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Citation: Chin. Phys. B, 2023, 32 (**12**): 120304. DOI: 10.1088/1674-1056/acf492
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Performance of entanglement-assisted quantum codes with noisy ebits over asymmetric and memory channels

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(Received 25 May 2023; revised manuscript received 28 August 2023; accepted manuscript online 29 August 2023)

Entanglement-assisted quantum error correction codes (EAQECCs) play an important role in quantum communications with noise. Such a scheme can use arbitrary classical linear code to transmit qubits over noisy quantum channels by consuming some ebits between the sender (Alice) and the receiver (Bob). It is usually assumed that the preshared ebits of Bob are error free. However, noise on these ebits is unavoidable in many cases. In this work, we evaluate the performance of EAQECCs with noisy ebits over asymmetric quantum channels and quantum channels with memory by computing the exact entanglement fidelity of several EAQECCs. We consider asymmetric errors in both qubits and ebits and show that the performance of EAQECCs in entanglement fidelity gets improved for qubits and ebits over asymmetric channels. In quantum memory channels, we compute the entanglement fidelity of several EAQECCs over Markovian quantum memory channels and show that the performance of EAQECCs is lowered down by the channel memory. Furthermore, we show that the performance of EAQECCs is diverse when the error probabilities of qubits and ebits are different. In both asymmetric and memory quantum channels, we show that the performance of EAQECCs is improved largely when the error probability of ebits is reasonably smaller than that of qubits.

Keywords: asymmetric quantum channel, entanglement fidelity, entanglement-assisted quantum error correction code, quantum memory channel

PACS: 03.67.Lx, 03.67.Pp **DOI:** 10.1088/1674-1056/acf492

1. Introduction

Quantum error correction (QEC) is crucial to realization of quantum communications and the building of quantum computers.[1,2] The stabilizer code formalism[3] is one of the most successful schemes in quantum information theory for realizing QEC. Stabilizer codes can be constructed by classical error correction codes with certain dual-containing conditions. Such constraint is not easy to be satisfied in many situations.[4] The entanglement-assisted (EA) quantum error correction codes (EAQECCs)[5–8] generalize the standard stabilizer codes, and EAQECCs can be constructed by any classical linear code without the dual-containing constraint. The price is that some ebits need to be preshared between the sender (Alice) and the receiver (Bob) before the transmission. In general, the preshared ebits need to be carefully preserved and thus they are assumed to exist without error. However, maintaining a number of ebits in perfect condition is usually difficult in practical applications. Such a phenomenon is crucial to the entanglement-assisted quantum communication formalism since imperfect ebits may degrade the performance of EAQECCs largely.[9] In Refs. [9,10], performance of EAQECCs with imperfect ebits was evaluated in entanglement fidelity. In Ref. [11], the entanglement fidelity of entanglement-assisted concatenated quantum codes was computed based on the concatenation scheme.

In most quantum channels, the probabilities of different types of quantum noise usually exhibit a large asymmetry. It was shown that the phase-flip errors (Z errors) happen much more frequently than the bit-flip (X errors).[12] Asymmetric quantum codes (AQCs) which have a more biased error correction towards Z errors than X errors, were constructed to better cope the significant asymmetry in quantum channels.[13–17] Moreover, in fault-tolerant quantum computation, asymmetric errors have been further explored to help improve the fault-tolerant thresholds.[18–22] What's more, the standard discretization error model[23] supposed that quantum errors are independent of each other. However, correlated errors are more practical in reality.[24–26] So far, the asymmetry and memory effect on the performance of EAQECCs with noisy ebtis are still unknown.

In this work, we study the performance of EAQECCs with noisy ebits over asymmetric quantum channels and quantum memory channels by computing the entanglement fidelity of several EAQECCs. For a $Q_{\mathrm{E}} = [[n,k,d;c]]$ EAQECC, we consider that asymmetric and correlated noises exist not only in the n physical qubits but also in the c ebits. We compute the entanglement fidelity of the $[[3,1,3;2]]$ entanglement-assisted repetition code and Bowen's $[[3,1,3;2]]$ EAQECC over asymmetric quantum channels. We know that the $[[3,1,3;2]]$ repetition code is also an asymmetric quantum code,[13] which is assumed to be more powerful than the standard stabilizer codes. Our computation results suggest that the performance of the two EAQECCs is quite diverse and is related not only to the channel asymmetry but also to the ratio (denoted by ω) of

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the error probabilities of ebits and qubits. Specifically, if the asymmetry of quantum channels is sufficiently large and the error probability of ebits is much smaller than that of qubits, then the $[[3,1,3;2]]$ repetition code will perform much better than Bowen's $[[3,1,3;2]]$ EAQECC and the $[[5,1,3]]$ stabilizer code.

In quantum memory channels, the numerical results suggest that the entanglement fidelity of the two EAQECCs is lowered down in most range of depolarizing probabilities. Moreover, the performance of the two EAQECCs is related not only to the channel memory μ but also to the ratio ω. We show that the two EAQECCs can outperform the $[[5,1,3]]$ stabilizer code over quantum memory channels as the ratio ω becomes smaller and smaller. For the two EAQECCs, we show that Bowen's $[[3,1,3;2]]$ EAQECC can outperform the $[[3,1,3;2]]$ repetition code over quantum memory channels if the ratio ω is relatively large, e.g., $\omega \geq 0.5$. However, if the ratio is small, e.g., $\omega = 0.01$, the performance of the two EAQECCs is diverse and is effected largely by the channel memory. In such a case, the numerical results suggest that the $[[3,1,3;2]]$ repetition code can outperform Bowen's $[[3,1,3;2]]$ EAQECC if the channel memory μ is small, e.g., $\mu \leq 0.1$. However, Bowen's $[[3,1,3;2]]$ EAQECC can beat the $[[3,1,3;2]]$ repetition code if the channel memory μ is large, e.g., $\mu = 0.5$.

The rest of this article is given as follows. In Section 2, we present the needed background about quantum codes. In Section 3, we compute and plot the entanglement fidelity of several EAQECCs over asymmetric quantum channels. In Section 4, we compute and plot the entanglement fidelity of several EAQECCs over Markovian quantum memory channels. The conclusion is given in Section 5.

2. Preliminaries

In this section, we present some background knowledge about quantum channels and entanglement-assisted quantum codes.

