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RAPID COMMUNICATION

Direct measurement of nonlocal quantum states without approximation

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Efficient acquiring information from a quantum state is important for research in fundamental quantum physics and quantum information applications. Instead of using standard quantum state tomography method with reconstruction algorithm, weak values were proposed to directly measure density matrix elements of quantum state. Recently, similar to the concept of weak value, modular values were introduced to extend the direct measurement scheme to nonlocal quantum wavefunction. However, this method still involves approximations, which leads to inherent low precision. Here, we propose a new scheme which enables direct measurement for ideal value of the nonlocal density matrix element without taking approximations. Our scheme allows more accurate characterization of nonlocal quantum states, and therefore has greater advantages in practical measurement scenarios.

Keywords: direct measurement, quantum tomography, nonlocal quantum state

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1. Introduction

Quantum states are the cornerstone of quantum information. An efficient method for characterizing quantum states is of importance for quantum information applications as well as quantum fundamentals. Quantum state tomography (QST)[1–12] is a conventional method for reconstructing the density matrix of a state, which typically involves a large number of projection measurements on several basis states, followed by a complex reconstruction algorithm.[13]

Alternatively, Lundeen *et al.*[14] proposed an efficient direct measurement scheme to determine a quantum wavefunction through weak measurements. This approach eliminates the complex reconstruction algorithm in previous methods, allowing for direct measurement of the wavefunction. Subsequently, this technique aroused great interest[25,26] and was developed to mixed states,[15–19] discrete two-dimensional[20] and higher-dimensional quantum systems.[21–24] Moreover, these schemes can be extended to arbitrary coupling strength, even to strong measurements.[27–31]

Recently, the direct measurement of wavefunctions with nonlocal entanglement was demonstrated in Ref. [32]. This scheme utilizes the modular values[34] to obtain the weak value of the observable product, which allows for direct measurement of nonlocal states. Afterward, this approach was further extended to the nonlocal density matrix.[35] Despite the advantages of these schemes, such methods can only obtain the approximated values rather than precise ideal ones.

In this paper, we propose a new scheme to directly measure arbitrary two-party nonlocal quantum states. We consider the scenario that the two parties are spatially separated over a long distance. And the interaction of the two parties is physically forbidden. We use a specialized interaction between the system and a meter to avoid the influence of redundant terms on the measurement of the density matrix elements. It enables the direct measurement of the ideal values of the matrix elements without any approximations. Our approach overcomes the low precision problem in weak-coupling schemes, thus enabling a more accurate characterization of nonlocal systems.

The paper is arranged as follows. In Section 2 we give an introduction to the direct measurement using weak values and modular values. Then in Section 3 we present our scheme and analyze the precision numerically. Discussions and conclusions are made in Section 4.

2. Direct measurement using weak values and modular values

Given that a quantum system is prepared in $|\psi_{\rm in}\rangle$ and post-selected by $|\psi_{\rm fin}\rangle$, the weak value of an observable $\hat{A}$ is defined as $\langle A\rangle_{\rm w} = \langle\psi_{\rm fin}|\hat{A}|\psi_{\rm in}\rangle/\langle\psi_{\rm fin}|\psi_{\rm in}\rangle$. Typically, weak values are obtained through weak measurements, which involve interaction between the quantum system and measuring meter described by the evolution operator $\hat{U} = {\rm e}^{-{\rm i}g\hat{A}\otimes\hat{\mathcal{P}}_{\rm m}}$, where g is a dimensionless coupling constant and $\hat{\mathcal{P}}_{\rm m}$ is the Gaussian momentum operator of the meter. In the case $g \ll 1$, the weak value can be obtained by measuring the position and momentum expectation values of the meter's final state. Specifically, the weak value of $\hat{A}$ is given by $\langle A\rangle_{\rm w} = \frac{1}{g}\langle X\rangle_{\rm fin} +$

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$\mathrm{i}\frac{\sigma^2}{2g}\langle P\rangle_{\rm fin}$,[33] where σ^2 is the variance of the meter momentum. When the meter is chosen as a qubit, the weak value can be calculated as $\langle A\rangle_{\rm w}=\frac{1}{2g}(\langle\hat{\sigma}_x\rangle_{\rm fin}+\mathrm{i}\langle\hat{\sigma}_y\rangle_{\rm fin})$, where $\hat{\sigma}_x$ and $\hat{\sigma}_y$ are the Pauli operators acting on the meter qubit.

