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Nondegenerate solitons of the (2+1)-dimensional coupled nonlinear Schrödinger equations with variable coefficients in nonlinear optical fibers

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We derive the multi-hump nondegenerate solitons for the $(2+1)$-dimensional coupled nonlinear Schrödinger equations with propagation distance dependent diffraction, nonlinearity and gain (loss) using the developing Hirota bilinear method, and analyze the dynamical behaviors of these nondegenerate solitons. The results show that the shapes of the nondegenerate solitons are controllable by selecting different wave numbers, varying diffraction and nonlinearity parameters. In addition, when all the variable coefficients are chosen to be constant, the solutions obtained in this study reduce to the shape-preserving nondegenerate solitons. Finally, it is found that the nondegenerate two-soliton solutions can be bounded to form a double-hump two-soliton molecule after making the velocity of one double-hump soliton resonate with that of the other one.

Keywords: nondegenerate solitons, variable coefficients coupled nonlinear Schrödinger equations, Hirota bilinear method

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1. Introduction

It is well known that one of the dominating application areas of solitons is the optical fiber communication. As the carriers of information, solitons can well solve the problem of nonlinear pulse distortions in the fiber optic channels. In 1973, Hasegawa and Tappert first predicted the existence of solitons in nonlinear optical fibers.[1] Later in 1980, Mollenauer *et al.* demonstrated the propagation of soliton in optical fibers.[2] Since then, an enormous amount of attention has been paid in this topic due to its versatile applications by both experimental and theoretical physicists.[3–8] In addition to single-hump solitons, it has been reported more recently that dispersion-managed communication systems can also admit double-hump solitons or multi-hump solitons, which may offer potential applications in improving capacity of optical fiber communication.[9,10] Especially, among these multi-hump solitons, the nondegenerate soliton solutions,[11] which are considered as the solitons in multi-mode fibers traveling with non-identical wave numbers, have been put forward for modeling the propagation of multi-hump solitons in optical fibers. When two nondegenerate solitons collide with each other, their amplitudes may change but their intensities remain unchanged. Later, inspired by nondegenerate solitons, more general nondegenerate Kuznetsov–Ma breathers[12] and nondegenerate Akhmediev breathers[13,14] have also been studied. In recent years, based on several different mathematical physics methods such as the Hirota bilinear method, Kadomtsev–Petviashvili hierarchy reduction and developing Darboux transformation method, nondegenerate solitons have been reported in various multi-mode fiber systems, such as the Manakov systems,[15–17] generalized coupled nonlinear Schrödinger systems,[18–22] long-wave short-wave resonant interaction systems,[23] and coupled Fokas–Lenells systems.[24]

Generally speaking, to describe the propagation of beams in two polarized directions, coupled nonlinear Schrödinger (CNLS) equations[25]

$$\mathrm{i}u_z+\frac{1}{2}u_{tt}+(|u|^2+|v|^2)u=0, \tag{1a}$$

$$\mathrm{i}v_z+\frac{1}{2}v_{tt}+(|u|^2+|v|^2)v=0, \tag{1b}$$

with u and v referring to two complex envelopes of the electrical field, z and t representing the longitudinal propagation coordinate and temporal transverse coordinates respectively, are one of the most commonly used models.[26,27] While the beams traveling in higher dimensions are taken into account, $(1+1)$-dimensional CNLS equations may be extended into $(2+1)$-dimensional ones[28,29]

$$\mathrm{i}u_z+u_{tt}+u_{yy}+(|u|^2+|v|^2)u=0, \tag{2a}$$

$$\mathrm{i}v_z+v_{tt}+v_{yy}+(|u|^2+|v|^2)v=0, \tag{2b}$$

where y represents the spatial transverse coordinate. Furthermore, in real fibers, owing to the inhomogeneity of fiber mediums, which may be caused by imperfections of manufacture, variations in the lattice parameters of the fiber media, fluctuations in the fiber diameters, and so on, the fiber mediums, such as dispersion, nonlinear effect and gain (loss) parameters, become the parameters of transmission distance, then the

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pulse propagations in such mediums should be described by variable coefficient coupled nonlinear Schrödinger (vcCNLS) equations.[30–33] Motivated by the practical significance of vcCNLS equations and the wide applications of nondegenerate solitons, we intend to investigate the multi-hump nondegenerate solitons in the following system of the $(2+1)$-dimensional CNLS equations with variable coefficients:

$$\mathrm{i}u_z+\frac{\beta(z)}{2}(u_{tt}+u_{yy})+R(z)(|u|^2+|v|^2)u-\mathrm{i}\rho(z)u=0, \quad (3a)$$

$$\mathrm{i}v_z+\frac{\beta(z)}{2}(v_{tt}+v_{yy})+R(z)(|u|^2+|v|^2)v-\mathrm{i}\rho(z)v=0, \quad (3b)$$

