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Parameterized monogamy and polygamy relations of multipartite entanglement

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Monogamy and polygamy relations are important properties of entanglement, which characterize the entanglement distribution of multipartite systems. We explore monogamy and polygamy relations of entanglement in multipartite systems by using two newly derived parameterized mathematical inequalities, and establish classes of parameterized monogamy and polygamy relations of multiqubit entanglement in terms of concurrence and entanglement of formation. We show that these new parameterized monogamy and poelygamy inequalities are tighter than the existing ones by detailed examples.

Keywords: monogamy, polygamy, concurrence, entanglement of formation

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1. Introduction

Entanglement is one of the most remarkable phenomena in quantum mechanics, which is also the most important resources in quantum information processing and communication.[1–4] In recent years, great efforts have been made toward the understanding of the role played by the entanglement in quantum information theory such as quantum teleportation,[5] quantum key distribution,[6] and quantum computing.[7]

An important problem in the theory of quantum entanglement is the quantification of entanglement.[8] The entanglement for bipartite systems has been intensively studied, while the characterization and quantification of multipartite entanglement is far from being satisfied. An essential property of quantum entanglement is that a quantum system entangled with one of the other subsystems limits its entanglement with the remaining ones, known as the monogamy of entanglement.[9,10] Monogamy describes the distribution of entanglement among the multipartite systems. It is an important way to study the structure and properties of multipartite entanglement. In Ref. [11] the authors showed that any entanglement measure given by a strictly concave function of the reduced density matrix for pure bipartite states is monogamous for pure tripartite states. Since the monogamy of entanglement restricts on the amount of information that an eavesdropper could potentially obtain about the secret key extraction, it plays a crucial role in the security of many information-theoretic protocols such as quantum key distribution protocols.[12,13] The monogamy of entanglement has also been widely used in, e.g., quantum information theory,[14] condensed-matter physics,[15] and even black-hole physics.[16]

Coffman, Kundu and Wootters first characterized the monogamy of entanglement for three-qubit states ρ_{ABC},[17]

$$\mathcal{E}(\rho_{A|BC}) \geq \mathcal{E}(\rho_{AB}) + \mathcal{E}(\rho_{AC}), \tag{1}$$

where $\rho_{AB} = \mathrm{tr}_C(\rho_{ABC})$ and $\rho_{AC} = \mathrm{tr}_B(\rho_{ABC})$ are the reduced density matrices and $\mathcal{E}$ is an entanglement measure. It is shown that not all entanglement measures satisfy such monogamy relations. Although the concurrence C does not satisfy such monogamy inequality, it has been proved that the squared concurrence C^2[18] and the squared entanglement of formation E^2[19,20] do satisfy the monogamy relations for multipartite qubit systems. With respect to the inequality (1), the monogamy relations have also been explored for other entanglement measures such as the convex-roof extended negativity (CREN) $\widetilde{N}$,[21–23] Tsallis-q entropy,[24–26] Rényi-α entropy,[27,28] unified entropy,[29,30] and geometric measure.[31,32]

Another important concept is the entanglement of assistance, which is a dual amount to bipartite entanglement measure. It has a dually monogamous property in multipartite quantum systems and gives rise to polygamy relations. As another entanglement constraint in multipartite entanglement, polygamy can be described by assisted entanglement. For arbitrary tripartite systems, Gour *et al.* in Ref. [33] established the first polygamy inequality by using the squared concurrence of assistance C_a^2. For a tripartite state ρ_{ABC}, the usual polygamy relation is characterized through the follow-

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ing form:

$$\mathcal{E}_a(\rho_{A|BC}) \leq \mathcal{E}_a(\rho_{AB}) + \mathcal{E}_a(\rho_{AC}), \tag{2}$$

where $\mathcal{E}_a$ is the corresponding entanglement measure of assistance associated to $\mathcal{E}$. It was later generalized to multiqubit systems by using various assisted entanglements.[24,29,33] For the arbitrary dimensional quantum systems, using entanglement of assistance, general polygamy inequalities of multipartite entanglement were also proposed in Refs. [34–36]. In Ref. [37] the author showed that any entanglement of assistance is polygamous but not monogamous, and any faithful entanglement measure is not polygamous.

Recently, generalized classes of monogamy inequalities related to the α-th power of entanglement measures were proposed. In Refs. [38–40], the authors proved that the squared concurrence and the convex-roof extended negativity satisfy the monogamy inequalities in multiqubit systems for $\alpha \geq 2$. It has also been shown that the entanglement of formation (EoF) satisfies monogamy relations when $\alpha \geq \sqrt{2}$.[38–40] In addition, the Tsallis-q entanglement and Rényi-α entanglement satisfy monogamy relations when $\alpha \geq 1$.[24,27,39,40] Moreover, the corresponding polygamy relations have also been established.[25,33,35,41]

In this paper, we present the monogamy inequalities in terms of the concurrence C and entanglement of formation E, we also provide the polygamy inequalities in terms of the concurrence of assistance C_a and the entanglement of assistance E_a. Furthermore, we show that our optimized monogamy and polygamy relations are tighter inequalities than the results in Refs. [38–40,42,43].

2. Tighter monogamy and polygamy relations for concurrence

Let $\mathcal{H}_X$ denote a finite-dimensional complex vector space associated to a quantum subsystem X. Given a bipartite pure state $|\phi\rangle_{AB}$ in Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the concurrence is given by

$$C(|\phi\rangle_{AB}) = \sqrt{2[1-\mathrm{Tr}(\rho_A^2)]}, \tag{3}$$

where $\rho_A = \mathrm{Tr}_B(|\phi\rangle_{AB}\langle\phi|)$ is the reduced density matrix obtained by tracing over the subsystem B.[44–46] The concurrence for a bipartite mixed state ρ_{AB} is defined by the convex roof extension as follows:

$$C(\rho_{AB}) = \min_{\{p_i,|\phi_i\rangle\}} \sum_i p_i C(|\phi_i\rangle), \tag{4}$$

where the minimum is taken over all possible decompositions of $\rho_{AB} = \sum_i p_i |\phi_i\rangle\langle\phi_i|$, with $p_i \geq 0$, $\sum_i p_i = 1$, and $|\phi_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. It has been proved that the concurrence of a mixed state ρ_{AB} can be calculated by an analytical expression as[47]

$$C(\rho_{AB}) = \max\{\mu_1 - \mu_2 - \mu_3 - \mu_4, 0\},$$

where $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$ are the eigenvalues of the matrix $\sqrt{\sqrt{\rho}\widetilde{\rho}\sqrt{\rho}}$. Here, $\widetilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, and ρ^* is the complex conjugation of ρ.

For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$ acting on $\mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, the concurrence $C(\rho_{A|B_1\cdots B_{N-1}})$ of the state $\rho_{AB_1\cdots B_{N-1}}$ under bipartition A and $B_1, B_2, \ldots, B_{N-1}$ satisfies

$$C^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \geq C^\alpha(\rho_{AB_1}) + C^\alpha(\rho_{AB_2}) + \cdots + C^\alpha(\rho_{AB_{N-1}}), \tag{5}$$

for all $\alpha \geq 2$, where $\rho_{AB_i} = \mathrm{Tr}_{B_1\cdots Bi-1Bi+1\cdots B_{N-1}}(\rho_{AB_1\cdots B_{N-1}})$.[38] The relation (5) is improved to be

$$\begin{aligned} C^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \geq\ & C^\alpha(\rho_{AB_1}) + \frac{\alpha}{2}C^\alpha(\rho_{AB_2}) + \cdots \\ & + \left(\frac{\alpha}{2}\right)^{m-1} C^\alpha(\rho_{AB_m}) \\ & + \left(\frac{\alpha}{2}\right)^{m+1} [C^\alpha(\rho_{AB_{m+1}}) + \cdots \\ & + C^\alpha(\rho_{AB_{N-2}})] \\ & + \left(\frac{\alpha}{2}\right)^{m} C^\alpha(\rho_{AB_{N-1}}) \end{aligned} \tag{6}$$

in Ref. [39] and then to be

$$\begin{aligned} C^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \geq\ & C^\alpha(\rho_{AB_1}) + \left(2^{\frac{\alpha}{2}} - 1\right) C^\alpha(\rho_{AB_2}) + \cdots \\ & + (2^{\frac{\alpha}{2}} - 1)^{m-1} C^\alpha(\rho_{AB_m}) \\ & + (2^{\frac{\alpha}{2}} - 1)^{m+1} [C^\alpha(\rho_{AB_{m+1}}) + \cdots \\ & + C^\alpha(\rho_{AB_{N-2}})] \\ & + \left(2^{\frac{\alpha}{2}} - 1\right)^{m} C^\alpha(\rho_{AB_{N-1}}) \end{aligned} \tag{7}$$

in Ref. [40] for all $\alpha \geq 2$, where $C(\rho_{AB_i}) \geq C(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i = 1, 2, \ldots, m$, and $C(\rho_{AB_j}) \leq C(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m+1, \ldots, N-2$, $1 \leq m \leq N-3$ and $N \geq 4$.