If the observable $\hat{A}$ is chosen as a projection operator $\hat{P}_n=|n\rangle\langle n|$ and the post-selected state is set as $|\psi_{\rm fin}\rangle=\frac{1}{\sqrt{d}}\sum_n|n\rangle$, the resulting weak value is proportional to the wavefunction, with $\psi_n\propto\langle P_n\rangle_{\rm w}$. As a result, by measuring the weak values through a traversal of the basis states $\{|n\rangle\}$, we can directly characterize the quantum state.

However, implementing this method on nonlocal quantum systems is not straightforward. Consider a nonlocal state composed of two space-like separated subsystems A and B, with a basis given by the tensor product of local bases of the two subsystems, denoted as $|n\rangle=|j,l\rangle=|j\rangle^A|l\rangle^B$ ($|j\rangle^A\in\mathbb{H}_A$, $|l\rangle^B\in\mathbb{H}_B$). In this case, the direct weak measurement scheme needs to introduce a nonlocal Hamiltonian $\hat{H}\propto\hat{P}^A\hat{P}^B$. Such a Hamiltonian implies an instantaneous interaction between the two distant subsystems, which is contradictory to the principle of causality in relativity. In fact, under certain coupling strengths, weak values of observable products can also be obtained by measuring local modular values.[34] In analogy to the measurement of weak values, the modular value of $\hat{A}$ can be defined as $\langle A\rangle_{\rm m}=\langle\psi_{\rm fin}|\mathrm{e}^{-\mathrm{i}g\hat{A}}|\psi_{\rm in}\rangle/\langle\psi_{\rm fin}|\psi_{\rm in}\rangle$. When the relevant observable is a projection, it satisfies the corresponding relationship with weak values as

$$\langle P_n\rangle_{\rm m}=1+(\mathrm{e}^{-\mathrm{i}g}-1)\langle P_n\rangle_{\rm w}. \tag{1}$$

If we choose the coupling strength to be $g=\pi$, we have $\langle P_n\rangle_{\rm w}=(1-\langle P_n\rangle_{\rm m})/2$, and the modulus of the sum of two commuting projection operators is given by $\langle P_j^A+P_l^B\rangle_{\rm m}=1-2\big(\langle P_j^A\rangle_{\rm w}+\langle P_l^B\rangle_{\rm w}\big)+4\langle P_j^AP_l^B\rangle_{\rm w}$. Hence, the weak values of the observable products can be transformed into the modular values of the local observables according to the transformation relationship given by

$$\begin{aligned}\Psi_{j,l}&\propto\langle P_j^AP_l^B\rangle_{\rm w}\\&=\frac{1}{4}\big(\langle P_j^A+P_l^B\rangle_{\rm m}-\langle P_j^A\rangle_{\rm m}-\langle P_l^B\rangle_{\rm m}+1\big).\end{aligned} \tag{2}$$

By utilizing the relationship in Eq. (2), the weak values of observable product can be measured, allowing for the direct measurement of nonlocal states.[32]

In a more general scenario, we consider the case of nonlocal quantum states in density matrix formalism, namely, $\rho=\sum_{n,n'}\rho_{n,n'}|n\rangle\langle n'|=\sum_{i,j}\sum_{i',j'}\rho_{iji'j'}\ |i\rangle|j\rangle\langle i'|\langle j'|$ or by Dirac distribution $S=\sum_{i,j}\sum_{f_1,f_2}\frac{1}{\langle f_1,f_2|i,j\rangle}S_{ijf_1f_2}|i\rangle|j\rangle\langle f_1|\langle f_2|$. Here $|f_1\rangle$ ($|f_2\rangle$) is the complementary basis of $|i\rangle$ ($|j\rangle$), which is inter-converted with discrete Fourier transform, given by $|f_1\rangle=\frac{1}{\sqrt{d}}\sum_i^d\mathrm{e}^{+\mathrm{i}\frac{2\pi if_1}{d}}|i\rangle$ ($|f_2\rangle=\frac{1}{\sqrt{d}}\sum_j^d\mathrm{e}^{+\mathrm{i}\frac{2\pi jf_2}{d}}|j\rangle$). With the more general definition of weak values $\langle A\rangle_{\rm w}=\langle\psi_{\rm fin}|\hat{A}\rho|\psi_{\rm fin}\rangle/\langle\psi_{\rm fin}|\rho|\psi_{\rm fin}\rangle$, we set the observable to be the projection operator $\hat{A}=\hat{P}_i^A\hat{P}_j^B=|i,j\rangle\langle i,j|$, and choose the unbiased basis as the post-selection operation $|\psi_{\rm fin}\rangle\langle\psi_{\rm fin}|=|f_1,f_2\rangle\langle f_1,f_2|$. The resulting weak value $\left\langle P_i^AP_j^B\right\rangle_{\rm w}$ exhibits a one-to-one correspondence with the Dirac distribution of the quantum system similar to Eq. (2) as