We use Pauli channels as the basic channel models to perform the computation of entanglement fidelity. Denote a two-dimensional Hilbert space by $\mathbb{C}^2$. For a qubit $|x\rangle$ in $\mathbb{C}^2$, we suppose $|x\rangle = a|0\rangle + b|1\rangle$, where a and b are two complex numbers and $a^2 + b^2 = 1$. Define the Pauli operators

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \ \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (1)$$

where $\mathrm{i} = \sqrt{-1}$. Denote the nth tensor product of $\mathbb{C}^2$ by $\mathbb{C}^{2^n} \equiv \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$, $\mathbb{H}_n = \mathbb{C}^{2^n}$. An n-qubit $|\alpha\rangle$ is an element in the nth tensor product of $\mathbb{C}^2$, i.e., $|\alpha\rangle \in \mathbb{C}^{2^n}$. Let u and v be two bits. Define two unitary operators on a qubit $|x\rangle$ as $\boldsymbol{X}(u)|x\rangle = |u+x\rangle$ and $\boldsymbol{Z}(v)|x\rangle = (-1)^{v\cdot x}|x\rangle$, respectively. For two bit vectors $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$, we define $\boldsymbol{X}(\boldsymbol{u}) = (\boldsymbol{X}(u_1), \ldots, \boldsymbol{X}(u_n))$ and $\boldsymbol{Z}(\boldsymbol{v}) = (\boldsymbol{Z}(v_1), \ldots, \boldsymbol{Z}(v_n))$. In Pauli channel models,[23] we only need to consider a discrete set of quantum errors, i.e., the bit-flip error (X error), the phase-flip error (Z error), and the combined bit-phase-flip error (Y error). For a single qubit, the discretization of errors can be characterized by the Pauli matrices in Eq. (1). For $1 \leq i \leq n$, we define $\varepsilon_i \in \{I, \sigma_x, \sigma_y, \sigma_z\}$. Then the quantum errors in an n-qubit quantum state can be described by the following group:

$$\mathcal{E}_n = \{\mathrm{i}^t \varepsilon_1 \otimes \cdots \otimes \varepsilon_n | 0 \leq t \leq 3\}. \quad (2)$$

In most situations, it is enough to use $\overline{\mathcal{E}}_n = \mathcal{E}_n / \{\pm 1, \pm \mathrm{i}\}$ since the global phase in $\mathcal{E}_n$ is not important. For an operator $\mathcal{E} = \varepsilon_1 \otimes \cdots \otimes \varepsilon_n \in \overline{\mathcal{E}}_n$, we define two bit vectors $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ with

$$(a_i, b_i) = \begin{cases} (0,0), \text{ if } \varepsilon_i = I, \\ (1,0), \text{ if } \varepsilon_i = \sigma_x, \\ (0,1), \text{ if } \varepsilon_i = \sigma_z, \\ (1,1), \text{ if } \varepsilon_i = \sigma_y, \end{cases} \quad (3)$$

where $1 \leq i \leq n$. Then we can write $\mathcal{E}$ as

$$\mathcal{E} = \boldsymbol{X}(\boldsymbol{a})\boldsymbol{Z}(\boldsymbol{b}). \quad (4)$$

The quantum stabilizer code $Q = [[n,k]]$ is a 2^k-dimensional subspace of the Hilbert space $\mathbb{C}^{2^n}$, and Q is specified by a stabilizer group S with stabilizer generators S_i ($1 \leq i \leq n-k$), i.e., $S = \langle S_1, \ldots, S_{n-k} \rangle$. Denote the parity check matrix of Q by $H = (H_X | H_Z)$. Then H can be obtained from Eq. (3) accordingly. Q has minimum distance d if and only if it can detect any error of weight less than or equal to $d-1$, but cannot detect some error of weight d.

Define $Q_{\mathrm{E}} = [[n,k,d;c]]$ as an entanglement-assisted quantum stabilizer code (EAQSC) which can detect any qubit error of weight less than or equal to $d-1$. The number of ebits is c. Consider the Pauli operators given by

$$E^{\mathrm{A}} \otimes E^{\mathrm{B}} \in \overline{\mathcal{E}}_{n+c}, \quad (5)$$

where $E^{\mathrm{A}} \in \overline{\mathcal{E}}_n$, $E^{\mathrm{B}} \in \overline{\mathcal{E}}_c$, and the superscripts A and B denote Alice and Bob, respectively. We take the following denotations:

$$\begin{cases} g_1' = g_1^{\mathrm{A}} \otimes Z_1^{\mathrm{B}}, \ldots, g_c' = g_c^{\mathrm{A}} \otimes Z_c^{\mathrm{B}}, \\ g_{c+1}' = g_{c+1}^{\mathrm{A}} \otimes I^{\mathrm{B}}, \ldots, g_{n-k}' = g_{n-k}^{\mathrm{A}} \otimes I^{\mathrm{B}}, \\ h_1' = h_1^{\mathrm{A}} \otimes X_1^{\mathrm{B}}, \ldots, h_c' = h_c^{\mathrm{A}} \otimes X_c^{\mathrm{B}}, \end{cases} \quad (6)$$

where $Z_i^{\mathrm{B}}, X_i^{\mathrm{B}} \in \overline{\mathcal{E}}_c$, $h_i^{\mathrm{A}} \in \overline{\mathcal{E}}_n$ for $1 \leq i \leq c$, and $g_j^{\mathrm{A}} \in \overline{\mathcal{E}}_n$ for $1 \leq j \leq n-k$. Moreover, g_i ($1 \leq i \leq n-k$) and h_j ($1 \leq j \leq c$) satisfy the following commutation or anticommutation relations:

$$\begin{cases} [g_i, g_j] = 0, \ i \neq j, \\ [h_i, h_j] = 0, \ i \neq j, \\ [g_i, h_j] = 0, \ i \neq j, \\ \{g_i, h_i\} = 0, \ \text{for all } i. \end{cases} \quad (7)$$

Denote $S_{\rm S} = \langle g'_1,\ldots,g'_c,h'_1,\ldots,h'_c\rangle \subseteq \overline{\mathcal{E}}_{n+c}$ and denote $S_{\rm I} = \langle g'_{c+1},\ldots,g'_{n-k}\rangle \subseteq \overline{\mathcal{E}}_{n+c}$. Denote $S_{\rm E} = S_{\rm S}\times S_{\rm I} = \langle g'_1,\ldots,g'_{n-k},h'_1,\ldots,h'_c\rangle$ which is generated by the generators of $S_{\rm S}$ and $S_{\rm I}$. Then the $Q_{\rm E} = [[n,k,d;c]]$ EAQSC is stabilized by $S_{\rm E}$, i.e.,

$$Q_{\rm E} = \{|\alpha\rangle \in \mathbb{H}_n \mid \ell|\alpha\rangle = |\alpha\rangle, \quad \text{for all } \ell \in S_{\rm E}\}. \tag{8}$$

Combining Eqs. (3) and (6), we can derive the parity check matrix of $Q_{\rm E}$. We denote the parity check matrix of $Q_{\rm E}$ by $H_{\rm E} = (H_X|H_Z)$. Moreover, we denote the set of syndrome representatives that correspond to the parity check matrix of $H_{\rm E} = (H_X|H_Z)$ by $T_{\rm E}$. Then $T_{\rm E}$ is a set of correctable error operators of $Q_{\rm E}$ and $|T_{\rm E}| = 2^{n-k+c}$.