In Ref. [42] the relation (7) is improved for all $\alpha \geq 2$ as

$$\begin{aligned} & C^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \\ \geq\ & C^\alpha(\rho_{AB_1}) + \left(\left((1+k)^{\frac{\alpha}{2}} - 1\right)/k^{\frac{\alpha}{2}}\right) C^\alpha(\rho_{AB_2}) \\ & + \cdots + \left(\left((1+k)^{\frac{\alpha}{2}} - 1\right)/k^{\frac{\alpha}{2}}\right)^{m-1} C^\alpha(\rho_{AB_m}) \\ & + \left(\left((1+k)^{\frac{\alpha}{2}} - 1\right)/k^{\frac{\alpha}{2}}\right)^{m+1} [C^\alpha(\rho_{AB_{m+1}}) \\ & + \cdots + C^\alpha(\rho_{AB_{N-2}})] \\ & + \left(\left((1+k)^{\frac{\alpha}{2}} - 1\right)/k^{\frac{\alpha}{2}}\right)^{m} C^\alpha(\rho_{AB_{N-1}}), \end{aligned} \tag{8}$$

with $kC^2(\rho_{AB_i}) \geq C^2(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i = 1, 2, \ldots, m$, and $C^2(\rho_{AB_j}) \leq kC^2(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m+1, \ldots, N-2$, $1 \leq m \leq N-3$, $N \geq 4$ and $0 < k \leq 1$. The relation (8) is further improved for all $\alpha \geq 2$ as

$$\begin{aligned} C^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \geq\ & C^\alpha(\rho_{AB_1}) + \mathcal{K}_1 C^\alpha(\rho_{AB_2}) + \cdots \\ & + \mathcal{K}_1 \cdots \mathcal{K}_{m-1} C^\alpha(\rho_{AB_m}) \end{aligned}$$

$$
\begin{aligned}
&+\mathcal{K}_1\cdots\mathcal{K}_m(\mathcal{K}_{m+1}C^\alpha(\rho_{AB_{m+1}})+\cdots\\
&+\mathcal{K}_{N-2}C^\alpha)(\rho_{AB_{N-2}})\\
&+\mathcal{K}_1\cdots\mathcal{K}_mC^\alpha(\rho_{AB_{N-1}})
\end{aligned}\tag{9}
$$

with $C^2(\rho_{AB_i})\geq l_iC^2(\rho_{A|B_{i+1}\cdots B_{N-1}})$, $C^2(\rho_{A|B_i\cdots B_{N-1}})\geq C^2(\rho_{AB_i})+\mu_iC^2(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i=1,2,\ldots,m$, $C^2(\rho_{A|B_{j+1}\cdots B_{N-1}})\geq l_jC^2(\rho_{AB_j})$, $C^2(\rho_{A|B_j\cdots B_{N-1}})\geq \mu_jC^2(\rho_{AB_j})+C^2(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j=m+1,\ldots,N-2$, $1\leq m\leq N-3, N\geq 4$, where $\mathcal{K}_r=(\mu_r+l_r)^{\frac{\alpha}{2}}-l_r^{\frac{\alpha}{2}}$ with $1\leq r\leq N-2$, $\mu_r\geq 1$ and $l_r\geq 1$ in Ref. [43].

Next, we show that the above monogamy relations can become even tighter under some conditions. For convenience, we denote $C_{AB_j}=C(\rho_{AB_j})$ for $j=1,2,\ldots,N-1$, and $C_{A|B_1B_2\cdots B_{N-1}}=C(\rho_{A|B_1B_2\cdots B_{N-1}})$. Before giving our result, we propose the following lemmas.

Lemma 1 Let $l\geq 1$ and $\delta\geq 1$ be any real numbers. For any $x\geq l^\delta\geq 1$, $1\leq p\leq 1+\frac{1}{x}$ and non-negative real numbers m,n, we have

$$
(1+x)^m-p^{m-1}x^m\geq(1+l^\delta)^m-p^{m-1}(l^\delta)^m,\ \text{for}\ m\geq 1,\tag{10}
$$

$$
(1+x)^n-p^{n-1}x^n\leq(1+l^\delta)^n-p^{n-1}(l^\delta)^n,\ \text{for}\ 0\leq n\leq 1.\tag{11}
$$

Proof We first prove the case for $m\geq 1$. Consider the function $f(x,m)=(1+x)^m-p^{m-1}x^m$ with $x\geq l^\delta\geq 1$, $1\leq p\leq 1+\frac{1}{x}$ and $m\geq 1$, $l\geq 1$ and $\delta\geq 1$. Since $\frac{\partial f(x,m)}{\partial x}=m(1+x)^{m-1}-mp^{m-1}x^{m-1}=m[(1+x)^{m-1}-(px)^{m-1}]\geq 0$, the function $f(x,m)$ increases with x. As $x\geq l^\delta\geq 1$, $f(x,m)\geq f(l^\delta,m)=(1+l^\delta)^m-p^{m-1}(l^\delta)^m$. Therefore, we get the inequality (10). The inequality (11) can be proved similarly to the proof of inequality (10).

For any $2\otimes 2\otimes 2^{N-2}$ mixed state $\rho_{ABC}\in\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_C$, we have the relation from inequality (5), $C^2_{A|BC}\geq C^2_{AB}+C^2_{AC}$. Therefore, there exists $\mu\geq 1$ such that

$$
C^2_{A|BC}\geq C^2_{AB}+\mu C^2_{AC}.\tag{12}
$$

Lemma 2 Suppose that μ, l and δ are any real numbers satisfying $\mu\geq 1$, $l\geq 1$ and $\delta\geq 1$. For any $2\otimes 2\otimes 2^{N-2}$ mixed state $\rho_{ABC}\in\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_C$, if $C^2_{AB}\geq l^\delta C^2_{AC}$ and $1\leq p\leq 1+\frac{\mu C^2_{AC}}{C^2_{AB}}$, we have

$$
C^\alpha_{A|BC}\geq p^{\frac{\alpha}{2}-1}C^\alpha_{AB}+((\mu+l^\delta)^{\frac{\alpha}{2}}-p^{\frac{\alpha}{2}-1}l^{\frac{\delta\alpha}{2}})C^\alpha_{AC}\tag{13}
$$

for all $\alpha\geq 2$.

Proof By straightforward calculation, we have

$$
\begin{aligned}
C^\alpha_{A|BC}&=(C^2_{A|BC})^{\frac{\alpha}{2}}\geq(C^2_{AB}+\mu C^2_{AC})^{\frac{\alpha}{2}}\\
&=\mu^{\frac{\alpha}{2}}C^\alpha_{AC}[(\mu^{-1}(C^2_{AB}/C^2_{AC})+1)^{\frac{\alpha}{2}}\\
&\quad-p^{\frac{\alpha}{2}-1}(\mu^{-1}(C^2_{AB}/C^2_{AC}))^{\frac{\alpha}{2}}]+p^{\frac{\alpha}{2}-1}C^\alpha_{AB}\\
&\geq\mu^{\frac{\alpha}{2}}C^\alpha_{AC}[(\mu^{-1}l^\delta+1)^{\frac{\alpha}{2}}-p^{\frac{\alpha}{2}-1}(\mu^{-1}l^\delta)^{\frac{\alpha}{2}}]\\
&\quad+p^{\frac{\alpha}{2}-1}C^\alpha_{AB}\\
&=p^{\frac{\alpha}{2}-1}C^\alpha_{AB}+((\mu+l^\delta)^{\frac{\alpha}{2}}-p^{\frac{\alpha}{2}-1}l^{\frac{\delta\alpha}{2}})C^\alpha_{AC},
\end{aligned}\tag{14}
$$

where the second inequality is due to Lemma 1. We can also see that if $C_{AB}=0$, then $C_{AC}=0$, and the lower bound becomes trivially zero.

For multiqubit systems, we have the following theorems.

Theorem 1 Let $\delta_r\geq 1$, $\mu_r\geq 1$ and $l_r\geq 1$ be real numbers, $1\leq r\leq N-2$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}\in\mathcal{H}_A\otimes\mathcal{H}_{B_1}\otimes\cdots\otimes\mathcal{H}_{B_{N-1}}$, if $C^2_{AB_i}\geq l_i^{\delta_i}C^2_{A|B_{i+1}\cdots B_{N-1}}$, $C^2_{A|B_i\cdots B_{N-1}}\geq C^2_{AB_i}+\mu_iC^2_{A|B_{i+1}\cdots B_{N-1}}$, $1\leq p_i\leq 1+\frac{\mu_iC^2_{A|B_{i+1}\cdots B_{N-1}}}{C^2_{AB_i}}$ for $i=1,2,\ldots,m$, and $C^2_{A|B_{j+1}\cdots B_{N-1}}\geq l_j^{\delta_j}C^2_{AB_j}$, $C^2_{A|B_j\cdots B_{N-1}}\geq\mu_jC^2_{AB_j}+C^2_{A|B_{j+1}\cdots B_{N-1}}$, $1\leq p_j\leq\frac{\mu_jC^2_{AB_j}}{C^2_{A|B_{j+1}\cdots B_{N-1}}}$ for $j=m+1,\ldots,N-2$, $1\leq m\leq N-3$ and $N\geq 4$, we have

$$
\begin{aligned}
&C^\alpha_{A|B_1\cdots B_{N-1}}\\
&\geq p_1^{\frac{\alpha}{2}-1}C^\alpha_{AB_1}+\sum_{i=2}^{m}\prod_{h=1}^{i-1}\Gamma_hp_i^{\frac{\alpha}{2}-1}C^\alpha_{AB_i}\\
&\quad+\Gamma_1\cdots\Gamma_m\Big(\Gamma_{m+1}C^\alpha_{AB_{m+1}}+\sum_{j=m+2}^{N-2}\prod_{h=m+1}^{j-1}p_h^{\frac{\alpha}{2}-1}\Gamma_jC^\alpha_{AB_j}\Big)\\
&\quad+\Gamma_1\cdots\Gamma_m(p_{m+1}\cdots p_{N-2})^{\frac{\alpha}{2}-1}C^\alpha_{AB_{N-1}},
\end{aligned}\tag{15}
$$

for all $\alpha\geq 2$, where $\Gamma_r=(\mu_r+l_r^{\delta_r})^{\frac{\alpha}{2}}-p_r^{\frac{\alpha}{2}-1}l_r^{\frac{\delta_r\alpha}{2}}$ with $1\leq r\leq N-2$.