$$S_{ijf_1f_2}=\langle f_1,f_2|i,j\rangle\langle i,j|\rho|f_1,f_2\rangle=p_{f_1,f_2}\left\langle P_i^AP_j^B\right\rangle_{\rm w}, \tag{3}$$

where $p_{f_1,f_2}=\langle f_1,f_2|\rho|f_1,f_2\rangle$ represents the probability of successful post-selection.

In order to measure the corresponding modular values, the initial state of the meter is set as $|\Phi_{\rm in}^{\rm m}\rangle=(|00\rangle+\varepsilon|11\rangle)/\sqrt{1+\varepsilon^2}$, with $\varepsilon\ll1$, and the corresponding quantum circuits are shown in Fig. 1. Take the measurement of $\langle P_i^A+P_j^B\rangle_{\rm m}$ as an example. After interaction $\mathrm{e}^{-\mathrm{i}\pi(\hat{P}_i^A\hat{P}_1^{a_1}+\hat{P}_j^B\hat{P}_1^{a_2})}$ between the system and the meter in Fig. 1(a), a post-selection operation $|f_1,f_2\rangle\langle f_1,f_2|$ is performed on the system. Then, the measurement of the meter only requires projecting the final state of the meter onto $|\phi_1\rangle|+\rangle^{a_1}|+\rangle^{a_2}$ and $|\phi_2\rangle=|+\rangle^{a_1}|R\rangle^{a_2}$ respectively. The resulting probabilities of projection are

$$\begin{aligned}\mathcal{P}_1&=\frac{1}{2}+\varepsilon\,\mathrm{Re}\left[\left\langle P_i^A+P_j^B\right\rangle_{\rm m}\right]+O\left(\varepsilon^2\right),\\\mathcal{P}_2&=\frac{1}{2}+\varepsilon\,\mathrm{Im}\left[\left\langle P_i^A+P_j^B\right\rangle_{\rm m}\right]+O\left(\varepsilon^2\right).\end{aligned} \tag{4}$$

As $\varepsilon\ll1$, the above equation can be simplified by neglecting higher-order terms and taking the first-order approximation. The real and imaginary parts of the weak value of $\left\langle P_i^A+P_j^B\right\rangle_{\rm m}$ can be obtained from the two measurements of the meter. The measurements of $\left\langle P_i^A\right\rangle_{\rm m}$ and $\left\langle P_j^B\right\rangle_{\rm m}$ follow the same procedure as depicted in Figs. 1(b) and 1(c). The only difference is the interaction between the system and the meter, which is changed to $\mathrm{e}^{-\mathrm{i}\pi\hat{P}_i^A\hat{P}_1^{a_1}}$ and $\mathrm{e}^{-\mathrm{i}\pi\hat{P}_j^B\hat{P}_1^{a_2}}$, respectively. Afterwards, the matrix elements of Dirac distribution of the system can be obtained using these modular values based on the relations in Eqs. (2) and (3), which enables a direct measurement of nonlocal mixed states.

However, the direct measurement schemes using weak values and modular values are both not theoretically exact in characterizing the system state. In these schemes, a small parameter is employed, and the first-order approximation on the parameter is implemented with all the higher-order terms discarded from the measurement of the final state of the meter. However, if we could avoid the contribution of the higher-order terms, we would directly extract the precise modular information. With such an idea, we propose a direct measurement scheme for nonlocal quantum states, which can directly read out the theoretical value of the density matrix elements without relying on approximations.

