[0]} = 0, \ P_{11}^{1} = -\mathrm{i}, \ P_{22}^{1} = \mathrm{i}. \tag{26}$$

Substituting Eq. (26) into Eq. (24) yields

$$P(\lambda)=\begin{pmatrix}-\mathrm{i}\lambda & u^{[1]}\\ v^{[1]} & \mathrm{i}\lambda\end{pmatrix}=U^{[1]}. \tag{27}$$

Proposition 1 is proved.

Proposition 2 According to Eq. (10) it can be determined that $V^{[1]}$ and V have the same form. Since $h(\lambda_i)$ and $b(\lambda_i)$ also satisfy Eq. (9), we can prove Proposition 2 using the same method of proof as in Proposition 1. $V^{[1]}$ has the following form:

$$V=\begin{pmatrix}H^{[1]} & J^{[1]}\\ K^{[1]} & -H^{[1]}\end{pmatrix}, \tag{28}$$

where

$$\begin{aligned}H^{[1]}=&\delta(t)(-4\mathrm{i}\rho\lambda^3-2\mathrm{i}\gamma\lambda^2-2\mathrm{i}\rho\lambda u^{[1]}v^{[1]}-\mathrm{i}\gamma u^{[1]}v^{[1]}\\&+\rho(v^{[1]}u_x^{[1]}-u^{[1]}v_x^{[1]})),\\ J^{[1]}=&\delta(t)(4\rho\lambda^2u^{[1]}+(2\mathrm{i}\rho u_x^{[1]}+2\gamma u^{[1]})\lambda+\mathrm{i}\gamma u_x^{[1]}\\&-\rho(u_{xx}^{[1]}-2(u^{[1]})^2v^{[1]})),\\ K^{[1]}=&\delta(t)(4\rho\lambda^2v^{[1]}-(2\mathrm{i}\rho v_x^{[1]}-2\gamma v^{[1]})\lambda-\mathrm{i}\gamma v_x^{[1]}\\&-\rho(v_{xx}^{[1]}-2(v^{[1]})^2u^{[1]})).\end{aligned} \tag{29}$$

4. Exact solutions of the nonlocal Hirota equation with variable coefficients

In this section, we utilize the Darboux transformation method to construct exact solutions for the nonlocal variable coefficient Hirota equation. To begin, let us assume two sets of solutions for the spectral problem (3):

$$h(\lambda)=(h_1(\lambda),h_2(\lambda))^{\mathrm{T}},\ \ b(\lambda)=(-h_2^*(\lambda^*),h_1^*(\lambda^*))^{\mathrm{T}}, \tag{30}$$

where the parameters satisfy

$$\lambda_{2j}=\lambda_{2j-1}^*,\ \ \mu_{2j}=\mu_{2j-1}^{*-1}. \tag{31}$$

We can get $\omega_j^{-1}=-\omega_{2j-1}^*$, $D_i=A_i^*$, $C_i=-B_i^*$, $(0\le k\le N-1)$, with

$$\omega_{2j-1}=\frac{h_2(\lambda_{2j-1})-\mu_{2j-1}b_2(\lambda_{2j-1})}{h_1(\lambda_{2j-1})-\mu_{2j-1}b_1(\lambda_{2j-1})},\ \ 1\le j\le N, \tag{32}$$

where A_i and B_i can be determined by the following algebraic system:

$$\begin{aligned}&\sum_{i=0}^{N-1}(A_i+\omega_{2j-1}B_i)\lambda_{2j-1}^i=-\lambda_{2j-1}^N,\\&\sum_{i=0}^{N-1}(\omega_{2j-1}^*A_i-B_i)\lambda_{2j-1}^{*i}=-\omega_{2j-1}\lambda_{2j-1}^{*N}.\end{aligned} \tag{33}$$

We take the trivial solution $u=0$, $v=0$ as the seed solution and bring them into the Lax pair to obtain

$$\begin{aligned}&\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}-\mathrm{i}\lambda & 0\\ 0 & \mathrm{i}\lambda\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\\&\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t=\begin{pmatrix}-2\mathrm{i}\delta(t)\lambda^2(2\lambda\rho+\gamma) & 0\\ 0 & 2\mathrm{i}\delta(t)\lambda^2(2\lambda\rho+\gamma)\end{pmatrix}\\&\qquad\times\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.\end{aligned} \tag{34}$$