Proof From Lemma 2, if $C^2_{AB_i}\geq l_i^{\delta_i}C^2_{A|B_{i+1}\cdots B_{N-1}}$, $C^2_{A|B_i\cdots B_{N-1}}\geq C^2_{AB_i}+\mu_iC^2_{A|B_{i+1}\cdots B_{N-1}}$, $1\leq p_i\leq 1+\frac{\mu_iC^2_{A|B_{i+1}\cdots B_{N-1}}}{C^2_{AB_i}}$ for $i=1,2,\ldots,m$, we have

$$
\begin{aligned}
C^\alpha_{A|B_1\cdots B_{N-1}}&\geq p_1^{\frac{\alpha}{2}-1}C^\alpha_{AB_1}+\Gamma_1C^\alpha_{A|B_2\cdots B_{N-1}}\\
&\geq p_1^{\frac{\alpha}{2}-1}C^\alpha_{AB_1}+\Gamma_1p_2^{\frac{\alpha}{2}-1}C^\alpha_{AB_2}\\
&\quad+\Gamma_1\Gamma_2C^\alpha_{A|B_3\cdots B_{N-1}}\geq\cdots\\
&\geq p_1^{\frac{\alpha}{2}-1}C^\alpha_{AB_1}+\Gamma_1p_2^{\frac{\alpha}{2}-1}C^\alpha_{AB_2}\\
&\quad+\cdots+\Gamma_1\cdots\Gamma_{m-1}p_m^{\frac{\alpha}{2}-1}C^\alpha_{AB_m}\\
&\quad+\Gamma_1\cdots\Gamma_mC^\alpha_{A|B_{m+1}\cdots B_{N-1}}.
\end{aligned}\tag{16}
$$

If $C^2_{A|B_{j+1}\cdots B_{N-1}}\geq l_j^{\delta_j}C^2_{AB_j}$, $C^2_{A|B_j\cdots B_{N-1}}\geq\mu_jC^2_{AB_j}+C^2_{A|B_{j+1}\cdots B_{N-1}}$, $1\leq p_j\leq\frac{\mu_jC^2_{AB_j}}{C^2_{A|B_{j+1}\cdots B_{N-1}}}$ for $j=m+1,\ldots,N-2$, we get

$$
\begin{aligned}
C^\alpha_{A|B_{m+1}\cdots B_{N-1}}&\geq\Gamma_{m+1}C^\alpha_{AB_{m+1}}+p_{m+1}^{\frac{\alpha}{2}-1}C^\alpha_{A|B_{m+2}\cdots B_{N-1}}\\
&\geq\Gamma_{m+1}C^\alpha_{AB_{m+1}}+p_{m+1}^{\frac{\alpha}{2}-1}\Gamma_{m+2}C^\alpha_{AB_{m+2}}\\
&\quad+(p_{m+1}p_{m+2})^{\frac{\alpha}{2}-1}C^\alpha_{A|B_{m+3}\cdots B_{N-1}}\geq\cdots\\
&\geq\Gamma_{m+1}C^\alpha_{AB_{m+1}}+p_{m+1}^{\frac{\alpha}{2}-1}\Gamma_{m+2}C^\alpha_{AB_{m+2}}+\cdots
\end{aligned}
$$

$$+(p_{m+1}\cdots p_{N-3})^{\frac{\alpha}{2}-1}\Gamma_{N-2}C^{\alpha}_{AB_{N-2}}$$
$$+(p_{m+1}\cdots p_{N-2})^{\frac{\alpha}{2}-1}C^{\alpha}_{AB_{N-1}}. \quad (17)$$

Combining inequalities (16) and (17), we complete the proof.

In Theorem 1 we have assumed that C_{AB_i} for $i=1,2,\ldots,m$. If $C_{AB_1}=0$, then $C_{AB_i}=0$ for $i=2,\ldots,m$, and the lower bound becomes trivially zero.

In particular, if $C^2_{AB_i}\geq l_i^{\delta_i}C^2_{A|B_{i+1}\cdots B_{N-1}}$, $C^2_{A|B_i\cdots B_{N-1}}\geq C^2_{AB_i}+\mu_i C^2_{A|B_{i+1}\cdots B_{N-1}}$, $1\leq p_i\leq 1+\frac{\mu_i C^2_{A|B_{i+1}\cdots B_{N-1}}}{C^2_{AB_i}}$ for $i=1,2,\ldots,N-2$, then the following monogamy relation holds.

Theorem 2 Let $\delta_r\geq 1$, $\mu_r\geq 1$ and $l_r\geq 1$ be real numbers, $1\leq r\leq N-2$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}\in\mathcal{H}_A\otimes\mathcal{H}_{B_1}\otimes\cdots\otimes\mathcal{H}_{B_{N-1}}$, if $C^2_{AB_i}\geq l_i^{\delta_i}C^2_{A|B_{i+1}\cdots B_{N-1}}$, $C^2_{A|B_i\cdots B_{N-1}}\geq C^2_{AB_i}+\mu_i C^2_{A|B_{i+1}\cdots B_{N-1}}$, $1\leq p_i\leq 1+\frac{\mu_i C^2_{A|B_{i+1}\cdots B_{N-1}}}{C^2_{AB_i}}$ for $i=1,2,\ldots,N-2$, we have

$$\begin{aligned}&C^{\alpha}_{A|B_1\cdots B_{N-1}}\\ &\geq p_1^{\frac{\alpha}{2}-1}C^{\alpha}_{AB_1}+\sum_{i=2}^{N-2}\prod_{h=1}^{i-1}\Gamma_h p_i^{\frac{\alpha}{2}-1}C^{\alpha}_{AB_i}\\ &+\Gamma_1\cdots\Gamma_{N-2}C^{\alpha}_{AB_{N-1}},\end{aligned} \quad (18)$$

for all $\alpha\geq 2$, where $\Gamma_r=(\mu_r+l_r^{\delta_r})^{\frac{\alpha}{2}}-p_r^{\frac{\alpha}{2}-1}k_r^{\frac{\delta_r\alpha}{2}}$ with $1\leq r\leq N-2$.

Remark 1 We have provided a class of lower bounds of the α-th power of concurrence with three different parameters p, δ and μ, where $1\leq p\leq 1+\frac{\mu C^2_{AC}}{C^2_{AB}}$, $\mu\geq 1$ and $\delta\geq 1$. When $p=1$, $\delta=1$ and $C^2_{AB}\geq lC^2_{AC}$, for all $\alpha\geq 2$, we have

$$C^{\alpha}_{A|BC}\geq C^{\alpha}_{AB}+((\mu+l)^{\frac{\alpha}{2}}-l^{\frac{\alpha}{2}})C^{\alpha}_{AC}. \quad (19)$$

Using concurrence as the quantum entanglement measure, it is obvious that inequality (19) in Ref. [43] is just a special case of our Lemma 2. Moreover, the lower bound of the α-th power of concurrence becomes tighter when p (or δ) increases as

$$\begin{aligned}C^{\alpha}_{A|BC}&\geq p^{\frac{\alpha}{2}-1}C^{\alpha}_{AB}+((\mu+l^{\delta})^{\frac{\alpha}{2}}-p^{\frac{\alpha}{2}-1}l^{\frac{\delta\alpha}{2}})C^{\alpha}_{AC}\\ &\geq C^{\alpha}_{AB}+((\mu+l^{\delta})^{\frac{\alpha}{2}}-l^{\frac{\delta\alpha}{2}})C^{\alpha}_{AC}\\ &\geq C^{\alpha}_{AB}+((\mu+l)^{\frac{\alpha}{2}}-l^{\frac{\alpha}{2}})C^{\alpha}_{AC}\end{aligned} \quad (20)$$

for all $\alpha\geq 2$, where the second and third equalities hold when $p=1$ and $\delta=1$, respectively. Then our results are seen to be tighter than that in Ref. [43]. Therefore, our new monogamy relation for concurrence is better than those in Refs. [38–40,42,43].

Example 1 Let us consider the three-qubit state $|\phi\rangle_{ABC}$ in the generalized Schmidt decomposition,[48,49]

$$|\phi\rangle_{ABC}=\lambda_0|000\rangle+\lambda_1 e^{i\varphi}|100\rangle+\lambda_2|101\rangle+\lambda_3|110\rangle+\lambda_4|111\rangle, \quad (21)$$

where $\lambda_i\geq 0$, $i=0,1,\ldots,4$, and $\sum_{i=0}^{4}\lambda_i^2=1$. One gets $C_{A|BC}=2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2}$, $C_{AB}=2\lambda_0\lambda_2$ and $C_{AC}=2\lambda_0\lambda_3$. Setting $\lambda_0=\lambda_3=\lambda_4=1/\sqrt{5}$, $\lambda_2=\sqrt{2/5}$ and $\lambda_1=0$, we have $C_{A|BC}=4/5$, $C_{AB}=2\sqrt{2}/5$ and $C_{AC}=2/5$. Therefore, with respect to inequality (13) we have

$$\begin{aligned}&p^{\frac{\alpha}{2}-1}C^{\alpha}_{AB}+((\mu+l^{\delta})^{\frac{\alpha}{2}}-p^{\frac{\alpha}{2}-1}l^{\frac{\delta\alpha}{2}})C^{\alpha}_{AC}\\ &=p^{\frac{\alpha}{2}-1}(2\sqrt{2}/5)^{\alpha}+((\mu+l^{\delta})^{\frac{\alpha}{2}}-p^{\frac{\alpha}{2}-1}l^{\frac{\delta\alpha}{2}})(2/5)^{\alpha},\end{aligned} \quad (22)$$

$$C^{\alpha}_{AB}+(2^{\frac{\alpha}{2}}-1)C^{\alpha}_{AC}=(2\sqrt{2}/5)^{\alpha}+(2^{\frac{\alpha}{2}}-1)(2/5)^{\alpha}, \quad (23)$$

$$\begin{aligned}&C^{\alpha}_{AB}+(((1+k)^{\frac{\alpha}{2}}-1)/k^{\frac{\alpha}{2}})C^{\alpha}_{AC}\\ &=(2\sqrt{2}/5)^{\alpha}+((1+k)^{\frac{\alpha}{2}}-1)/k^{\frac{\alpha}{2}})(2/5)^{\alpha},\end{aligned} \quad (24)$$

$$\begin{aligned}&C^{\alpha}_{AB}+((\mu+l)^{\frac{\alpha}{2}}-l^{\frac{\alpha}{2}})C^{\alpha}_{AC}\\ &=(2\sqrt{2}/5)^{\alpha}+((\mu+l)^{\frac{\alpha}{2}}-l^{\frac{\alpha}{2}})(2/5)^{\alpha}.\end{aligned} \quad (25)$$

When $l=1/k$, $\mu=1$, the lower bound Eq. (25) gives the same result as Eq. (24). Let $k=2/3$, $l=3/2$, $\mu=2$, $\delta=1$ and $p=2$. It can be seen that our result Eq. (22) is better than the result Eq. (25) in Ref. [43] for $\alpha\geq 2$, hence also better than Eq. (24) given in Ref. [42] and Eq. (23) given in Ref. [40], see Fig. 1.