The Pauli channel on a single qubit can be written as

$$\phi \mapsto p_0\phi + p_xX\phi X + p_yY\phi Y + p_zZ\phi Z, \tag{9}$$

where ϕ is a single qubit, $0 \le p_0,p_x,p_y,p_z \le 1$, and $p_0+p_x+p_y+p_z = 1$. Let $p = 1-p_0$ be the error probability of a Pauli channel. If $p_x = p_y = p_z = p/3$, then the standard depolarizing channel is given by

$$\phi \mapsto (1-p)\phi + \frac{p}{3}X\phi X + \frac{p}{3}pY\phi Y + \frac{p}{3}pZ\phi Z. \tag{10}$$

Moreover, quantum channels usually have memories in practice.[24] The standard discretization error model in Ref. [12] supposes that quantum errors are independent of each other. However, correlated errors are more practical in reality. Similarly to classical information theory, an important family of quantum memory channels is specified by the channels with Markovian correlated errors. Thus we use a Markovian correlated depolarizing quantum channel model (see Refs. [17,24]) to present the behavior of quantum channels with memory. For an n-qubits quantum system, we denote

$$\begin{aligned}\Psi^{(n)}(\rho) = &\sum_{k_1,\ldots,k_n=0}^{3} p_{k_n|k_{n-1}}\cdots p^{(2)}_{k_2|k_1}p_{k_1}\\ &\times A_{k_n}\cdots A_{k_1}\rho A^{\dagger}_{k_1}\cdots A^{\dagger}_{k_n},\end{aligned} \tag{11}$$

where $\{A_{k_i}\}_{k,i}$ are the Pauli operators. The conditional probabilities in Eq. (11) follow the normalization condition as

$$\sum_{k_1,\ldots,k_n=0}^{3} p_{k_n|k_{n-1}}\cdots p_{k_2|k_1}p_{k_1} = 1, \tag{12}$$

where $p_0 = 1-p$, $p_{1,2,3} = p/3$, $p_{l|i} = (1-\mu)p_l + \mu\delta_{(i,l)}$ for $0 \le i,l \le 3$ and $1 \le j \le n$, and $\mu \in [0,1]$ is the degree of memory.

3. Fidelity of entanglement-assisted concatenated quantum codes over asymmetric quantum channels

Entanglement fidelity of quantum codes is a reasonable measure of how well the qubits and entanglement of a subsystem of a larger quantum system are preserved. In practice, quantum codes with very high entanglement fidelity are preferred. In Ref. [11], entanglement fidelity of several EAQSCs over depolorizing channels was computed. In this section, we compute and compare the entanglement fidelity of several EACQCs over asymmetric quantum channels.

Let $\mathbb{H}$ be a Hilbert space of dimension n. Let Δ be a quantum operator acting on a mixed state $\rho = \sum_i p_i\rho_i = {\rm Tr}_{\mathbb{H}_{\rm R}}|\alpha\rangle\langle\alpha|$ in terms of a purification $|\alpha\rangle \in \mathbb{H}\otimes\mathbb{H}_{\rm R}$, where $\mathbb{H}_{\rm R}$ is a reference system. The output state $\Delta(\rho)$ also lies in the Hilbert space $\mathbb{H}$. Then the entanglement fidelity is defined as

$$F(\rho,\Delta) = \langle\alpha|(\Delta\otimes I|\alpha\rangle\langle\alpha|)|\alpha\rangle. \tag{13}$$

In addition, we can write the quantum operator Δ in terms of Kraus operators, i.e., $\Delta(\rho) = \sum_i E_i\rho E_i^{\dagger}$, where $E_i^{\dagger}E_i = I$. Then the entanglement fidelity can be further expressed by a useful computational formula

$$F(\rho,\Delta) = \sum_i |{\rm Tr}(\rho E_i)|^2. \tag{14}$$

For a $Q_{\rm E} = [[n,k,d;c]]$ EAQSC, denote the encoding and decoding operations of $Q_{\rm E}$ by U and $U^{\dagger}$, respectively. Denote $\mathcal{T}$ and $\mathcal{D}$ by the encoding and decoding channels, respectively. Let $\{M_j\}$ be the syndrome measurement operations and let $\{D_j\}$ be the correction operations. Let the channel $\mathcal{N}$ be the encoded quantum information transmitted. During the decoding $U^{\dagger}$, denote the basis of the $n-k-c$ ancillas and c ebits by $\{|l\rangle\}$ and $\{|s\rangle\}$, respectively. Let $|\Phi_+\rangle_{\rm AB}$ be the Bell state shared between Alice and Bob. We define

$$\begin{aligned}\{V_{i,j,l,s} \equiv &(I^{\otimes k}\otimes\langle s|\langle l|)U^{\dagger}D_jM_j\sqrt{p_i}E_iU\\ &\times(I^{\otimes k}\otimes|0\rangle^{\otimes n-k-c}|\Phi_+\rangle_{\rm AB}^{\otimes c})\}\end{aligned}$$

as the set of operators on the composite channel $\mathcal{TNDT}$, where the subscripts i,j,l,s are taken accordingly. According to Ref. [9], the entanglement fidelity of $Q_{\rm E}$ over depolarizing channels is given by

$$F_{Q_{\rm E}} = \frac{1}{2^{2k}}\sum_{i,j,l,s}|V_{i,j,l,s}|^2. \tag{15}$$

Moreover, there is a result which can facilitate the computation of entanglement fidelity in Eq. (15).

Lemma 1[9]

$$\frac{1}{2^{2k}}\sum_{l,s}|V_{i,j,l,s}|^2 = \begin{cases} p_i, & \text{if } E_i \in T_{\rm E}\times S_{\rm E}\\ 0, & \text{otherwise,}\end{cases} \tag{16}$$

where $T_{\rm E}$ is a set of syndrome representatives and $S_{\rm E}$ is the stabilizer group.

For EAQSCs over asymmetric quantum channels, we have the following result about the computation of entanglement fidelity.

Theorem 1 Let $Q_{\rm E}$ be an an EAQSC code, with a stabilizer group $S_{\rm E}$ and a set of syndrome representatives $T_{\rm E}$. For q_{w_x,w_z} and r_{v_x,v_z} defined in Eqs. (18) and (19), denote $\Lambda=\{q_{w_x,w_z}r_{v_x,v_z}\}$ by the probability distribution of elements in $T_{\rm E}\times S_{\rm E}$. The entanglement fidelity of $Q_{\rm E}$ over asymmetric depolarizing channels $N_{\rm A}\otimes N_{\rm B}$ is the weight enumerator of Λ.