3. Direct measurement of nonlocal quantum states without approximation

Looking back at the process of measuring the modular values, the purpose of choosing the meter state $|\Phi_{\rm in}^{\rm m}\rangle = (|00\rangle + \varepsilon|11\rangle)/\sqrt{1+\varepsilon^2}$ and the system-meter interaction $\mathrm{e}^{-\mathrm{i}\pi\left(\hat{P}_i^A\hat{P}_1^{a_1}+\hat{P}_j^B\hat{P}_1^{a_2}\right)}$ is to concentrate the information related to the modular values onto the $|11\rangle$ component with the coefficient $\varepsilon/\sqrt{1+\varepsilon^2} \ll 1$. This inevitably introduces complex quadratic terms $O(\varepsilon^2)$ in the measurement as shown in Eq. (4). With this setting, the scheme can only give an approximate value from the measurement results while obtaining the modular values.

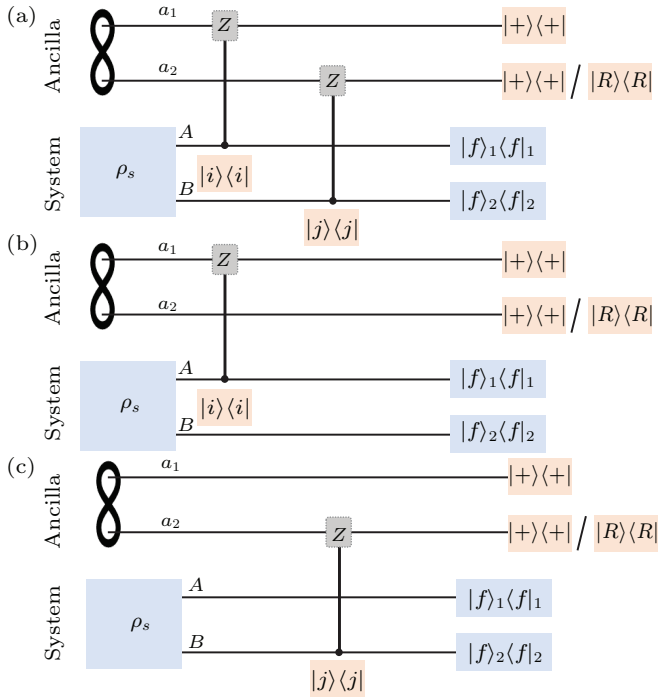


Fig. 1. Direct measurement of nonlocal states. (a) The measurement of $\mathcal{P}_{AB}$ in Section 3, or $\langle P_i^A + P_j^B\rangle_{\rm m}$ in Section 2. (b) The measurement of $\mathcal{P}_A$ in Section 3 or $\langle P_i^A\rangle_{\rm m}$ in Section 2. (c) The measurement of $\mathcal{P}_B$ in Section 3 or $\langle P_j^B\rangle_{\rm m}$ in Section 2. The system and the meter interact via controlled-Z operations using the basis $|i\rangle\langle i|$ and $|j\rangle\langle j|$ (or $|i\rangle\langle i|/|j\rangle\langle j|$) as control bits for addressing, followed by post-selection on the system using $|f\rangle\langle f|$, where $|f\rangle = |f_1\rangle^A|f_2\rangle^B$.

Our method builds upon the above measurement scheme, but with some important modifications to directly measure the nonlocal quantum states without relying on approximations. Taking the example of the two-body system discussed above, for convenience, let us assume that the two subsystems have the same dimension d. In our method, the initial state of the meter is chosen as the maximally entangled state $|\Phi_{\rm in}^{\rm m}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, which does not require a small parameter ε. The principle of our method can be illustrated by Fig. 1 but with different controlled-Z operations. Specifically, we use the interactions $\mathrm{e}^{-\mathrm{i}\pi\left(\hat{P}_i^A\hat{P}_1^{a_1}+\hat{P}_j^B\hat{P}_0^{a_2}\right)}$, $\mathrm{e}^{-\mathrm{i}\pi\hat{P}_i^A\hat{P}_1^{a_1}}$, and $\mathrm{e}^{-\mathrm{i}\pi\hat{P}_j^B\hat{P}_0^{a_2}}$ for Figs. 1(a), 1(b), and 1(c), respectively, where $\hat{P}_1^{a_1}$ and $\hat{P}_1^{a_2}$ represent the projection operators that project the a_1 and a_2 parts of the meter to $|1\rangle$, respectively. In the following, we first introduce the process of Fig. 1(a). Here, we no longer focus on the modular values and write the interaction as

$$\mathrm{e}^{-\mathrm{i}\pi\left(\hat{P}_i^A\hat{P}_1^{a_1}+\hat{P}_j^B\hat{P}_1^{a_0}\right)} = (1-2\hat{P}_j^B\hat{P}_0^{a_2})(1-2\hat{P}_i^A\hat{P}_1^{a_1}). \tag{5}$$