By solving the spectral problem (34), we can get

$$\begin{aligned}h(x,t;\lambda)&=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\lambda_i(4\int\delta(t)\mathrm{d}t\rho\lambda_i^2+2\int\delta(t)\mathrm{d}t\gamma\lambda_i+x)}\\ 0\end{pmatrix},\\ b(x,t;\lambda)&=\begin{pmatrix}0\\ \mathrm{e}^{\mathrm{i}\lambda_i(4\int\delta(t)\mathrm{d}t\rho\lambda_i^2+2\int\delta(t)\mathrm{d}t\gamma\lambda_i+x)}\end{pmatrix}.\end{aligned} \tag{35}$$

Substituting Eq. (35) into Eq. (32) yields

$$\omega_{2j-1}=\mu_{2j-1}\mathrm{e}^{-2\mathrm{i}\lambda_{2j-1}(4\rho\lambda_{2j-1}^2\int\delta(t)\,\mathrm{d}t+2\gamma\lambda_{2j-1}\int\delta(t)\,\mathrm{d}t+x)}. \tag{36}$$

Here, Eq. (13) determines that it must satisfy the following constraints:

$$\mu_{2j-1}^2=1,\ \ \mu_{2j-1}^{*2}=1. \tag{37}$$

Thus, the N-fold Darboux transformation of Eq. (1) is

$$u^{[j]}(x,t)=2\mathrm{i}B_{N-1}+u, \tag{38}$$

where $B_{N-1}=\Omega_{B_{N-1}}/\Omega$.

Here, Ω can be obtained from the coefficients of the linear algebraic system (33):

$$\Omega=\begin{vmatrix}1 & \omega_1 & \lambda_1 & \omega_1\lambda_1 & \cdots & \lambda_1^{N-1} & \omega_1\lambda_1^{N-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & \omega_{2N-1} & \lambda_{2N-1} & \omega_{2N-1}\lambda_{2N-1} & \cdots & \lambda_{2N-1}^{N-1} & \omega_{2N-1}\lambda_{2N-1}^{N-1}\\ \omega_1^* & -1 & \omega_1^*\lambda_1^* & -\lambda_1^* & \cdots & \omega_1^*\lambda_1^{*N-1} & -\lambda_1^{*N-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \omega_{2N-1}^* & -1 & \omega_{2N-1}^*\lambda_{2N-1}^* & -\lambda_{2N-1}^* & \cdots & \omega_{2N-1}^*\lambda_{2N-1}^{*N-1} & -\lambda_{2N-1}^{*N-1}\end{vmatrix}. \tag{39}$$

The $\Omega_{B_{N-1}}$ is obtained by replacing the $2N$-th row of determinant Ω with $(-\lambda_1^N,\ldots,-\lambda_{2N-1}^N,-\omega_1^*\lambda_1^*,\ldots,-\omega_{2N-1}^*\lambda_{2N-1}^*)^{\mathrm{T}}$. As a result, by employing Eq. (38), N-soliton solutions for the nonlocal variable coefficient Hirota equation can be constructed.

Next, we analyze the 1-soliton solution and 2-soliton solution of the nonlocal Hirota equation with variable coefficients.

Case I When $N=1$, using Eqs. (38) and (39), we can obtain

$$B_0=-\frac{\omega_1{}^*(\lambda_1-\lambda_1{}^*)}{\omega_1\omega_1{}^*+1}. \tag{40}$$

Therefore, we can derive the 1-soliton solution of the nonlocal Hirota equation with variable coefficients as follows:

$$u=\frac{2\mathrm{i}\mu_1{}^*\mathrm{e}^{-2\mathrm{i}\lambda_1{}^*(4\int\delta(t)\mathrm{d}t\rho\lambda_1{}^{*2}+2\int\delta(t)\mathrm{d}t\gamma\lambda_1{}^*+x)}(\lambda_1-\lambda_1{}^*)}{\mu_1\mathrm{e}^{2\mathrm{i}\lambda_1(4\int\delta(t)\mathrm{d}t\rho\lambda_1{}^2+2\int\delta(t)\mathrm{d}t\gamma\lambda_1+x)}\mu_1{}^*\mathrm{e}^{-2\mathrm{i}\lambda_1{}^*(4\int\delta(t)\mathrm{d}t\rho\lambda_1{}^{*2}+2\int\delta(t)\mathrm{d}t\gamma\lambda_1{}^*+x)}+1}, \tag{41}$$

where $\lambda_1=\sigma_1+\tau_1\mathrm{i}$.