Proof According to Eq. (15) and Lemma 1, the entanglement fidelity of $Q_{\rm E}=[[n,k,d;c]]$ is given by

$$\begin{aligned}F_{Q_{\rm E}} &= \frac{1}{2^{2k}}\sum_{i,j,l,s}|V_{i,j,l,s}|^2=\sum_{{\rm E}_i\in T_{\rm E}\otimes S_{\rm E}}p_i\\ &=\sum_{w_x,w_z,v_x,v_z}\lambda_{w_x,w_z,v_x,v_z}q_{w_x,w_z}r_{w_x,w_z},\end{aligned}\tag{17}$$

where $\lambda_{w_x,w_z,v_x,v_z}$ is the number of error operators in $T_{\rm E}\otimes S_{\rm E}$ of weight distribution $\{w_x,w_z,v_x,v_z\}$.

In most quantum systems, there exists a large asymmetry between the occurrence probabilities of X errors, Y errors, and Z errors.[13,14,16] Furthermore, it was shown that the occurrence probability of Z errors is much larger than that of X errors. Since the Y operator can be formulated by $Y={\rm i}XZ$, a Y error can be seen as a combination of an X error and a Z error. Therefore we let $p_y=p_x$. Moreover, we suppose that Alice and Bob transmit quantum information through two independent quantum channels $N_{\rm A}$ and $N_{\rm B}$.[9] Suppose that both $N_{\rm A}$ and $N_{\rm B}$ are depolarizing channels. Let p_x, p_y, and p_z be the occurrence probabilities of X, Y, and Z errors in $N_{\rm A}$, respectively. Denote $p_a=p_x+p_y+p_z=2p_x+p_z$. Let r_x, r_y, and r_z be the occurrence probabilities of X, Y, and Z errors in $N_{\rm B}$, respectively. Denote $r_a=r_x+r_y+r_z=2r_x+r_z$. We have

$$q_{w_x,w_z}=(1-p_a)^{n-w_x-w_z}p_x^{w_x}p_z^{w_z},\tag{18}$$

where $0\le w_x+w_z\le n$, and denote by

$$r_{v_x,v_z}=(1-r_a)^{c-v_x-v_z}r_x^{v_x}r_z^{v_z},\tag{19}$$

where $0\le v_x+v_z\le c$. Define $\omega=r_a/p_a$ as the ratio of the error probabilities of ebits and qubits. In asymmetric quantum channels, we take $\eta=p_z/p_x=r_z/r_x$ as the channel asymmetry. It is reasonable to set the same channel asymmetry for the qubit and ebit channels. There may exist quantum channels that qubtis and ebits suffer from different channel asymmetry. But in this paper we do not consider that case. Therefore we have $\omega=r_a/p_a=r_x/p_x=r_z/p_z$.

Let $\mathcal{Q}_{{\rm E}_1}=[[3,1,3;2]]$ be an entanglement-assisted quantum repetition code. The parity check matrix of $\mathcal{Q}_{{\rm E}_1}=[[3,1,3;2]]$ (see Ref. [9]) is given by

$$H_{\mathcal{Q}_{{\rm E}_1}}=\begin{pmatrix}1&1&0&1&0&0&0&0&0&0\\0&1&1&0&1&0&0&0&0&0\\0&0&0&0&0&1&1&0&0&1\\0&0&0&0&0&0&1&1&1&0\end{pmatrix}.\tag{20}$$

According to Eqs. (3) and (4), the stabilizer group of $\mathcal{Q}_{{\rm E}_1}$ is given as follows:

$$S_{{\rm E}_1}=\langle X_1X_2X_4,X_2X_3X_5,Z_1Z_2Z_5,Z_2Z_3Z_4\rangle\tag{21}$$

and the syndrome representives of $Q_{{\rm E}_1}$ is given as

$$\begin{aligned}T_{{\rm E}_1}=\{&I,X_1,\ldots,X_3,Z_1,\ldots,Z_3,Y_1,\ldots,Y_3,\\&X_1Z_2,X_1Z_3,Z_1X_2,X_2Z_3,Z_1X_3,Z_2X_3\}.\end{aligned}$$

In asymmetric Pauli channels, we rewrite $T_{{\rm E}_1}$ as

$$\begin{aligned}T_{{\rm E}_1}=\{&I,X_1,\ldots,X_3,Z_1,\ldots,Z_3,X_1Z_1,\ldots,X_3Z_3,\\&X_1Z_2,X_1Z_3,Z_1X_2,X_2Z_3,Z_1X_3,Z_2X_3\}.\end{aligned}$$

Let $\mathcal{Q}_{{\rm E}_2}=[[3,1,3;2]]$ be an EAQSC given by Bowen in Ref. [27]. The parity check matrix of $\mathcal{Q}_{{\rm E}_2}=[[3,1,3;2]]$ is given by

$$H_{\mathcal{Q}_{{\rm E}_2}}=\begin{pmatrix}1&0&0&1&0&0&1&1&0&0\\0&0&1&0&1&1&1&0&0&0\\0&1&1&0&0&1&1&1&1&0\\1&1&0&0&0&1&1&1&0&1\end{pmatrix}.\tag{22}$$

According to Eqs. (3) and (4), the stabilizer group of $\mathcal{Q}_{{\rm E}_2}$ is given as follows:

$$S_{{\rm E}_2}=\langle X_1Z_2Z_3X_4,Z_1Z_2X_3X_5,Z_1Y_2Y_3Z_4,Y_1Y_2Z_3Z_5\rangle.\tag{23}$$

and the syndrome representives of $\mathcal{Q}_{{\rm E}_2}$ is given as

$$T_{{\rm E}_2}=\{I,X_1,\ldots,X_5,Z_1,\ldots,Z_5,Y_1,\ldots,Y_5\}.$$

In asymmetric Pauli channels, we rewrite $S_{{\rm E}_2}$ and $T_{{\rm E}_1}$ as

$$\begin{aligned}S_{{\rm E}_2}=\langle&X_1Z_2Z_3X_4,Z_1Z_2X_3X_5,Z_1X_2Z_2X_3Z_3Z_4,\\&X_1Z_1X_2Z_2Z_3Z_5\rangle,\\T_{{\rm E}_2}=\{&I,X_1,\ldots,X_5,Z_1,\ldots,Z_5,X_1Z_1,\ldots,X_5Z_5\}.\end{aligned}$$