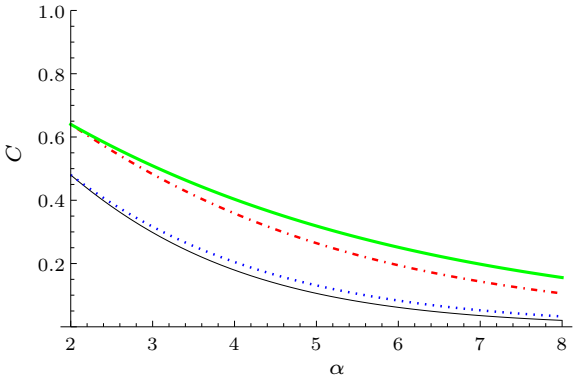


Fig. 1. From top to bottom, the green thick (black thin, blue dotted and red dot-dashed) line represents the lower bound taken from our previous results (Refs. [40,42,43], respectively).

For a tripartite state $|\phi\rangle_{ABC}$, the concurrence of assistance (CoA) is defined by[50,51]

$$\mathcal{C}_a(\rho_{AB})=\max_{\{p_i,|\phi_i\rangle\}}\sum_i p_i\mathcal{C}(|\phi_i\rangle), \quad (26)$$

where the maximum is taken over all possible pure state decompositions of ρ_{AB}. For pure states $|\phi\rangle_{AB}$ one has $\mathcal{C}(|\phi\rangle_{AB})=\mathcal{C}_a(|\phi\rangle_{AB})$. The generalized polygamy relation based on the concurrence of assistance was established in Refs. [33,52],

$$\begin{aligned}\mathcal{C}^2(|\phi\rangle_{A|B_1\cdots B_{N-1}})&=\mathcal{C}_a^2(|\phi\rangle_{A|B_1\cdots B_{N-1}})\\ &\leq\mathcal{C}_a^2(\rho_{AB_1})+\mathcal{C}_a^2(\rho_{AB_2})\\ &+\cdots+\mathcal{C}_a^2(\rho_{AB_{N-1}}).\end{aligned} \quad (27)$$

For convenience, we denote by $C_{aAB_j}=C_a(\rho_{AB_j})$ for $j=1,2,\ldots,N-1$, and $C_{aA|B_1B_2\cdots B_{N-1}}=C_a(\rho_{A|B_1B_2\cdots B_{N-1}})$. Similar to the proof of Theorem 1, by using inequality (11) we

have the following tighter polygamy relations for the concurrence of assistance.

Theorem 3 Let $\delta_r \geq 1$, $0 \leq \mu_r \leq 1$ and $l_r \geq 1$ be real numbers, $1 \leq r \leq N-2$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $C^2_{aAB_i} \geq l_i^{\delta_i} C^2_{aA|B_{i+1}\cdots B_{N-1}}$, $C^2_{aA|B_i\cdots B_{N-1}} \leq C^2_{aAB_i} + \mu_i C^2_{aA|B_{i+1}\cdots B_{N-1}}$, $1 \leq p_i \leq 1 + \frac{\mu_i C^2_{aA|B_{i+1}\cdots B_{N-1}}}{C^2_{aAB_i}}$ for $i = 1,2,\ldots,m$, and $C^2_{aA|B_{j+1}\cdots B_{N-1}} \geq l_j^{\delta_j} C^2_{aAB_j}$, $C^2_{aA|B_j\cdots B_{N-1}} \leq \mu_j C^2_{aAB_j} + C^2_{aA|B_{j+1}\cdots B_{N-1}}$, $1 \leq p_j \leq \frac{\mu_j C^2_{aAB_j}}{C^2_{aA|B_{j+1}\cdots B_{N-1}}}$ for $j = m+1,\ldots,N-2$, $1 \leq m \leq N-3, N \geq 4$, we have

$$\begin{aligned} & C^{\alpha}_{aA|B_1\cdots B_{N-1}} \\ \leq\ & p_1^{\frac{\alpha}{2}-1} C^{\alpha}_{aAB_1} + \sum_{i=2}^{m}\prod_{h=1}^{i-1} \Gamma_h p_i^{\frac{\alpha}{2}-1} C^{\alpha}_{aAB_i} \\ & + \Gamma_1\cdots\Gamma_m \Big(\Gamma_{m+1} C^{\alpha}_{aAB_{m+1}} + \sum_{j=m+2}^{N-2}\prod_{h=m+1}^{j-1} p_h^{\frac{\alpha}{2}-1} \Gamma_j C^{\alpha}_{aAB_j} \Big) \\ & + \Gamma_1\cdots\Gamma_m (p_{m+1}\cdots p_{N-2})^{\frac{\alpha}{2}-1} C^{\alpha}_{aAB_{N-1}} \end{aligned} \tag{28}$$

for all $0 \leq \alpha \leq 2$, where $\Gamma_r = (\mu_r + l_r^{\delta_r})^{\frac{\alpha}{2}} - p_r^{\frac{\alpha}{2}-1} l_r^{\frac{\delta_r \alpha}{2}}$ with $1 \leq r \leq N-2$.

In particular, if $C^2_{aAB_i} \geq l_i^{\delta_i} C^2_{aA|B_{i+1}\cdots B_{N-1}}$, $C^2_{aA|B_i\cdots B_{N-1}} \geq C^2_{aAB_i} + \mu_i C^2_{aA|B_{i+1}\cdots B_{N-1}}$, $1 \leq p_i \leq 1 + \frac{\mu_i C^2_{aA|B_{i+1}\cdots B_{N-1}}}{C^2_{aAB_i}}$ for $i = 1,2,\ldots,N-2$, we have the following polygamy relations.

Theorem 4 Let $\delta_r \geq 1$, $0 \leq \mu_r \leq 1$ and $l_r \geq 1$ be real numbers, $1 \leq r \leq N-2$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $C^2_{aAB_i} \geq l_i^{\delta_i} C^2_{aA|B_{i+1}\cdots B_{N-1}}$, $C^2_{aA|B_i\cdots B_{N-1}} \leq C^2_{aAB_i} + \mu_i C^2_{aA|B_{i+1}\cdots B_{N-1}}$, $1 \leq p_i \leq 1 + \frac{\mu_i C^2_{aA|B_{i+1}\cdots B_{N-1}}}{C^2_{aAB_i}}$ for $i = 1,2,\ldots,N-2$, we have

$$\begin{aligned} C^{\alpha}_{aA|B_1\cdots B_{N-1}} \leq\ & p_1^{\frac{\alpha}{2}-1} C^{\alpha}_{aAB_1} + \sum_{i=2}^{N-2}\prod_{h=1}^{i-1} \Gamma_h p_i^{\frac{\alpha}{2}-1} C^{\alpha}_{aAB_i} \\ & + \Gamma_1\cdots\Gamma_{N-2} C^{\alpha}_{aAB_{N-1}}, \end{aligned} \tag{29}$$

for all $0 \leq \alpha \leq 2$, where $\Gamma_r = (\mu_r + l_r^{\delta_r})^{\frac{\alpha}{2}} - p_r^{\frac{\alpha}{2}-1} l_r^{\frac{\delta_r \alpha}{2}}$ with $1 \leq r \leq N-2$.

Remark 2 We have provided a class of upper bounds of the α-th power of CoA with three different parameters p, μ and δ, where $1 \leq p \leq 1 + \frac{\mu C^2_{aAC}}{C^2_{aAB}}$, $0 < \mu \leq 1$ and $\delta \geq 1$. The upper bounds of the α-th power of CoA get tighter when p (or δ) decreases, or μ increases, namely,

$$\begin{aligned} C^{\alpha}_{aA|BC} &\leq p^{\frac{\alpha}{2}-1} C^{\alpha}_{aAB} + \Big((\mu + l^{\delta})^{\frac{\alpha}{2}} - p^{\frac{\alpha}{2}-1} l^{\frac{\delta\alpha}{2}} \Big) C^{\alpha}_{aAC} \\ &\leq C^{\alpha}_{aAB} + \Big((\mu + l^{\delta})^{\frac{\alpha}{2}} - l^{\frac{\delta\alpha}{2}} \Big) C^{\alpha}_{aAC} \\ &\leq C^{\alpha}_{aAB} + \Big((1 + l^{\delta})^{\frac{\alpha}{2}} - l^{\frac{\delta\alpha}{2}} \Big) C^{\alpha}_{aAC} \\ &\leq C^{\alpha}_{aAB} + \Big((1 + l)^{\frac{\alpha}{2}} - l^{\frac{\alpha}{2}} \Big) C^{\alpha}_{aAC} \\ &\leq C^{\alpha}_{aAB} + (2^{\frac{\alpha}{2}} - 1) C^{\alpha}_{aAC} \end{aligned} \tag{30}$$

for all $0 \leq \alpha \leq 2$, where the second, third, fourth and fifth equalities hold when $p = 1$, $\mu = 1$, $\delta = 1$ and $l = 1$, respectively. From inequality (30), we get $(1+l)^{\frac{\alpha}{2}} - l^{\frac{\alpha}{2}} = \frac{(1+k)^{\frac{\alpha}{2}} - 1}{k^{\frac{\alpha}{2}}}$ if $l = \frac{1}{k}$ with $0 \leq k \leq 1$, as well as $\frac{(1+k)^{\frac{\alpha}{2}} - 1}{k^{\frac{\alpha}{2}}} \leq 2^{\frac{\alpha}{2}} - 1$ for $0 \leq \alpha \leq 2$. Obviously, the new polygamy inequality of CoA is better than the inequality in Ref. [42]. Therefore, our inequality is also tighter than the inequalities in Refs. [38–40].