In the following, we analyze the dynamic behavior to the solutions of the variable coefficient nonlocal Hirota equation, as shown in Fig. 1. Firstly, we examine the effect of the coefficient function $\delta(t)$ on the solutions. We choose the parameters as $\sigma_1=0.1$, $\tau_1=0.5$, $\mu_1=1$, $\gamma=2$, $\rho=2$ and observe the dynamic behavior of the solutions by choosing the coefficient function. When $\delta(t)=1$, we can obtain a set of bright soliton solutions. When $\delta(t)=\sin t$, a set of soliton solutions with periodic properties are obtained. When $\delta(t)=\mathrm{sech}t$, a set of kinked solitons can be obtained. We can easily find from Fig. 1 that different expressions of coefficient function will have different effects on the soliton solutions. The coefficient function can change the shape of the soliton, rather than the amplitude of the soliton. The peak size of the soliton is 1.

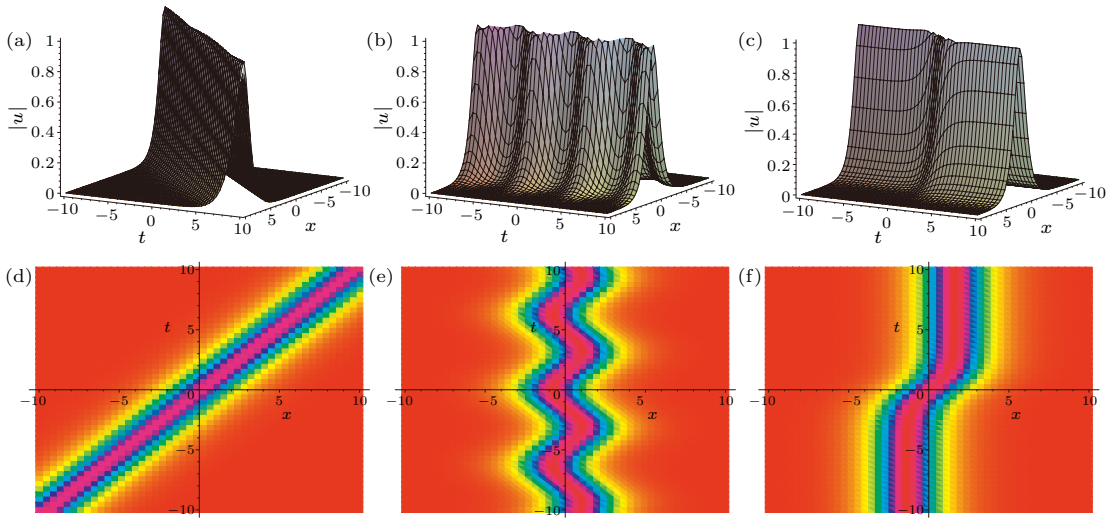


Fig. 1. Effect of the coefficient function on the solutions in the (x,t) plane: (a) a soliton, (b) a periodic soliton, (c) a kinked soliton. (d)–(f) Density maps of the solutions.

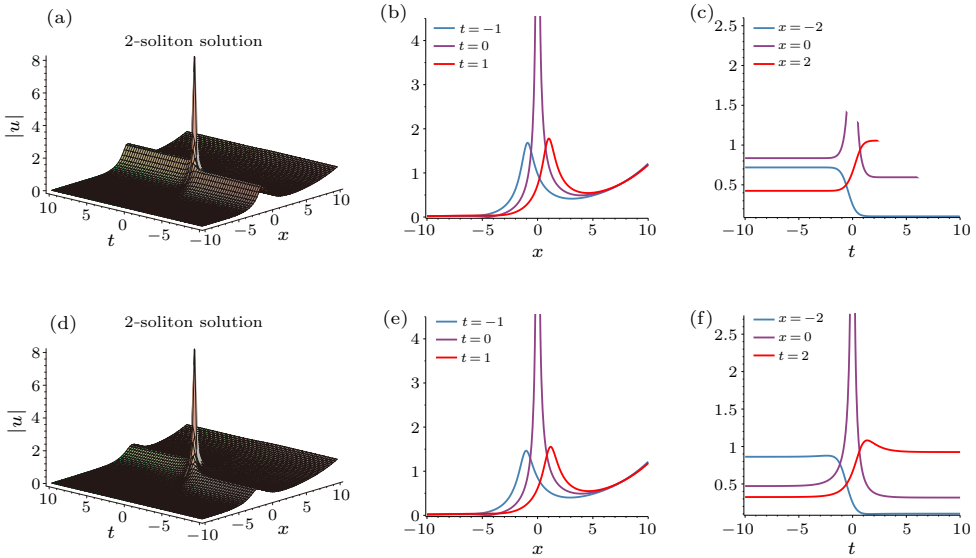


Fig. 2. Effect of the coefficient function on the solutions in the (x,t) plane: [(a), (d)] peakon soliton and kink soliton interaction solutions, [(b), (c), (e), (f)] two-dimensional plots of the solution in the x and t directions.