Let $\mathcal{Q}_{\rm E}=[[5,1,3]]$ be the quantum stabilizer code.[3,9] The stabilizer group of $\mathcal{Q}_{\rm E}$ is given by

$$S_{{\rm E}_1}=\langle X_1Z_2Z_3X_4,X_2Z_3Z_4X_5,X_1X_3Z_4Z_5,Z_1X_2X_4Z_5\rangle.\tag{24}$$

Since the $\mathcal{Q}_{\rm E}=[[5,1,3]]$ stabilizer code is equivalent to Bowen's $\mathcal{Q}_{{\rm E}_2}=[[3,1,3;2]]$ EAQSC, the syndrome representive of $\mathcal{Q}_{\rm E}$ is the same as that of Bowen's code.

$$T_{\rm E}=\{I,X_1,\ldots,X_5,Z_1,\ldots,Z_5,Y_1,\ldots,Y_5\}.$$

According to Theorem 1, the entanglement fidelity of $\mathcal{Q}_{{\rm E}_1}$ over asymmetric quantum Pauli channels is given by

$$\begin{aligned}F_{\mathcal{A}_{{\rm E}_1}}=&q_{0,0}r_{0,0}+2q_{0,0}r_{0,1}+4q_{0,0}r_{1,0}+3q_{0,1}r_{0,0}+4q_{0,1}r_{1,1}\\&+2q_{0,1}r_{2,0}+4q_{0,2}r_{1,0}+6q_{0,2}r_{1,1}+3q_{0,2}r_{2,0}\\&+2q_{0,3}r_{1,0}+q_{0,3}r_{2,0}+6q_{1,0}r_{0,0}+2q_{1,0}r_{0,2}\\&+4q_{1,0}r_{1,1}+6q_{1,0}r_{2,0}+8q_{1,1}r_{0,1}+6q_{1,1}r_{0,2}\\&+8q_{1,1}r_{1,0}+12q_{1,1}r_{1,1}+18q_{1,1}r_{2,0}+2q_{1,2}r_{0,0}\\&+6q_{1,2}r_{0,1}+6q_{1,2}r_{1,0}+6q_{1,2}r_{1,1}+6q_{1,2}r_{2,0}\end{aligned}$$

$$
\begin{aligned}
&+4q_{2,0}r_{0,1}+3q_{2,0}r_{0,2}+12q_{2,0}r_{1,0}+18q_{2,0}r_{1,1}\\
&+15q_{2,0}r_{2,0}+2q_{2,1}r_{0,0}+6q_{2,1}r_{0,1}+3q_{2,1}r_{0,2}\\
&+18q_{2,1}r_{1,0}+12q_{2,1}r_{1,1}+3q_{2,1}r_{2,0}+2q_{3,0}r_{0,0}\\
&+6q_{3,0}r_{0,1}+2q_{3,0}r_{0,2}+10q_{3,0}r_{1,0}\\
&+2q_{3,0}r_{1,1}+10q_{3,0}r_{2,0}. \qquad (25)
\end{aligned}
$$

The entanglement fidelity of $\mathcal{Q}_{\mathrm{E}_2}$ over asymmetric quantum Pauli channels is given by

$$
\begin{aligned}
F_{\mathcal{A}_{\mathrm{E}_2}} =&q_{0,0}r_{0,0}+3q_{0,1}r_{0,0}+4q_{0,1}r_{0,1}+2q_{0,1}r_{0,2}\\
&+2q_{0,2}r_{0,1}+q_{0,2}r_{0,2}+2q_{0,3}r_{0,1}+q_{0,3}r_{0,2}\\
&+6q_{1,0}r_{0,0}+4q_{1,0}r_{0,1}+2q_{1,0}r_{0,2}+10q_{1,0}r_{1,0}\\
&+6q_{1,0}r_{1,1}+8q_{1,0}r_{2,0}+6q_{1,1}r_{0,0}+12q_{1,1}r_{0,1}\\
&+6q_{1,1}r_{0,2}+16q_{1,1}r_{1,0}+16q_{1,1}r_{1,1}+16q_{1,1}r_{2,0}\\
&+8q_{1,2}r_{0,1}+4q_{1,2}r_{0,2}+6q_{1,2}r_{1,0}+10q_{1,2}r_{1,1}\\
&+8q_{1,2}r_{2,0}+8q_{2,0}r_{1,0}+8q_{2,0}r_{1,1}+8q_{2,0}r_{2,0}\\
&+8q_{2,1}r_{1,0}+8q_{2,1}r_{1,1}+8q_{2,1}r_{2,0}+16q_{3,0}r_{1,0}\\
&+16q_{3,0}r_{1,1}+16q_{3,0}r_{2,0}. \qquad (26)
\end{aligned}
$$

In Fig. 1, we plot the entanglement fidelity of several EAQSCs over asymmetric quantum channels with different asymmetry η, and we choose different ratios ω. For all the three codes in Fig. 1, it is shown that the entanglement fidelity of each code gets improved as the channel asymmetry η grows. In particular, the effect of channel asymmetry is more obvious to the $\mathcal{Q}_{\mathrm{E}_1}=[[3,1,3;2]]$ repetition code. For example, the entanglement fidelity of $\mathcal{Q}_{\mathrm{E}_1}$ with $\eta=100$ is much higher than that of $\mathcal{Q}_{\mathrm{E}_2}$ with $\eta=10$ and $\eta=1$. Moreover, the three codes perform diversely under different ratios ω. In particular, the $\mathcal{Q}_{\mathrm{E}_1}=[[3,1,3;2]]$ repetition code performs much better than Bowen's $\mathcal{Q}_{\mathrm{E}_2}=[[3,1,3;2]]$ EAQSC and the $[[5,1,3]]$ stabilizer code when $\omega=0.01$. It should be noted that the curve of Bowen's $\mathcal{Q}_{\mathrm{E}_2}=[[3,1,3;2]]$ EAQSC coincides with that of the $[[5,1,3]]$ stabilizer code when $\omega=1$ in Fig. 1. The reason is that the stabilizer group of Bowen's $\mathcal{Q}_{\mathrm{E}_2}=[[3,1,3;2]]$ EAQSC is equivalent to that of the $[[5,1,3]]$ stabilizer code. When $\omega=1$, the entanglement fidelity of Bowen's $\mathcal{Q}_{\mathrm{E}_2}=[[3,1,3;2]]$ EAQSC is equal to that of the $[[5,1,3]]$ stabilizer code.