Example 2 Let us consider the three-qubit state $|\phi\rangle_{ABC}$ in Eq. (21) again. We have $C_{aA|BC} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$, $C_{aAB} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_4^2}$ and $C_{aAC} = 2\lambda_0\sqrt{\lambda_3^2 + \lambda_4^2}$. Setting $\lambda_0 = \lambda_3 = 1/2$, $\lambda_2 = \sqrt{2}/2$ and $\lambda_1 = \lambda_4 = 0$, we have $C_{aA|BC} = \sqrt{3}/2$, $C_{aAB} = \sqrt{2}/2$ and $C_{aAC} = 1/2$. Therefore, with respect to inequality (29), when $N = 3$ we have

$$\begin{aligned} & p^{\frac{\alpha}{2}-1} C^{\alpha}_{aAB} + \Big((\mu + l^{\delta})^{\frac{\alpha}{2}} - p^{\frac{\alpha}{2}-1} l^{\frac{\delta\alpha}{2}} \Big) C^{\alpha}_{aAC} \\ =\ & p^{\frac{\alpha}{2}-1} \Big(\frac{\sqrt{2}}{2} \Big)^{\alpha} + \Big((\mu + l^{\delta})^{\frac{\alpha}{2}} - p^{\frac{\alpha}{2}-1} l^{\frac{\delta\alpha}{2}} \Big) \Big(\frac{1}{2} \Big)^{\alpha}, \end{aligned} \tag{31}$$

$$C^{\alpha}_{aAB} + \Big(2^{\frac{\alpha}{2}} - 1 \Big) C^{\alpha}_{aAC} = \Big(\frac{\sqrt{2}}{2} \Big)^{\alpha} + (2^{\frac{\alpha}{2}} - 1) \Big(\frac{1}{2} \Big)^{\alpha}, \tag{32}$$

$$\begin{aligned} & C^{\alpha}_{aAB} + \Big(\big((1+k)^{\frac{\alpha}{2}} - 1 \big) / k^{\frac{\alpha}{2}} \Big) C^{\alpha}_{aAC} \\ =\ & \Big(\frac{\sqrt{2}}{2} \Big)^{\alpha} + \big((1+k)^{\frac{\alpha}{2}} - 1 \big) / k^{\frac{\alpha}{2}} \Big(\frac{1}{2} \Big)^{\alpha}. \end{aligned} \tag{33}$$

Let $k = 4/7$, $l = 7/4$, $\mu = 3/4$, $\delta = 1$, $p = 11/8$. It can be seen that our result Eq. (31) is better than the result Eq. (33) in Ref. [42] for $0 < \alpha \leq 2$, thus also better than Eq. (32) given in Ref. [40], see Fig. 2.

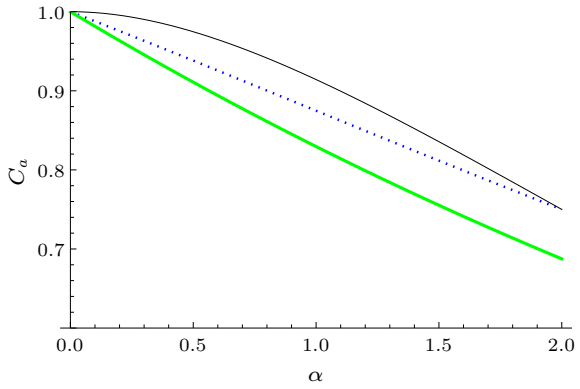


Fig. 2. From bottom to top, the green thick (black thin and blue dotted) line represents the upper bound from our result (Refs. [40,42], respectively).

3. Tighter monogamy and polygamy relations for EoF

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be m- and n-dimensional $(m \leq n)$ vector spaces, respectively. The entanglement of formation of a pure state $|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is defined by $E(|\phi\rangle) = S(\rho_A)$, where

$\rho_A = \mathrm{Tr}_B(|\phi\rangle_{AB}\langle\phi|)$ and $S(\rho) = -\mathrm{Tr}(\rho\log_2\rho)$.[53,54] For a bipartite mixed state $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, the EoF is given by

$$E(\rho_{AB}) = \min_{\{p_i,|\phi_i\rangle\}} \sum_i p_i E(|\phi_i\rangle), \tag{34}$$

with the minimum taking over all possible pure state decomposition of ρ_{AB}.

Denote $f(x) = H(\frac{1+\sqrt{1-x}}{2})$, where $H(x) = -x\log_2(x) - (1-x)\log_2(1-x)$. It is obvious that $f(x)$ is a monotonically increasing function for $0 \le x \le 1$ which satisfies

$$f^{\sqrt{2}}(x^2+y^2) \ge f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2), \tag{35}$$

$$f(x^2+y^2) \le f(x^2) + f(y^2), \tag{36}$$

where $f^{\sqrt{2}}(x^2+y^2) = [f(x^2+y^2)]^{\sqrt{2}}$. It is shown in Ref. [47] that $E(|\varphi\rangle) = f(C^2(|\varphi\rangle))$ for any $2\otimes m$ ($m \ge 2$) pure state $|\varphi\rangle$, and $E(\rho) = f(C^2(\rho))$ for any two-qubit mixed state ρ.

EoF does not satisfy the inequality $E(\rho_{A|BC}) \ge E(\rho_{AB}) + E(\rho_{AC})$.[17] In Ref. [19] it is shown that EoF is a monotonic function: $E^2(C^2(\rho_{A|B_1B_2\cdots B_{N-1}})) \ge E^2(\sum_{i=1}^{N-1} C^2(\rho_{AB_i}))$. It is further proved that for N-qubit systems,

$$\begin{aligned} E^\alpha(\rho_{A|B_1B_2\cdots B_{N-1}}) \ge\ & E^\alpha(\rho_{AB_1}) + E^\alpha(\rho_{AB_2}) \\ & + \cdots + E^\alpha(\rho_{AB_{N-1}}) \end{aligned} \tag{37}$$

for all $\alpha \ge \sqrt{2}$, where $E(\rho_{A|B_1B_2\cdots B_{N-1}})$ is the EoF of ρ under bipartition $A|B_1B_2\cdots B_{N-1}$, and $E(\rho_{AB_i})$, $i = 1,2,\ldots,N-1$, is the EoF of the bipartite state $\rho_{AB_i} = \mathrm{Tr}_{B_1B_2\cdots B_{i-1}B_{i+1}\cdots B_{N-1}}(\rho)$.[38]

The relation (37) is improved to be

$$\begin{aligned} E^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \ge\ & E^\alpha(\rho_{AB_1}) + \frac{\alpha}{\sqrt{2}}E^\alpha(\rho_{AB_2}) \\ & + \cdots + \Big(\frac{\alpha}{\sqrt{2}}\Big)^{m-1} E^\alpha(\rho_{AB_m}) \\ & + \Big(\frac{\alpha}{\sqrt{2}}\Big)^{m+1} (E^\alpha(\rho_{AB_{m+1}}) \\ & + \cdots + E^\alpha(\rho_{AB_{N-2}})) \\ & + \Big(\frac{\alpha}{\sqrt{2}}\Big)^{m} E^\alpha(\rho_{AB_{N-1}}) \end{aligned} \tag{38}$$

in Ref. [39], and then to be

$$\begin{aligned} E^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \ge\ & E^\alpha(\rho_{AB_1}) + \Big(2^{\frac{\alpha}{\sqrt{2}}} - 1\Big)E^\alpha(\rho_{AB_2}) \\ & + \cdots + \Big(2^{\frac{\alpha}{\sqrt{2}}} - 1\Big)^{m-1} E^\alpha(\rho_{AB_m}) \\ & + \Big(2^{\frac{\alpha}{\sqrt{2}}} - 1\Big)^{m+1} (E^\alpha(\rho_{AB_{m+1}}) \\ & + \cdots + E^\alpha(\rho_{AB_{N-2}})) \\ & + \Big(2^{\frac{\alpha}{\sqrt{2}}} - 1\Big)^{m} E^\alpha(\rho_{AB_{N-1}}) \end{aligned} \tag{39}$$

in Ref. [40] for $\alpha \ge \sqrt{2}$, with the conditions $E(\rho_{AB_i}) \ge E(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i = 1,2,\ldots,m$, and $E(\rho_{AB_j}) \le E(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m+1,\ldots,N-2$, $1 \le m \le N-3$ and $N \ge 4$.

The relation (39) is further improved for all $\alpha \ge \sqrt{2}$ as

$$\begin{aligned} & E^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \\ \ge\ & E^\alpha(\rho_{AB_1}) + \Big(\Big((1+k)^{\frac{\alpha}{\sqrt{2}}} - 1\Big)/k^{\frac{\alpha}{\sqrt{2}}}\Big)E^\alpha(\rho_{AB_2}) \\ & + \cdots + \Big(\Big((1+k)^{\frac{\alpha}{\sqrt{2}}} - 1\Big)/k^{\frac{\alpha}{\sqrt{2}}}\Big)^{m-1} E^\alpha(\rho_{AB_m}) \\ & + \Big(\Big((1+k)^{\frac{\alpha}{\sqrt{2}}} - 1\Big)/k^{\frac{\alpha}{\sqrt{2}}}\Big)^{m+1} (E^\alpha(\rho_{AB_{m+1}}) \\ & + \cdots + E^\alpha(\rho_{AB_{N-2}})) \\ & + \Big(\Big((1+k)^{\frac{\alpha}{\sqrt{2}}} - 1\Big)/k^{\frac{\alpha}{\sqrt{2}}}\Big)^{m} E^\alpha(\rho_{AB_{N-1}}), \end{aligned} \tag{40}$$

with $kE^{\sqrt{2}}(\rho_{AB_i}) \ge E^{\sqrt{2}}(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i = 1,2,\ldots,m$, and $E^{\sqrt{2}}(\rho_{AB_j}) \le kE^{\sqrt{2}}(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m+1,\ldots,N-2$, $1 \le m \le N-3$, $N \ge 4$ and $0 < k \le 1$ in Ref. [42]. Then inequality (40) is further improved as

$$\begin{aligned} & E^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \\ \ge\ & E^\alpha(\rho_{AB_1}) + \mathcal{K}_1E^\alpha(\rho_{AB_2}) + \cdots + \mathcal{K}_1\cdots\mathcal{K}_{m-1}E^\alpha(\rho_{AB_m}) \\ & + \mathcal{K}_1\cdots\mathcal{K}_m(\mathcal{K}_{m+1}E^\alpha(\rho_{AB_{m+1}}) + \cdots + \mathcal{K}_{N-2}E^\alpha(\rho_{AB_{N-2}})) \\ & + \mathcal{K}_1\cdots\mathcal{K}_mE^\alpha(\rho_{AB_{N-1}}) \end{aligned} \tag{41}$$

for all $\alpha \ge \sqrt{2}$, with $E^{\sqrt{2}}(\rho_{AB_i}) \ge l_iE^{\sqrt{2}}(\rho_{A|B_{i+1}\cdots B_{N-1}})$, $E^{\sqrt{2}}(\rho_{A|B_i\cdots B_{N-1}}) \ge E^{\sqrt{2}}(\rho_{AB_i}) + \mu_iE^{\sqrt{2}}(\rho_{A|B_{i+1}\cdots B_{N-1}})$ for $i = 1,2,\ldots,m$, $E^{\sqrt{2}}(\rho_{A|B_{j+1}\cdots B_{N-1}}) \ge l_jE^{\sqrt{2}}(\rho_{AB_j})$, $E^{\sqrt{2}}(\rho_{A|B_j\cdots B_{N-1}}) \ge \mu_jE^{\sqrt{2}}(\rho_{AB_j}) + E^{\sqrt{2}}(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j = m+1,\ldots,N-2$, $1 \le m \le N-3, N \ge 4$, where $\mathcal{K}_r = (\mu_r + l_r)^{\frac{\alpha}{\sqrt{2}}} - l_r^{\frac{\alpha}{\sqrt{2}}}$ with $1 \le r \le N-2$, $\mu_r \ge 1$ and $l_r \ge 1$ in Ref. [43].