In the scheme in Section 1, the projection operators $\hat{P}_1^{a_1}$ and $\hat{P}_1^{a_2}$ of the meter are coupled with the system observables $\hat{P}_i^A$ and $\hat{P}_j^B$ involved in addressing. In our scheme, we use $\hat{P}_1^{a_1}$ and $\hat{P}_0^{a_2}$ of the meter as the control qubits for the controlled-Z operations, and apply the projection operators directly on the system and meter states through Eq. (5) to eliminate the contribution of redundant terms. For convenience, the density matrix of the initial state of the system and meter is denoted as $\rho_{\rm s}\otimes\rho_{\rm m}$, where $\rho_{\rm m} = |\Phi_{\rm in}^{\rm m}\rangle\langle\Phi_{\rm in}^{\rm m}|$. After interaction with the system and choosing $|f\rangle\langle f| = |f_1,f_2\rangle\langle f_1,f_2|$, we perform a projection measurement $|\phi\rangle\langle\phi|$ on the meter, where $|\phi\rangle$ includes two cases, $|\phi_1\rangle$ and $|\phi_2\rangle$. The resulting probability is

$$\begin{aligned}
\mathcal{P}_{AB} \;=\; & \langle f|\rho_{\rm s}|f\rangle\langle\phi|\rho_{\rm m}|\phi\rangle \\
& -2\langle f|\hat{P}_j^B\rho_{\rm s}|f\rangle\langle\phi|\hat{P}_0^{a_2}\rho_{\rm m}|\phi\rangle \\
& -2\langle f|\hat{P}_i^A\rho_{\rm s}|f\rangle\langle\phi|\hat{P}_1^{a_1}\rho_{\rm m}|\phi\rangle \\
& -2\langle f|\rho_{\rm s}\hat{P}_i^A|f\rangle\langle\phi|\rho_{\rm m}\hat{P}_1^{a_1}|\phi\rangle \\
& -2\langle f|\rho_{\rm s}\hat{P}_j^B|f\rangle\langle\phi|\rho_{\rm m}\hat{P}_0^{a_2}|\phi\rangle \\
& +4\langle f|\hat{P}_i^A\rho_{\rm s}\hat{P}_i^A|f\rangle\langle\phi|\hat{P}_1^{a_1}\rho_{\rm m}\hat{P}_1^{a_1}|\phi\rangle \\
& +4\langle f|\hat{P}_j^B\rho_{\rm s}\hat{P}_i^A|f\rangle\langle\phi|\hat{P}_0^{a_2}\rho_{\rm m}\hat{P}_1^{a_1}|\phi\rangle \\
& +4\langle f|\hat{P}_j^B\rho_{\rm s}\hat{P}_j^B|f\rangle\langle\phi|\hat{P}_0^{a_2}\rho_{\rm m}\hat{P}_0^{a_2}|\phi\rangle \\
& +4\langle f|\hat{P}_i^A\rho_{\rm s}\hat{P}_j^B|f\rangle\langle\phi|\hat{P}_1^{a_1}\rho_{\rm m}\hat{P}_0^{a_2}|\phi\rangle .
\end{aligned} \tag{6}$$