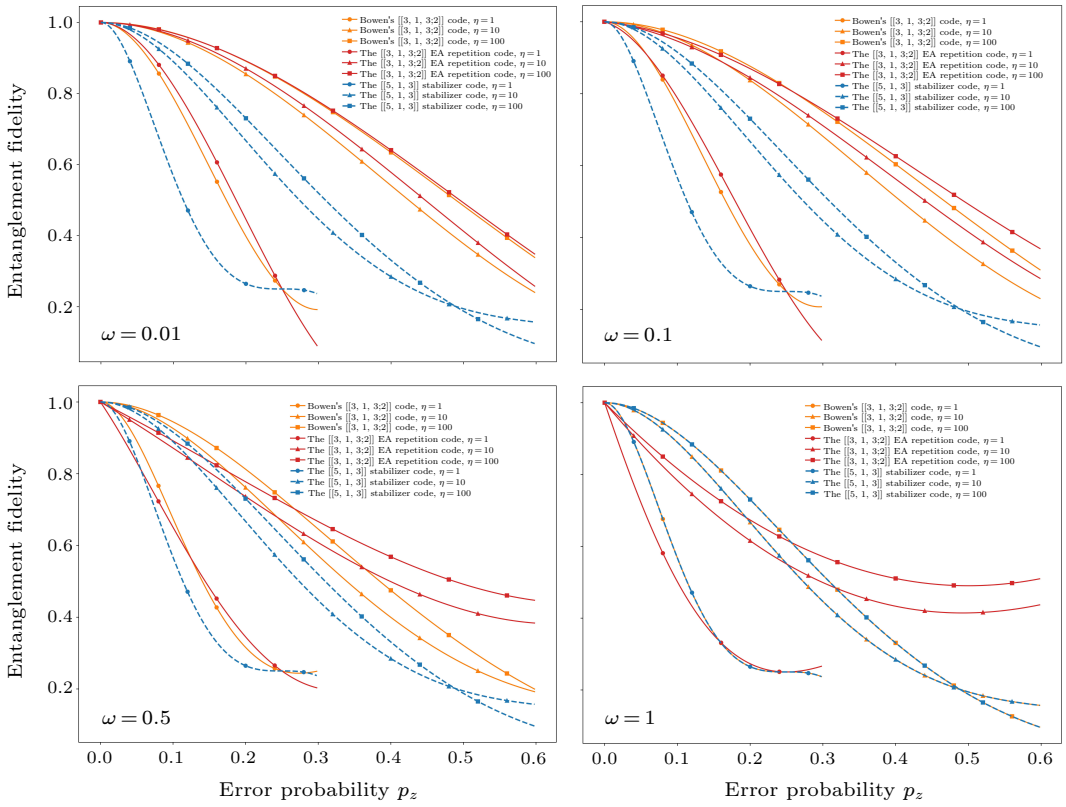


Fig. 1. Entanglement fidelity of EAQSCs and the $[[5,1,3]]$ stabilizer code over asymmetric quantum channels. The channel asymmetry is denoted by $\eta=p_z/p_x=r_z/r_x$. The ratio of the error probabilities of ebits and qubits is denoted by $\omega=r_a/p_a=r_x/p_x=r_z/p_z$. We use the error probability p_z and the entanglement fidelity as the X and Y axes, respectively.

4. Fidelity of entanglement-assisted quantum codes over quantum memory channels

In practical quantum channels, errors in qubits are usually not absolutely independent of each other. In this section, we compute and compare the entanglement fidelity of several entanglement-assisted quantum codes over Markovian quantum memory channels. We not only consider the memory in qubits but also consider the memory in ebits. Since qubits and ebits are usually stored in different space or in different time, we assume that errors in qubits are independent of those

in ebits. Therefore Alice and Bob transmit quantum information through two independent quantum channels $M_{\rm A}$ and $M_{\rm B}$.[9] Suppose that $M_{\rm A}$ and $M_{\rm B}$ are two independent Markovian depolarizing channels. The conditional probability of $M_{\rm A}$ satisfies

$$\sum_{k_1,\ldots,k_{n_{\rm A}}=0}^{3} p_{k_{n_{\rm A}}|k_{n_{\rm A}-1}}\cdots p_{k_2|k_1}p_{k_1}=1, \tag{27}$$

where $p_0=1-p$, $p_{1,2,3}=p/3$, $p_{l|i}=(1-\mu_{\rm A})p_l+\mu_{\rm A}\delta_{(i,l)}$ for $0\le i,l\le 3$, and $\mu_{\rm A}\in[0,1]$ is the degree of memory of channel $M_{\rm A}$. The conditional probabilities of $M_{\rm B}$ satisfy

$$\sum_{k_1,\ldots,k_{n_{\rm B}}=0}^{3} t_{k_{n_{\rm B}}|k_{n_{\rm B}-1}}\cdots t_{k_2|k_1}t_{k_1}=1, \tag{28}$$

where $t_0=1-t$, $t_{1,2,3}=t/3$, $t_{l|i}=(1-\mu_{\rm B})t_l+\mu_{\rm B}\delta_{(i,l)}$ for $0\le i,l\le 3$, and $\mu_{\rm B}\in[0,1]$ is the degree of memory of channel $M_{\rm B}$.

Let $S_{\rm E}$ and $T_{\rm E}$ be the stabilizer group and the set of syndrome representatives of an EAQSC, respectively. According to Lemma 1 and Theorem 1, the entanglement fidelity of EAQSCs over Markovian depolarizing channels is the weight enumerator of the corresponding probability distribution of elements in $T_{\rm E}\times S_{\rm E}$. We not only consider the memory in the qubits but also consider that in the ebits. For simplicity, we let the degree of channel $M_{\rm A}$ be the same as the degree of channel $M_{\rm B}$, i.e., we let $\mu=\mu_{\rm A}=\mu_{\rm B}$. Moreover, from Eqs. (27) and (28), we have

$$\begin{aligned}
&p_{01}=p_{02}=p_{03}=(1-\mu)(1-p),\\
&p_{11}=p_{22}=p_{33}=(1-\mu)\frac{p}{3}+\mu,\\
&p_{10}=p_{12}=p_{13}=p_{20}=p_{21}=p_{23}\\
&\quad=p_{30}=p_{31}=p_{32}=(1-\mu)\frac{p}{3},
\end{aligned} \tag{29}$$

$$\begin{aligned}
&t_{01}=t_{02}=t_{03}=(1-\mu)(1-t),\\
&t_{11}=t_{22}=t_{33}=(1-\mu)\frac{t}{3}+\mu,\\
&t_{10}=t_{12}=t_{13}=t_{20}=t_{21}=t_{23}=t_{30}\\
&\quad=t_{31}=t_{32}=(1-\mu)\frac{t}{3}.
\end{aligned} \tag{30}$$