In the following, we show that these monogamy relations for EoF can become even tighter under some conditions. For convenience, we denote $E_{AB_j} = E(\rho_{AB_j})$ for $j = 1,2,\ldots,N-1$, and $E_{A|B_1B_2\cdots B_{N-1}} = E(\rho_{A|B_1B_2\cdots B_{N-1}})$.

Theorem 5 Let $\delta_r \ge 1$, $\mu_r \ge 1$ and $l_r \ge 1$ be real numbers, $1 \le r \le N-2$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$, if $E^{\sqrt{2}}_{AB_i} \ge l_i^{\delta_i}E^{\sqrt{2}}_{A|B_{i+1}\cdots B_{N-1}}$, $E^{\sqrt{2}}_{A|B_i\cdots B_{N-1}} \ge E^{\sqrt{2}}_{AB_i} + \mu_iE^{\sqrt{2}}_{A|B_{i+1}\cdots B_{N-1}}$, $1 \le p_i \le 1 + \frac{\mu_iE^{\sqrt{2}}_{A|B_{i+1}\cdots B_{N-1}}}{E^{\sqrt{2}}_{AB_i}}$ for $i = 1,2,\ldots,m$, and $E^{\sqrt{2}}_{A|B_{j+1}\cdots B_{N-1}} \ge l_j^{\delta_j}E^{\sqrt{2}}_{AB_j}$, $E^{\sqrt{2}}_{A|B_j\cdots B_{N-1}} \ge \mu_jE^{\sqrt{2}}_{AB_j} + E^{\sqrt{2}}_{A|B_{j+1}\cdots B_{N-1}}$, $1 \le p_j \le \frac{\mu_jE^{\sqrt{2}}_{AB_j}}{E^{\sqrt{2}}_{A|B_{j+1}\cdots B_{N-1}}}$ for $j = m+1,\ldots,N-2$, $1 \le m \le N-3, N \ge 4$, then

$$\begin{aligned} & E^\alpha_{A|B_1\cdots B_{N-1}} \\ \ge\ & p_1^{\frac{\alpha}{\sqrt{2}}-1}E^\alpha_{AB_1} + \sum_{i=2}^{m}\prod_{h=1}^{i-1}\Gamma_h p_i^{\frac{\alpha}{\sqrt{2}}-1}E^\alpha_{AB_i} \\ & + \Gamma_1\cdots\Gamma_m\Big(\Gamma_{m+1}E^\alpha_{AB_{m+1}} + \sum_{j=m+2}^{N-2}\prod_{h=m+1}^{j-1} p_h^{\frac{\alpha}{\sqrt{2}}-1}\Gamma_jE^\alpha_{AB_j}\Big) \end{aligned}$$

$$+\Gamma_1\cdots\Gamma_m(p_{m+1}\cdots p_{N-2})^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{AB_{N-1}}, \tag{42}$$

for all $\alpha\geq\sqrt{2}$, where $\Gamma_r=(\mu_r+l_r^{\delta_r})^{\frac{\alpha}{\sqrt{2}}}-p_r^{\frac{\alpha}{\sqrt{2}}-1}l_r^{\frac{\delta_r\alpha}{\sqrt{2}}}$ with $1\leq r\leq N-2$.

Proof Since $f^{\sqrt{2}}(x^2)\geq l^{\delta}f^{\sqrt{2}}(y^2)$, $1\leq p\leq 1+\frac{\mu f^{\sqrt{2}}(y^2)}{f^{\sqrt{2}}(x^2)}$, due to inequality (35) there exists $\mu\geq 1$ such that $f^{\sqrt{2}}(x^2+y^2)\geq f^{\sqrt{2}}(x^2)+\mu f^{\sqrt{2}}(y^2)$. Hence for $\alpha\geq\sqrt{2}$ we have

$$\begin{aligned}
&f^{\alpha}(x^2+y^2)\\
&=[f^{\sqrt{2}}(x^2+y^2)]^{\frac{\alpha}{\sqrt{2}}}\geq[f^{\sqrt{2}}(x^2)+\mu f^{\sqrt{2}}(y^2)]^{\frac{\alpha}{\sqrt{2}}}\\
&=\mu^{\frac{\alpha}{\sqrt{2}}}f^{\alpha}(y^2)[(1+\mu^{-1}(f^{\sqrt{2}}(x^2)/f^{\sqrt{2}}(y^2)))^{\frac{\alpha}{\sqrt{2}}}\\
&\quad-p^{\frac{\alpha}{\sqrt{2}}-1}(\mu^{-1}(f^{\sqrt{2}}(x^2)/f^{\sqrt{2}}(y^2)))^{\frac{\alpha}{\sqrt{2}}}]+p^{\frac{\alpha}{\sqrt{2}}-1}f^{\alpha}(x^2)\\
&\geq\mu^{\frac{\alpha}{\sqrt{2}}}f^{\alpha}(y^2)[(\mu^{-1}l+l^{\delta})^{\frac{\alpha}{\sqrt{2}}}-p^{\frac{\alpha}{\sqrt{2}}-1}(\mu^{-1}l^{\delta})^{\frac{\alpha}{\sqrt{2}}}]\\
&\quad+p^{\frac{\alpha}{\sqrt{2}}-1}f^{\alpha}(x^2)\\
&=[(\mu+l^{\delta})^{\frac{\alpha}{\sqrt{2}}}-p^{\frac{\alpha}{\sqrt{2}}-1}l^{\frac{\alpha\delta}{\sqrt{2}}}]f^{\alpha}(y^2)+p^{\frac{\alpha}{\sqrt{2}}-1}f^{\alpha}(x^2),
\end{aligned}\tag{43}$$

where the second inequality is obtained from inequality (10). Let $\rho_{A|B_1\cdots B_{N-1}}=\sum_i p_i|\phi_i\rangle_{A|B_1\cdots B_{N-1}}\langle\phi_i|\in\mathcal{H}_A\otimes\mathcal{H}_{B_1}\otimes\cdots\otimes\mathcal{H}_{B_{N-1}}$ be the optimal decomposition such that $E(\rho_{A|B_1\cdots B_{N-1}})=\sum_i p_iE(|\phi_i\rangle_{A|B_1\cdots B_{N-1}})$ for the N-qubit mixed state ρ. Then we have[40]

$$\begin{aligned}
E(\rho_{A|B_1\cdots B_{N-1}})&=\sum_i p_iE(|\phi_i\rangle_{A|B_1\cdots B_{N-1}})\\
&=\sum_i p_if\Big(C^2(|\phi_i\rangle_{A|B_1\cdots B_{N-1}})\Big)\\
&\geq f\Big(\sum_i p_iC^2(|\phi_i\rangle_{A|B_1\cdots B_{N-1}})\Big)\\
&=f\Big(\sum_i(\sqrt{p_i}^2)\sum_i(\sqrt{p_i}C(|\phi_i\rangle_{A|B_1\cdots B_{N-1}}))^2\Big)\\
&\geq f\Big(\Big[\sum_i p_iC(|\phi_i\rangle_{A|B_1\cdots B_{N-1}})\Big]^2\Big)\\
&\geq f(C^2(\rho_{A|B_1\cdots B_{N-1}})),
\end{aligned}\tag{44}$$

where the first inequality is due to $f(x)$ that is a convex function. The second inequality is due to the Cauchy-Schwarz inequality: $(\sum_i x_i^2)^{\frac{1}{2}}(\sum_i y_i^2)^{\frac{1}{2}}\geq\sum_i x_iy_i$, with $x_i=\sqrt{p_i}$ and $y_i=\sqrt{p_i}C(|\phi_i\rangle_{A|B_1\cdots B_{N-1}})$. Due to the definition of concurrence and because $f(x)$ is a monotonically increasing function, we obtain the third inequality. Therefore,

$$\begin{aligned}
E^{\alpha}_{A|B_1\cdots B_{N-1}}&\geq f^{\alpha}(C^2_{A|B_1\cdots B_{N-1}})\\
&\geq p_1^{\frac{\alpha}{\sqrt{2}}-1}f^{\alpha}(C^2_{AB_1})+\sum_{i=2}^{m}\prod_{h=1}^{i-1}\Gamma_hp_i^{\frac{\alpha}{\sqrt{2}}-1}f^{\alpha}(C^2_{AB_i})\\
&\quad+\Gamma_1\cdots\Gamma_m\Big(\Gamma_{m+1}f^{\alpha}(C^2_{AB_{m+1}})\\
&\quad+\sum_{j=m+2}^{N-2}\prod_{h=m+1}^{j-1}p_h^{\frac{\alpha}{\sqrt{2}}-1}\Gamma_jf^{\alpha}(C^2_{AB_j})\Big)\\
&\quad+\Gamma_1\cdots\Gamma_m(p_{m+1}\cdots p_{N-2})^{\frac{\alpha}{\sqrt{2}}-1}f^{\alpha}(C^2_{AB_{N-1}})\\
&=p_1^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{AB_1}+\sum_{i=2}^{m}\prod_{h=1}^{i-1}\Gamma_hp_i^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{AB_i}\\
&\quad+\Gamma_1\cdots\Gamma_m\Big(\Gamma_{m+1}E^{\alpha}_{AB_{m+1}}\\
&\quad+\sum_{j=m+2}^{N-2}\prod_{h=m+1}^{j-1}p_h^{\frac{\alpha}{\sqrt{2}}-1}\Gamma_jE^{\alpha}_{AB_j}\Big)\\
&\quad+\Gamma_1\cdots\Gamma_m(p_{m+1}\cdots p_{N-2})^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{AB_{N-1}},
\end{aligned}\tag{45}$$

where the first inequality is due to inequality (44), the second inequality is obtained similarly to the proof of Theorem 1 by using inequality (43). The last equality holds since for any $2\otimes 2$ quantum state ρ_{AB_i}, $E(\rho_{AB_i})=f[C^2(\rho_{AB_i})]$.