Similarly, we can obtain the measurement results for the interactions $\mathrm{e}^{-\mathrm{i}\pi\hat{P}_i^A\hat{P}_1^{a_1}}$ and $\mathrm{e}^{-\mathrm{i}\pi\hat{P}_j^B\hat{P}_0^{a_2}}$ as

$$\begin{aligned}
\mathcal{P}_A \;=\; & \langle f|\rho_{\rm s}|f\rangle\langle\phi|\rho_{\rm m}|\phi\rangle - 2\langle f|\hat{P}_j^B\rho_{\rm s}|f\rangle\langle\phi|\hat{P}_0^{a_2}\rho_{\rm m}|\phi\rangle \\
& +4\langle f|\hat{P}_i^A\rho_{\rm s}\hat{P}_i^A|f\rangle\langle\phi|\hat{P}_1^{a_1}\rho_{\rm m}\hat{P}_1^{a_1}|\phi\rangle \\
& -2\langle f|\rho_{\rm s}\hat{P}_i^A|f\rangle\langle\phi|\rho_{\rm m}\hat{P}_1^{a_1}|\phi\rangle ,
\end{aligned} \tag{7}$$

$$\begin{aligned}
\mathcal{P}_B \;=\; & \langle f|\rho_{\rm s}|f\rangle\langle\phi|\rho_{\rm m}|\phi\rangle - 2\langle f|\hat{P}_i^A\rho_{\rm s}|f\rangle\langle\phi|\hat{P}_1^{a_1}\rho_{\rm m}|\phi\rangle \\
& +4\langle f|\hat{P}_j^B\rho_{\rm s}\hat{P}_j^B|f\rangle\langle\phi|\hat{P}_0^{a_2}\rho_{\rm m}\hat{P}_0^{a_2}|\phi\rangle \\
& -2\langle f|\rho_{\rm s}\hat{P}_j^B|f\rangle\langle\phi|\rho_{\rm m}\hat{P}_0^{a_2}|\phi\rangle .
\end{aligned} \tag{8}$$

Comparing the above two equations with Eq. (6), it can be found that after removing the duplicated terms, Eq. (6) only has two terms that are complex conjugates of each other: $\langle f|\hat{P}_i^A\rho_{\rm s}\hat{P}_j^B|f\rangle\langle\phi|\hat{P}_1^{a_1}\rho_{\rm m}\hat{P}_0^{a_2}|\phi\rangle$ and $\langle f|\hat{P}_j^B\rho_{\rm s}\hat{P}_i^A|f\rangle\langle\phi|\hat{P}_0^{a_2}\rho_{\rm m}\hat{P}_1^{a_1}|\phi\rangle$. Then we make the expansion below $\langle f|\hat{P}_i^A\rho_{\rm s}\hat{P}_j^B|f\rangle$, and have

$$\begin{aligned}
\langle f|\hat{P}_i^A\rho_{\rm s}\hat{P}_j^B|f\rangle \;&=\; \langle f_1|i\rangle\langle j|f_2\rangle\langle f_2|^B\langle i|^A\rho_{\rm s}|f_1\rangle^A|j\rangle^B \\
&=\; \frac{1}{d}\mathrm{e}^{\mathrm{i}\frac{2\pi}{d}(f_2 j - f_1 i)}\rho_{if_2f_1j},
\end{aligned} \tag{9}$$

where $\rho_{if_2f_1j} = \langle j|\langle f_1|i\rangle|f_2\rangle\langle f_2|\langle i|\rho_{\rm s}|f_1\rangle|j\rangle$. The term $\rho_{if_2f_1j}$ defined here is similar to the Dirac distribution and is equivalent to the density matrix in terms of information. The distribution matrix element $\rho_{if_2f_1i}$ also has a conversion relationship

with the density matrix element $\rho_{ij'i'j}$ as

$$\rho_{if_2f_1j} = \frac{1}{d^2}\sum_{i',j'}^{d}\rho_{ij'i'j}\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{d}[(i'-i)f_1+(j-j')f_2]},$$
$$\rho_{ij'i'j} = \frac{1}{d^2}\sum_{i',j'}^{d}\rho_{if_2f_1j}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{d}[(i'-i)f_1+(j-j')f_2]}. \quad (10)$$