Then the entanglement fidelity of the $\mathcal{Q}_{\rm E_1}=[[3,1,3;2]]$ entanglement-assisted repetition code is given by

$$\begin{aligned}
F_{\mathcal{M}_{\rm E_1}}=&t_0\{t_{00}[p_0(p_{00}^2+3p_{00}p_{12}+3p_{01}p_{12}+2p_{12}^2)\\
&+p_1(3p_{00}p_{01}+4p_{01}p_{12})]+t_{12}[p_0(2p_{00}p_{12}\\
&+5p_{01}p_{12}+p_{11}p_{12}+7p_{12}^2)+p_1(2p_{00}p_{01}\\
&+p_{01}(p_{11}+10p_{12})+3p_{11}^2+12p_{11}p_{12}+5p_{12}^2)]\}\\
&+t_1\{t_{01}[p_0(2p_{00}p_{12}+5p_{01}p_{12}+p_{11}p_{12}+7p_{12}^2)\\
&+p_1(2p_{00}p_{01}+p_{01}(p_{11}+10p_{12})+3p_{11}^2\\
&+12p_{11}p_{12}+5p_{12}^2)]+t_{11}[p_0(5p_{00}p_{12}+2p_{12}^2)\\
&+p_1(5p_{00}p_{01}+19p_{01}p_{12}+3p_{11}^2+8p_{11}p_{12}+6p_{12}^2)]\\
&+t_{12}[p_0(3p_{00}p_{12}+2p_{01}p_{12}+p_{11}p_{12}+12p_{12}^2)\\
&+p_1(3p_{00}p_{01}+p_{01}(p_{11}+20p_{12})+6p_{11}^2\\
&+28p_{11}p_{12}+20p_{12}^2)]\}.
\end{aligned} \tag{31}$$

The entanglement fidelity of Bowen's $\mathcal{Q}_{\rm E_2}=[[3,1,3;2]]$ EAQSC is given by

$$\begin{aligned}
F_{\mathcal{M}_{\rm E_2}}=&t_0\{t_{00}[p_0(p_{00}^2+3p_{00}p_{12}+3p_{01}p_{12})\\
&+p_1(3p_{00}p_{01}+6p_{11}p_{12})]+t_{12}[p_0(3p_{00}^2+6p_{12}^2)\\
&+p_1(p_{01}(3p_{11}+9p_{12})+3p_{11}^2+15p_{11}p_{12}+9p_{12}^2)]\}\\
&+t_1\{t_{01}[p_0(3p_{00}^2+3p_{11}p_{12}+3p_{12}^2)+p_1(12p_{01}p_{12}\\
&+3p_{11}^2+15p_{11}p_{12}+9p_{12}^2)]+t_{11}[p_0(3p_{00}p_{12}\\
&+6p_{12}^2)+p_1(3p_{00}p_{01}+21p_{01}p_{12}+3p_{11}^2+6p_{11}p_{12}\\
&+6p_{12}^2)]+t_{12}[p_0(3p_{00}p_{12}+6p_{01}p_{12}+6p_{11}p_{12}\\
&+15p_{12}^2)+p_1(3p_{00}p_{01}+p_{01}(6p_{11}+27p_{12})\\
&+18p_{11}p_{12}+12p_{12}^2)]\}.
\end{aligned} \tag{32}$$

For comparison with standard quantum stabilizer codes, we also compute the entanglement fidelity of the $[[5,1,3]]$ stabilizer codes as follows:

$$\begin{aligned}
F_{\rm S}=&p_0[p_{00}^4+3p_{00}^3p_{12}+9p_{00}^2p_{01}p_{12}+6p_{00}p_{11}p_{12}^2\\
&+p_{01}(6p_{11}p_{12}^2+12p_{12}^3)+6p_{11}^2p_{12}^2+15p_{11}p_{12}^3+6p_{12}^4]\\
&+p_1[3p_{00}^3p_{01}+p_{00}p_{01}(12p_{11}p_{12}+6p_{12}^2)+18p_{01}^2p_{12}^2\\
&+p_{01}(12p_{11}^2p_{12}+57p_{11}p_{12}^2+39p_{12}^3)+6p_{11}^3p_{12}\\
&+9p_{11}^2p_{12}^2+24p_{11}p_{12}^3+6p_{12}^4].
\end{aligned} \tag{33}$$

In Fig. 2, we plot the entanglement fidelity of several EAQSCs over Markovian quantum memory channels. From Fig. 2, we know that the channel memory lowers down the performance of EAQSCs in most range of the depolarizing probability. Moreover, the $\mathcal{Q}_{\rm E_1}=[[3,1,3;2]]$ repetition code and Bowen's $\mathcal{Q}_{\rm E_2}=[[3,1,3;2]]$ EAQSC can outperform the $[[5,1,3]]$ stabilizer code provided that the error probability of ebits is sufficiently smaller than that of qubits. When $\omega=1$, the $[[5,1,3]]$ stabilizer code performs better than the two EAQSCs over quantum memory channels; while $\omega=0.5$, Bowen's $\mathcal{Q}_{\rm E_2}=[[3,1,3;2]]$ EAQSC can outperform the $\mathcal{Q}_{\rm E_1}=[[3,1,3;2]]$ repetition code and the $[[5,1,3]]$ stabilizer code. As the ratio ω is smaller, e.g., $\omega=0.1,0.01$, the performance of the two EAQSCs becomes much better than the $[[5,1,3]]$ stabilizer code over quantum memory channels. Specifically, when $\omega=0.01$, the performance of the two EAQSCs is diverse and is effected largely by the channel memory. In such a case, the $\mathcal{Q}_{\rm E_1}=[[3,1,3;2]]$ repetition code can outperform Bowen's $\mathcal{Q}_{\rm E_2}=[[3,1,3;2]]$ EAQSC if the channel memory μ is small, e.g., $\mu=0,0.1$. However, Bowen's $\mathcal{Q}_{\rm E_2}=[[3,1,3;2]]$ EAQSC can beat the $\mathcal{Q}_{\rm E_1}=[[3,1,3;2]]$ repetition code if the

channel memory μ is large, e.g., $\mu = 0.5$. It should be noted that the curve of Bowen's $\mathcal{Q}_{E_2} = [[3,1,3;2]]$ EAQSC does not coincides with that of the $[[5,1,3]]$ stabilizer code when $\omega = 1$ ($\mu \neq 0$) in Fig. 2. Here the situation is different from that in asymmetric quantum channels. For Bowen's $\mathcal{Q}_{E_2} = [[3,1,3;2]]$ EAQSC over Markovian quantum memory channels, we separate the three qubits and the two ebits. The two parts are independent of each other. However, for the $[[5,1,3]]$ stabilizer code over Markovian quantum memory channels, the five qubits are considered as a whole.