In particular, similar to the case of concurrence, we also have the following tighter monogamy relation for EoF.

Theorem 6 Let $\delta_r\geq 1$, $\mu_r\geq 1$ and $l_r\geq 1$ be real numbers, $1\leq r\leq N-2$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}\in\mathcal{H}_A\otimes\mathcal{H}_{B_1}\otimes\cdots\otimes\mathcal{H}_{B_{N-1}}$, if $E^{\sqrt{2}}_{AB_i}\geq l_i^{\delta_i}E^{\sqrt{2}}_{A|B_{i+1}\cdots B_{N-1}}$, $E^{\sqrt{2}}_{A|B_i\cdots B_{N-1}}\geq E^{\sqrt{2}}_{AB_i}+\mu_iE^{\sqrt{2}}_{A|B_{i+1}\cdots B_{N-1}}$, $1\leq p_i\leq 1+\frac{\mu_iE^{\sqrt{2}}_{A|B_{i+1}\cdots B_{N-1}}}{E^{\sqrt{2}}_{AB_i}}$ for $i=1,2,\ldots,N-2$, we have

$$\begin{aligned}
E^{\alpha}_{A|B_1\cdots B_{N-1}}&\geq p_1^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{AB_1}+\sum_{i=2}^{N-2}\prod_{h=1}^{i-1}\Gamma_hp_i^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{AB_i}\\
&\quad+\Gamma_1\cdots\Gamma_{N-2}E^{\alpha}_{AB_{N-1}}
\end{aligned}\tag{46}$$

for all $\alpha\geq\sqrt{2}$, where $\Gamma_r=(\mu_r+l_r^{\delta_r})^{\frac{\alpha}{\sqrt{2}}}-p_r^{\frac{\alpha}{\sqrt{2}}-1}l_r^{\frac{\delta_r\alpha}{\sqrt{2}}}$ with $1\leq r\leq N-2$.

Remark 3 We have presented a class of lower bounds of the α-th power of EoF with three different parameters p, δ and μ, where $1\leq p\leq 1+\frac{\mu E^{\sqrt{2}}_{AC}}{E^{\sqrt{2}}_{AB}}$, $\delta\geq 1$ and $\mu\geq 1$. The lower bound of the α-th power of EoF gets tighter when p (or δ) increases. Hence we have

$$\begin{aligned}
E^{\alpha}_{A|BC}&\geq p^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{AB}+\Big((\mu+l^{\delta})^{\frac{\alpha}{\sqrt{2}}}-p^{\frac{\alpha}{\sqrt{2}}-1}l^{\frac{\delta\alpha}{\sqrt{2}}}\Big)E^{\alpha}_{AC}\\
&\geq E^{\alpha}_{AB}+\Big((\mu+l^{\delta})^{\frac{\alpha}{\sqrt{2}}}-l^{\frac{\delta\alpha}{\sqrt{2}}}\Big)E^{\alpha}_{AC}\\
&\geq E^{\alpha}_{AB}+\Big((\mu+l)^{\frac{\alpha}{\sqrt{2}}}-l^{\frac{\alpha}{\sqrt{2}}}\Big)E^{\alpha}_{AC}
\end{aligned}$$

for all $\alpha\geq\sqrt{2}$, where the the second and third equalities hold when $p=1$ and $\delta=1$, respectively. Therefore, our inequalities are tighter than that in Ref. [43], and hence the ones in Refs. [38–40,42,43].

Example 3 Let us again consider the three-qubit state $|\phi\rangle_{ABC}$ defined in Example 1 with $\lambda_0=\lambda_3=\lambda_4=1/\sqrt{5}$, $\lambda_2=\sqrt{2/5}$ and $\lambda_1=0$. We have

$$\begin{aligned}
E_{A|BC}&=-(4/5)\log_2(4/5)-(1/5)\log_2(1/5)\approx 0.721928,\\
E_{AB}&=-((5+\sqrt{17})/10)\log_2((5+\sqrt{17})/10)\\
&\quad-((5-\sqrt{17})/10)\log_2((5-\sqrt{17})/10)\approx 0.428710,\\
E_{AC}&=-((5+\sqrt{21})/10)\log_2((5+\sqrt{21})/10)\\
&\quad-((5-\sqrt{21})/10)\log_2((5-\sqrt{21})/10)\approx 0.250225.
\end{aligned}$$

Therefore, with respect to inequality (46), when $N=3$ we have

$$\begin{aligned}
&p^{\frac{\alpha}{\sqrt{2}}-1}E_{AB}^{\alpha}+\left((\mu+l^{\delta})^{\frac{\alpha}{\sqrt{2}}}-p^{\frac{\alpha}{\sqrt{2}}-1}l^{\frac{\delta\alpha}{\sqrt{2}}}\right)E_{AC}^{\alpha}\\
&=p^{\frac{\alpha}{\sqrt{2}}-1}(0.428710)^{\alpha}\\
&\quad+((\mu+l^{\delta})^{\frac{\alpha}{\sqrt{2}}}-p^{\frac{\alpha}{\sqrt{2}}-1}l^{\frac{\delta\alpha}{\sqrt{2}}})(0.250225)^{\alpha}, \qquad (47)\\
&E_{AB}^{\alpha}+\left(2^{\frac{\alpha}{\sqrt{2}}}-1\right)E_{AC}^{\alpha}\\
&=(0.428710)^{\alpha}+\left(2^{\frac{\alpha}{\sqrt{2}}}-1\right)(0.250225)^{\alpha}, \qquad (48)\\
&E_{AB}^{\alpha}+\left(\frac{(1+k)^{\frac{\alpha}{\sqrt{2}}}-1}{k^{\frac{\alpha}{\sqrt{2}}}}\right)E_{AC}^{\alpha}\\
&=(0.428710)^{\alpha}+\left(\frac{(1+k)^{\frac{\alpha}{\sqrt{2}}}-1}{k^{\frac{\alpha}{\sqrt{2}}}}\right)(0.250225)^{\alpha}, \qquad (49)\\
&E_{AB}^{\alpha}+\left((\mu+l)^{\frac{\alpha}{\sqrt{2}}}-l^{\frac{\alpha}{\sqrt{2}}}\right)E_{AC}^{\alpha}\\
&=(0.428710)^{\alpha}+\left((\mu+l)^{\frac{\alpha}{\sqrt{2}}}-l^{\frac{\alpha}{\sqrt{2}}}\right)(0.250225)^{\alpha}. \qquad (50)
\end{aligned}$$

When $l=1/k$ and $\mu=1$, the lower bound Eq. (50) gives the same result as Eq. (49). Set $k=2/3$, $l=3/2$, $\mu=1.1$, $\delta=1$ and $p=1.8$. It can be seen that our result Eq. (47) is better than the result Eq. (50) in Ref. [43] for $\alpha\geq\sqrt{2}$, hence better than Eq. (49) given in Ref. [42] and Eq. (48) given in Ref. [40], see Fig. 3.

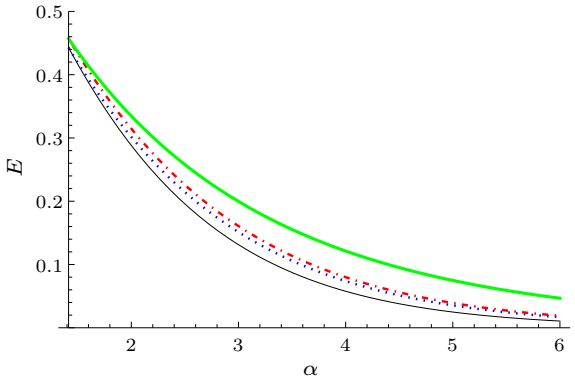


Fig. 3. From top to bottom, the green thick (black thin, blue dotted and red dotdashed) line represents the lower bound from our result (Refs. [40,42,43], respectively).

We can also provide tighter polygamy relations for the entanglement of assistance. The corresponding entanglement of assistance (EoA) of a bipartite mixed state ρ_{AB} is defined by

$$E_a(\rho_{AB})=\max_{\{p_i,|\phi_i\rangle\}}\sum_i p_i E(|\phi_i\rangle), \qquad (51)$$

where the maximum is taken over all possible pure state decompositions of $\rho_{AB}=\sum_i p_i|\phi_i\rangle_{AB}\langle\phi_i|$ with $p_i\geq 0$ and $\sum_i p_i=1$ in Ref. [55]. For an arbitrary dimensional multipartite quantum state $\rho_{AB_1B_2\cdots BN-1}$, a general polygamy inequality of multipartite quantum entanglement was established in Ref. [35], $E_a(\rho_{A|B_1B_2\cdots BN-1})\leq\sum_i^{N-1}E_a(\rho_{AB_i})$. In the following, for convenience we denote $E_{aAB_j}=E_a(\rho_{AB_j})$ for $j=1,2,\ldots,N-1$, and $E_{aA|B_1B_2\cdots B_{N-1}}=E_a(\rho_{A|B_1B_2\cdots B_{N-1}})$. Using the same approach as for CoA, we have the following theorems.