Therefore, the distribution $\rho_{if_2f_1j}$ expanded in the basis $|i\rangle|f_2\rangle\langle f_1|\langle j|$ can also be viewed as a representation of general quantum state. Through the above measurement process, we can directly obtain the distribution matrix element $\rho_{if_2f_1j}$ from the measurement results. Let $\mathcal{P}_0 = \langle f|\rho_{\mathrm{s}}|f\rangle\langle\phi|\rho_{\mathrm{m}}|\phi\rangle$ denote the resulting probability without performing any operations. In the first case, the meter projection measurement in the above measurement is taken as $|\phi\rangle\langle\phi| = |\phi_1\rangle\langle\phi_1|$, where $|\phi_1\rangle = |+\rangle^{a_1}|+\rangle^{a_2}$ and $|+\rangle = 1/\sqrt{2}(|0\rangle+|1\rangle)$. At this point, the real part of the distribution $\rho_{if_2f_1j}$ can be obtained from the results of the four measurements, given by

$$\mathcal{P}_{AB} - \mathcal{P}_A - \mathcal{P}_B + \mathcal{P}_0 = \mathrm{Re}[\rho_{if_2f_1j}]. \quad (11)$$

For the second case, the meter projection is taken as $|\phi\rangle\langle\phi| = |\phi_2\rangle\langle\phi_2|$, where $|\phi_2\rangle = |+\rangle^{a_1}|R\rangle^{a_2}$ and $|R\rangle = 1/\sqrt{2}(|0\rangle+\mathrm{i}|1\rangle)$. From the measurement results, the imaginary part of the distribution $\rho_{if_2f_1j}$ can be obtained as

$$\mathcal{P}_{AB} - \mathcal{P}_A - \mathcal{P}_B + \mathcal{P}_0 = -\mathrm{Im}[\rho_{if_2f_1j}]. \quad (12)$$

Therefore, traversing through all the basis $\{|n\rangle\langle n|\}$ allows us to obtain the complete distribution.

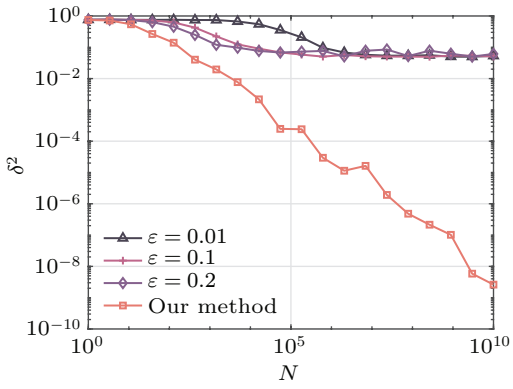


Fig. 2. The average precision of direct characterization of arbitrary two-particle two-dimensional states using both our method and approximated modular value schemes. Our method shows better precision as N increases.

To evaluate the precision improvement of our scheme, we numerically analyze the fidelity between states measured using different methods and the ideal states. We define the precision of characterization schemes as average infidelity

$$\delta^2 = 1 - E[F(\rho_{\mathrm{exp}},\rho_{\mathrm{ideal}})]$$
$$= 1 - E\left[\mathrm{trace}\left(\sqrt{\rho_{\mathrm{exp}}^{\frac{1}{2}}\rho_{\mathrm{ideal}}\rho_{\mathrm{exp}}^{\frac{1}{2}}}\right)\right], \quad (13)$$

where $E[F(\rho_{\mathrm{exp}},\rho_{\mathrm{ideal}})] = \frac{1}{S}\Sigma_S F(\rho_{\mathrm{exp}}^{(S)},\rho_{\mathrm{ideal}}^{(S)})$ is the average fidelity for S samples of arbitrary state. In Fig. 2, the mean precision of two-particle two-dimensional states is numerically estimated using our method and the approximated modular value method by assuming the Poisson distribution statistics. Here, the number of random input states is chosen as $S = 100$. The numerical results show the feasibility of our method. Compared to the approximated modular value method, the precision increases with the growth of number of input copies N.

4. Discussion and conclusions

We propose a new direct measurement scheme of nonlocal quantum state without any approximations. This scheme effectively avoids the impact of redundant terms on measuring the density matrix elements through a special system–meter coupling. Consequently, it enables the direct measurement of the ideal value of nonlocal density matrix element without taking approximations. Our numerical results show the advantage of higher precision compared to the approximation methods. Our approach overcomes the inherent low precision associated with weak coupling schemes, which facilitates a more accurate characterization of nonlocal systems and opens up new avenues for directly measuring nonlocal quantum states. Furthermore, it is possible to extend our method to direct characterization of multiparticle states by utilizing entangled ancillary states, such as GHZ state. The specific density matrix elements can be directly acquired from the measurement statistics with elaborate quantum circuits and post-selections.

Acknowledgements

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