Case II When $N=2$, let $\lambda_k=\sigma_k+\tau_k \mathrm{i}$, $(k=1,2)$, through the determinant (39) we get

$$\Omega_{B_1}=\begin{vmatrix} 1 & \omega_1 & \lambda_1 & -{\lambda_1}^2 \\ 1 & \omega_3 & \lambda_3 & -{\lambda_3}^2 \\ {\omega_1}^* & -1 & {\lambda_1}^*{\omega_1}^* & -{\omega_1}^*{\lambda_1}^{*2} \\ {\omega_3}^* & -1 & {\omega_3}^*{\lambda_3}^* & -{\omega_3}^*{\lambda_3}^{*2} \end{vmatrix},$$
$$\Omega=\begin{vmatrix} 1 & \omega_1 & \lambda_1 & \omega_1\lambda_1 \\ 1 & \omega_3 & \lambda_3 & \omega_3\lambda_3 \\ {\omega_1}^* & -1 & {\lambda_1}^*{\omega_1}^* & -{\lambda_1}^* \\ {\omega_3}^* & -1 & {\omega_3}^*{\lambda_3}^* & -{\lambda_3}^* \end{vmatrix}. \tag{42}$$

The 2-soliton solution of Eq. (1) can be obtained by using Eqs. (38) and (42):

$$u[2]=u+2\mathrm{i}\frac{\Omega_{B_1}}{\Omega}. \tag{43}$$

We choose the coefficient function as $\delta(t)=\mathrm{e}^{-t^2}$ and $\delta(t)=\mathrm{sech}t$, which yields two sets of peakon soliton and kink soliton interaction solutions. We select the parameters as $\sigma_1=0$, $\tau_1=0.3$, $\sigma_2=0$, $\tau_2=0.1$, $\mu_1=1$, and $\mu_2=-1$. Peakon solutions are a special case of solitary wave solutions. They are similar to solitary waves but with a crucial difference C, and their waveforms have peaks or crests. The presence of these peaks allows peakon solutions to maintain their shape during the time evolution without spreading or dispersing. Their dynamical behavior is illustrated in Fig. 2: as t approaches zero, peakon solutions exhibit discreteness, meaning their positions and velocities are discrete during the time evolution, and the peaks undergo sudden increases. However, the peakon solutions maintain their peak-shaped form throughout the time evolution, and this stability makes them highly significant in nonlinear wave equations. Peakon solitons interact with kink solitons, leading to changes in the kink soliton wave peak.

The N-soliton solutions of the nonlocal Hirota equation with variable coefficients can be constructed by Eq. (38) and the determinant (39).

5. Conservation laws of the variable coefficient nonlocal Hitota equation

In this section, we utilize Lax pairs to construct conservation laws for Eq. (1). The corresponding space part can be obtained from the spectral problem (2) as

$$\begin{aligned} \psi_{1,x}&=-\lambda\psi_1+u\psi_2, \\ \psi_{2,x}&=\lambda\psi_2+v\psi_1. \end{aligned} \tag{44}$$

The corresponding time development equation is

$$\begin{aligned} \psi_{1,t}&=H\psi_1+J\psi_2, \\ \psi_{2,t}&=K\psi_1-H\psi_2. \end{aligned} \tag{45}$$

By introducing a transformation $\varpi=\psi_1/\psi_2$, a Riccati equation related to ϖ can be obtained according to Eq. (44).