Theorem 7 Let $\delta_r\geq 1$, $0\leq\mu_r\leq 1$ and $l_r\geq 1$ be real numbers, $1\leq r\leq N-2$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}\in\mathcal{H}_A\otimes\mathcal{H}_{B_1}\otimes\cdots\otimes\mathcal{H}_{B_{N-1}}$, if $E^{\sqrt{2}}_{aAB_i}\geq l_i^{\delta_i}E^{\sqrt{2}}_{aA|B_{i+1}\cdots B_{N-1}}$, $E^{\sqrt{2}}_{aA|B_i\cdots B_{N-1}}\leq E^{\sqrt{2}}_{aAB_i}+\mu_i E^{\sqrt{2}}_{aA|B_{i+1}\cdots B_{N-1}}$, $1\leq p_i\leq 1+\frac{\mu_i E^{\sqrt{2}}_{aA|B_{i+1}\cdots B_{N-1}}}{E^{\sqrt{2}}_{aAB_i}}$ for $i=1,2,\ldots,m$, and $E^{\sqrt{2}}_{aA|B_{j+1}\cdots B_{N-1}}\geq l_j^{\delta_j}E^{\sqrt{2}}_{aAB_j}$, $E^{\sqrt{2}}_{aA|B_j\cdots B_{N-1}}\leq\mu_j E^{\sqrt{2}}_{aAB_j}+E^{\sqrt{2}}_{aA|B_{j+1}\cdots B_{N-1}}$, $1\leq p_j\leq\frac{\mu_j E^{\sqrt{2}}_{aAB_j}}{E^{\sqrt{2}}_{aA|B_{j+1}\cdots B_{N-1}}}$ for $j=m+1,\ldots,N-2$, $1\leq m\leq N-3, N\geq 4$, then

$$\begin{aligned}
E^{\alpha}_{aA|B_1\cdots B_{N-1}}\leq\ & p_1^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{aAB_1}+\sum_{i=2}^{m}\prod_{h=1}^{i-1}\Gamma_h p_i^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{aAB_i}\\
&+\Gamma_1\cdots\Gamma_m\Big(\Gamma_{m+1}E^{\alpha}_{aAB_{m+1}}\\
&+\sum_{j=m+2}^{N-2}\prod_{h=m+1}^{j-1}p_h^{\frac{\alpha}{\sqrt{2}}-1}\Gamma_j E^{\alpha}_{aAB_j}\Big)\\
&+\Gamma_1\cdots\Gamma_m(p_{m+1}\cdots p_{N-2})^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{aAB_{N-1}} \qquad (52)
\end{aligned}$$

for all $0\leq\alpha\leq\sqrt{2}$, where $\Gamma_r=(\mu_r+l_r^{\delta_r})^{\frac{\alpha}{\sqrt{2}}}-p_r^{\frac{\alpha}{\sqrt{2}}-1}l_r^{\frac{\delta_r\alpha}{\sqrt{2}}}$ with $1\leq r\leq N-2$.

In particular, we have the following polygamy relations.

Theorem 8 Let $\delta_r\geq 1$, $0\leq\mu_r\leq 1$ and $l_r\geq 1$ be real numbers, $1\leq r\leq N-2$. For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}\in\mathcal{H}_A\otimes\mathcal{H}_{B_1}\otimes\cdots\otimes\mathcal{H}_{B_{N-1}}$, if $E^{\sqrt{2}}_{aAB_i}\geq l_i^{\delta_i}E^{\sqrt{2}}_{aA|B_{i+1}\cdots B_{N-1}}$, $E^{\sqrt{2}}_{aA|B_i\cdots B_{N-1}}\leq E^{\sqrt{2}}_{aAB_i}+\mu_i E^{\sqrt{2}}_{aA|B_{i+1}\cdots B_{N-1}}$, $1\leq p_i\leq 1+\frac{\mu_i E^{\sqrt{2}}_{aA|B_{i+1}\cdots B_{N-1}}}{E^{\sqrt{2}}_{aAB_i}}$ for $i=1,2,\ldots,N-2$, we have

$$\begin{aligned}
E^{\alpha}_{aA|B_1\cdots B_{N-1}}\leq\ & p_1^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{aAB_1}+\sum_{i=2}^{N-2}\prod_{h=1}^{i-1}\Gamma_h p_i^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{aAB_i}\\
&+\Gamma_1\cdots\Gamma_{N-2}E^{\alpha}_{aAB_{N-1}} \qquad (53)
\end{aligned}$$

for all $0\leq\alpha\leq\sqrt{2}$, where $\Gamma_r=(\mu_r+l_r^{\delta_r})^{\frac{\alpha}{\sqrt{2}}}-p_r^{\frac{\alpha}{\sqrt{2}}-1}l_r^{\frac{\delta_r\alpha}{\sqrt{2}}}$ with $1\leq r\leq N-2$.

Remark 4 We have presented a class of upper bounds of the α-th power of EoA with three different parameters p, δ and μ, where $1\leq p\leq 1+\frac{\mu E^{\sqrt{2}}_{aAC}}{E^{\sqrt{2}}_{aAB}}$, $\delta\geq 1$ and $0\leq\mu\leq 1$. The upper bound of the α-th power of EoA is tighter when p (or δ) decreases. Hence, we have

$$\begin{aligned}
E^{\alpha}_{aA|BC}\ &\leq\ p^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{aAB}+\left((\mu+l^{\delta})^{\frac{\alpha}{\sqrt{2}}}-p^{\frac{\alpha}{\sqrt{2}}-1}l^{\frac{\delta\alpha}{\sqrt{2}}}\right)E^{\alpha}_{aAC}\\
&\leq\ E^{\alpha}_{aAB}+\left((\mu+l^{\delta})^{\frac{\alpha}{\sqrt{2}}}-l^{\frac{\delta\alpha}{\sqrt{2}}}\right)E^{\alpha}_{aAC}\\
&\leq\ E^{\alpha}_{aAB}+\left((\mu+l)^{\frac{\alpha}{\sqrt{2}}}-l^{\frac{\alpha}{\sqrt{2}}}\right)E^{\alpha}_{aAC}
\end{aligned}$$

for all $0\leq\alpha\leq\sqrt{2}$, where the second and third equalities hold when $p=1$ and $\delta=1$, respectively. Obviously our inequal-

ities are tighter than that in Ref. [43], and also than those in Refs. [38–40,42,43].

Example 4 Let us again consider the three-qubit state $|\phi\rangle_{ABC}$ given in Example 2 with $\lambda_0=\lambda_3=1/2$, $\lambda_2=\sqrt{2}/2$ and $\lambda_1=\lambda_4=0$. We have

$$\begin{aligned}
E_{aA|BC} &= -(3/4)\log_2(3/4)-(1/4)\log_2(1/4)\approx 0.811278,\\
E_{aAB} &= -((2+\sqrt{2})/4)\log_2((2+\sqrt{2})/4)\\
&\quad -((2-\sqrt{2})/4)\log_2((2-\sqrt{2})/4)\approx 0.600877,\\
E_{aAC} &= -((2+\sqrt{3})/4)\log_2((2+\sqrt{3})/4)\\
&\quad -((2-\sqrt{3})/4)\log_2((2-\sqrt{3})/4)\approx 0.354578.
\end{aligned}$$

Therefore, with respect to inequality (53), when $N=3$ we have

$$\begin{aligned}
&p^{\frac{\alpha}{\sqrt{2}}-1}E^{\alpha}_{aAB}+((\mu+l^{\delta})^{\frac{\alpha}{\sqrt{2}}}-p^{\frac{\alpha}{\sqrt{2}}-1}l^{\frac{\delta\alpha}{\sqrt{2}}})E^{\alpha}_{aAC}\\
&=p^{\frac{\alpha}{\sqrt{2}}-1}(0.600877)^{\alpha}+((\mu+l^{\delta})^{\frac{\alpha}{\sqrt{2}}}\\
&\quad -p^{\frac{\alpha}{\sqrt{2}}-1}l^{\frac{\delta\alpha}{\sqrt{2}}})(0.354578)^{\alpha}, \qquad (54)\\
&E^{\alpha}_{aAB}+(2^{\frac{\alpha}{\sqrt{2}}}-1)E^{\alpha}_{aAC}\\
&=(0.600877)^{\alpha}+(2^{\frac{\alpha}{\sqrt{2}}}-1)(0.354578)^{\alpha}, \qquad (55)\\
&E^{\alpha}_{aAB}+(\frac{(1+k)^{\frac{\alpha}{\sqrt{2}}}-1}{k^{\frac{\alpha}{\sqrt{2}}}})E^{\alpha}_{aAC}\\
&=(0.600877)^{\alpha}+(\frac{(1+k)^{\frac{\alpha}{\sqrt{2}}}-1}{k^{\frac{\alpha}{\sqrt{2}}}})(0.354578)^{\alpha}, \qquad (56)\\
&E^{\alpha}_{aAB}+((\mu+l)^{\frac{\alpha}{\sqrt{2}}}-l^{\frac{\alpha}{\sqrt{2}}})E^{\alpha}_{aAC}\\
&=(0.600877)^{\alpha}+((\mu+l)^{\frac{\alpha}{\sqrt{2}}}-l^{\frac{\alpha}{\sqrt{2}}})(0.354578)^{\alpha}. \qquad (57)
\end{aligned}$$

When $l=1/k$ and $\mu=1$, the lower bound Eq. (57) gives the same result as Eq. (56). Let $k=2/3$, $l=3/2$, $\mu=1/2$, $\delta=1$ and $p=5/4$. It can be seen that our result Eq. (54) is better than the result Eq. (57) in Ref. [43] for $0\le\alpha\le\sqrt{2}$, hence better than Eq. (56) given in Ref. [42] and Eq. (55) given in Ref. [40], see Fig. 4.

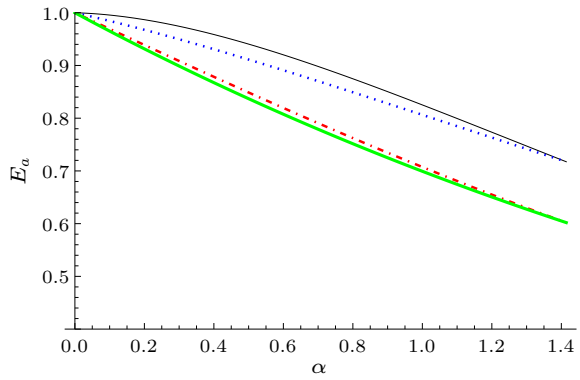


Fig. 4. From bottom to top, the green thick (black thin, blue dotted, and red dot-dashed) line represents the upper bound from our result (Refs. [40,42,43], respectively).

4. Conclusions

Entanglement monogamy and polygamy are the essential properties of multipartite qubit systems, which characterize the distributions of entanglement. We have presented monogamy relations related to the α-th power of concurrence and entanglement of formation. We have also provided polygamy relations related to the corresponding entanglement measures of assistance. All the relations we derived are tighter than the previous existing results. These tighter monogamy and polygamy inequalities provide finer characterizations of the entanglement distributions among the multiqubit systems. By using our approach the monogamy and polygamy properties with respect to other entanglement measures such as convex-roof extended negativity, Tsallis-q entanglement and Rényi-α entanglement can be similarly investigated.

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