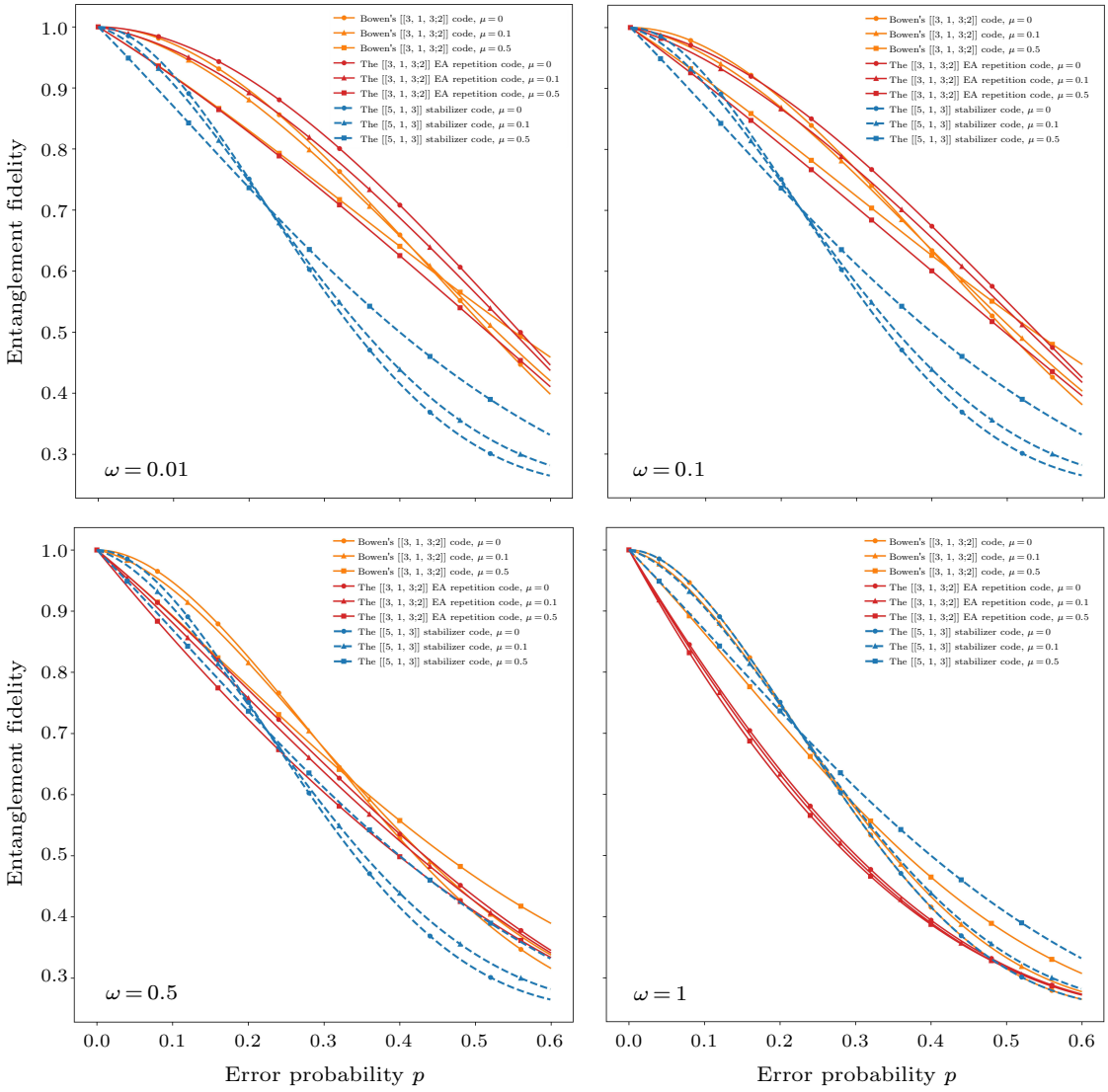


Fig. 2. Entanglement fidelity of EAQSCs and the $[[5,1,3]]$ stabilizer code over Markovian quantum memory channels. The channel memory is denoted by μ. The ratio of the error probabilities of ebits and qubits is denoted by $\omega = r_a/p_a = r_x/p_x = r_z/p_z$. We use the error probability p and the entanglement fidelity as the X axis and the Y axis, respectively.

5. Conclusion

In summary, we have computed the entanglement fidelity of two entanglement-assisted quantum stabilizer codes over asymmetric quantum channels and quantum memory channels. Moreover, we have not only considered asymmetric errors and correlated errors in qubits but also in ebits. In asymmetric quantum channels, we have shown that the performance of the two EAQSCs is quite diverse and is related not only to the channel asymmetry but also to the ratio of the error probabilities of ebits and qubits. We have shown that the $[[3,1,3;2]]$ repetition code performs much better than Bowen's $[[3,1,3;2]]$ EAQECCs and the $[[5,1,3]]$ stabilize code if the asymmetry is sufficiently large and the error probability of ebits is much smaller than that of qubits. In Markovian quantum memory channels, we have shown that the entanglement fidelity of the two EAQSCs is lowered down by the channel memory. We have shown that the performance of the two EAQSCs is not only related to the channel memory μ but also to the ratio of the error probabilities of ebits and qubits. We have shown that the two EAQSCs can outperform the $[[5,1,3]]$ stabilizer code over quantum memory channels if the ratio is small. We have shown that Bowen's $[[3,1,3;2]]$ EAQSC can outperform the $[[3,1,3;2]]$ repetition

code over quantum memory channels if the ratio of the error probabilities of ebits and qubits is relatively large. However, if the ratio is small, we have shown that the performance of the two EAQSCs is diverse and is effected greatly by the channel memory. We have shown that the $[[3,1,3;2]]$ repetition code can outperform Bowen's $[[3,1,3;2]]$ EAQSC if the channel memory is small. However, Bowen's $[[3,1,3;2]]$ EAQSC can beat the $[[3,1,3;2]]$ repetition code if the channel memory becomes larger. How to compute the entanglement fidelity of other EAQSCs over asymmetric quantum channels and quantum memory channels needs further research. However, the exact computation is extremely difficult as the code length becomes longer. One possible method is to approximate the computation of the entanglement fidelity. The other method is to use the entanglement-assisted concatenated quantum codes scheme. However, the exact computation of the entanglement fidelity of entanglement-assisted concatenated quantum codes is not a direct generalization and needs further research in the future work. Moreover, how to use entanglement-assisted quantum codes in quantum key distribution[28,29] to improve the entanglement fidelity is one important future work which needs further studied.

Acknowledgments

Project supported by the National Key R&D Program of China (Grant No. 2022YFB3103802), the National Natural Science Foundation of China (Grant Nos. 62371240 and 61802175), and the Fundamental Research Funds for the Central Universities (Grant No. 30923011014).

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