The corresponding time development equation is

$$\varpi_x=v+2\lambda\varpi-u\varpi^2. \tag{46}$$

Transformation of Eq. (46) is

$$(u\varpi)_x=uv+2\lambda u\varpi-(u\varpi)^2+\frac{u_x}{u}(u\varpi). \tag{47}$$

Then we construct the following expansions:

$$u\varpi=\sum_{i=0}^{\infty}\frac{\varpi_i}{(2\lambda)^{i+1}}. \tag{48}$$

Substituting Eq. (48) into Eq. (47), the following recursive formula can be obtained by comparing the coefficients of the same power of λ:

$$\begin{aligned} &\varpi_0=-uv, \\ &\varpi_1=-uv_x, \\ &\varpi_2=u^2v^2-uv_{xx}, \\ &\cdots \\ &\varpi_{i+1}=u\Big(\frac{\varpi_i}{u}\Big)_x+\sum_{j=0}^{i-1}\varpi_j\varpi_{i-j-1},\quad (i=1,2,\ldots). \end{aligned} \tag{49}$$

Based on the symmetric reduction (6), Eq. (49) can be transformed into the following nonlocal form:

$$\begin{aligned} &\varpi_0=u(x,t)u(-x,-t), \\ &\varpi_1=u(x,t)u(-x,-t)_x, \\ &\varpi_2=u(x,t)^2u(-x,-t)^2+u(x,t)u(-x,-t)_{xx}, \\ &\qquad\cdots \\ &\varpi_{i+1}=u(x,t)(\frac{\varpi_i}{u(x,t)})_x+\sum_{j=0}^{i-1}\varpi_j\varpi_{i-j-1},\quad (i=1,2,\ldots). \end{aligned} \tag{50}$$

In order to further construct the conservation laws for the variable coefficient nonlocal Hirota equation, we make ψ identically divisible by Eqs. (44) and (45):

$$(\ln\psi_1)_x=u\varpi-\lambda,\quad (\ln\psi_1)_t=H+J\varpi. \tag{51}$$

Using the compatibility condition $(\ln\psi_1)_{xt}=(\ln\psi_1)_{tx}$ of Eq. (51), we can obtain

$$(u\varpi-\lambda)_t=(H+J\varpi)_x. \tag{52}$$

Transforming Eq. (4) by symmetry reduction (6) leads to

$$\begin{aligned} H=&\delta(t)[-4\mathrm{i}\rho\lambda^3-2\mathrm{i}\gamma\lambda^2+2\mathrm{i}\rho\lambda u(x,t)u(-x,-t) \\ &+\mathrm{i}\gamma u(x,t)u(-x,-t)-\rho(u(-x,-t)u(x,t)_x \\ &-u(x,t)u(-x,-t)_x)], \\ J=&\delta(t)\{4\rho\lambda^2u(x,t)+[2\mathrm{i}\rho u(x,t)_x+2\gamma u(x,t)]\lambda \\ &+\mathrm{i}\gamma u(x,t)_x-\rho(u(x,t)_{xx}+2u(x,t)^2u(-x,-t))\}, \end{aligned}$$

$$K=\delta(t)\{-4\rho\lambda^2u(-x,-t)+(2\mathrm{i}\rho u(-x,-t)_x -2\gamma u(-x,-t))\lambda+\mathrm{i}\gamma u(-x,-t)_x+\rho[u(-x,-t)_{xx} +2u(-x,-t)^2u(x,t)]\}. \tag{53}$$

The infinite conservation laws for the variable coefficient nonlocal Hirota equation can be obtained by taking the nonlocal expansion of $u\varpi$ and Eq. (53) into Eq. (52):

$$\begin{aligned}
&[u(x,t)u(-x,-t)]_t\\
=&[3\delta(t)\rho u(x,t)^2u(-x,-t)^2-\mathrm{i}\delta(t)\gamma(u(x,t)_xu(-x,-t)\\
&-u(x,t)u(-x,-t)_x)-\delta(t)\rho(-u(x,t)_{xx}u(-x,-t)\\
&-u(x,t)u(-x,-t)_{xx}+u(x,t)_xu(-x,-t)_x)]_x,\\
&[u(x,t)u(-x,-t)_x]_t\\
=&[\mathrm{i}\delta(t)\gamma u(x,t)^2u(-x,-t)^2-\mathrm{i}\delta(t)\gamma(u(x,t)_xu(-x,-t)_x\\
&-u(x,t)u(-x,-t)_{xx})+\delta(t)\rho(u(x,t)u(-x,-t)_{xxx}\\
&-u(x,t)_xu(-x,-t)_{xx}+u(x,t)_{xx}u(-x,-t)_x)\\
&+6\delta(t)\rho u(-x,-t)u(x,t)^2u(-x,-t)_x]_x,\\
&\cdots
\end{aligned} \tag{54}$$

The existence of the infinite conservation laws for the above nonlocal equations shows that Eq. (1) is conservation integrable.

6. Conclusion

We have constructed the nonlocal Hirota equation with variable coefficients and its Darboux transformation. Utilizing the zero seed solution, 1-soliton and 2-soliton solutions of the equation are constructed through the Darboux transformation. Additionally, expressions for the N-soliton solutions of the equation are derived. The influence of the coefficient function $\delta(t)$ on the solutions is investigated. Furthermore, the dynamical behavior of the solutions is analyzed. The Lax pair in this study is employed to establish an infinite number of conservation laws, and this approach has been extended to nonlocal equations. The study of infinite conservation laws for non-local equations holds significant importance for the integrability of such equations.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 11505090), Liaocheng University Level Science and Technology Research Fund (Grant No. 318012018), Discipline with Strong Characteristics of Liaocheng University–Intelligent Science and Technology (Grant No. 319462208), Research Award Foundation for Outstanding Young Scientists of Shandong Province (Grant No. BS2015SF009), and the Doctoral Foundation of Liaocheng University (Grant No. 318051413).

References

[1] Yin Y H, Ma W X, Liu J G and Lü X 2018 *Comput. Appl. Math.* **76** 1275
[2] Zhou Q, Triki H, Xu J, Zeng Z L, Liu W J and Biswas A 2022 *Chaos, Solitons & Fractals* **160** 112198
[3] Kumar S, Kumar D and Kumar A 2021 *Chaos, Solitons & Fractals* **142** 110507
[4] McDonald A and Clerk A A 2022 *Phys. Rev. Lett.* **128** 033602
[5] Chen Y X and Xiao X 2022 *Nonlinear Dyn.* **109** 2003
[6] Xin X, Zhang L, Xia Y and Liu H 2019 *Appl. Math. Lett.* **94** 112
[7] Guan X, Liu W, Zhou Q and Biswas A 2019 *Nonlinear Dyn.* **98** 1491
[8] Wang R R, Wang Y Y and Dai C Q 2022 *Opt. Laser Technol.* **152** 108103
[9] Geng K L, Zhu B W, Cao Q H, Dai C Q and Wang Y Y 2023 *Nonlinear Dyn.* **111** 16483
[10] Zhao Z and He L 2021 *Appl. Math. Lett.* **111** 106612
[11] Chen J, Yan Q and Zhang H 2020 *Appl. Math. Lett.* **106** 106375
[12] Ali M R, Ma W X and Sadat R 2022 *Ocean. Eng Sci.* **7** 248
[13] Yang Y and Fan E 2021 *Physica D* **417** 132811
[14] Zhang X F, Tian S F and Yang J J 2021 *Anal. Math. Phys.* **11** 1
[15] Wen X K, Jiang J H, Liu W and Dai C Q 2023 *Nonlinear Dyn.* **111** 13343
[16] Wu X H, Gao Y T, Yu X, Ding C and Li L 2022 *Chaos, Solitons & Fractals* **162** 112399
[17] Zhai Y, Ji T and Geng X 2021 *Appl. Math. Comput.* **411** 126551
[18] Xin X P, Liu Y T, Xia Y R and Liu H Z 2021 *Appl. Math. Lett.* **119** 107209
[19] Wang M, Tian B, Hu C C and Liu S 2021 *Appl. Math. Lett.* **119** 106936
[20] Li B Q and Ma Y L 2020 *Appl. Math. Comput.* **386** 125469
[21] Yang D Y, Tian B, Qu Q X, Zhang C, Chen S and Wei C 2021 *Chaos, Solitons & Fractals* **150** 110487
[22] Jia T T, Gao Y T, Yu X and Li L 2021 *Appl. Math. Lett.* **114** 106702
[23] Liu L, Wang D S, Han K and Wen X 2018 *Commun. Nonlinear Sci.* **63** 57
[24] Liu L, Wen X Y and Wang D S 2019 *Appl. Math. Model.* **67** 201
[25] Chen J, Ma Z and Hu Y 2018 *J. MATH. ANAL. APPL.* **460** 987
[26] Li B Q and Ma Y L 2020 *Appl. Math. Comput.* **386** 125469
[27] Yang Y, Suzuki T and Cheng X 2020 *Appl. Math. Lett.* **99** 105998
[28] Zhang Y, Dong K and Jin R 2013 *Am. J. Phys.* **1562** 249
[29] Liu J G and Zhu W H 2020 *Nonlinear Dyn.* **100** 2739
[30] Hong B and Lu D 2012 *Appl. Math. Comput.* **219** 2732
[31] Kaur J, Gupta R K and Kumar S 2020 *Commun. Nonlinear Sci.* **83** 105108