where real analytic functions $\beta(z)$ and $\rho(z)$ denote the diffraction and gain (loss) parameters, respectively; $R(z)$ is the coefficient of self-focusing Kerr nonlinearity as $R(z)>0$, and $R(z)$ is the coefficient of self-defocusing Kerr nonlinearity as $R(z)<0$. Due to the non-integrability of vcCNLS equations, the studies of these systems were often realized by converting them into integrable constant coefficient CNLS equations. For example, the similarity transformation technique has been brought into vcCNLS equations to obtain the vector rogue wave solutions and bright soliton solutions.[34,35] Wu and Jiang constructed the $(2+1)$-dimensional breathers for vcCNLS equations through transforming them to the standard NLS equation.[36] However, we would like to construct the nondegenerate solitons for Eq. (3) in the frame of the developing Hirota bilinear method in this paper.

The plan of this work is organized as follows. In Section 2, the bilinear forms of the $(2+1)$-dimensional vcCNLS equations (3) are derived. The nondegenerate one-soliton and two-soliton solutions of Eq. (3) are obtained by the developing Hirota bilinear method in Section 3. The propagation characteristics of the nondegenerate solitons and the influences of variable coefficients on the shapes of nondegenerate solitons are analyzed in Section 4. Section 5 is allotted for some important conclusions.

2. Bilinear forms of (2+1)-dimensional vcCNLS equations

To seek the nondegenerate solitons for vcCNLS equations (3), we introduce the following dependent variable transformation[37]

$$u=\frac{g_1}{f}\mathrm{e}^{\int\rho(z)\,\mathrm{d}z}, \quad v=\frac{g_2}{f}\mathrm{e}^{\int\rho(z)\,\mathrm{d}z}, \quad (4)$$

where g_1 and g_2 are both complex functions of (y,z,t), while $f\equiv f(y,z,t)$ is a real function. Through transformation (4) and the constraints among $\rho(z)$, $\beta(z)$ and $R(z)$, i.e.,

$$\rho(z)=\frac{1}{2}\left(\ln\frac{\beta(z)}{R(z)}\right)_z, \quad (5)$$

the bilinear form for Eq. (3) can be written as

$$\left[\mathrm{i}D_z+\frac{\beta(z)}{2}(D_t^2+D_y^2)\right](g_1\cdot f)=0, \quad (6a)$$

$$\left[\mathrm{i}D_z+\frac{\beta(z)}{2}(D_t^2+D_y^2)\right](g_2\cdot f)=0, \quad (6b)$$

$$(D_t^2+D_y^2)(f\cdot f)=2(|g_1|^2+|g_2|^2), \quad (6c)$$

with D_y, D_z and D_t being bilinear derivative operators,[38] defined as

$$\begin{aligned}&D_y^{m_1}D_z^{m_2}D_t^{m_3}(a(y,z,t)\cdot b(y',z',t'))\\&=\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^{m_1}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z'}\right)^{m_2}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^{m_3}a(y,z,t)\\&\times b(y',z',t')|_{y'=y,z'=z,t'=t},\end{aligned} \quad (7)$$

where m_1, m_2 and m_3 are non-negative integers.

Inspired by the developing Hirota bilinear method applied to above-mentioned CNLS equations,[15,21] one may expand g_1, g_2 and f as the power series expansions

$$\begin{aligned}&g_1=\varepsilon g_{11}+\varepsilon^3 g_{31}+\cdots, \quad g_2=\varepsilon g_{12}+\varepsilon^3 g_{32}+\cdots,\\&f=1+\varepsilon^2 f_2+\varepsilon^4 f_4+\cdots.\end{aligned} \quad (8)$$

Here ε is a formal expansion parameter, g_{j1} and g_{j2} $(j=1,3,5,\ldots)$ are complex functions with respect to (y,z,t), and $f_j(y,z,t)$ $(j=2,4,6,\ldots)$ are all real functions. By inserting expansions (8) into the bilinear equations (6) and collecting the terms proportional to the same powers of ε, a system of linear partial differential equations to be solved can be acquired. The multi-hump nondegenerate soliton solution constituting by g_j $(j=1,2)$ and f for Eq. (3) can then be determined by solving these linear partial differential equations.

3. Nondegenerate soliton solutions

3.1. Nondegenerate one-soliton solutions for Eq. (3)

In order to explore the nondegenerate one-soliton solutions for $(2+1)$-dimensional vcCNLS equations (3), we truncate Eq. (8) to

$$\begin{aligned}&g_1=\varepsilon g_{11}+\varepsilon^3 g_{31}, \quad g_2=\varepsilon g_{12}+\varepsilon^3 g_{32},\\&f=1+\varepsilon^2 f_2+\varepsilon^4 f_4,\end{aligned} \quad (9)$$

where we assume

$$\begin{aligned}&g_{11}=a_{11}\mathrm{e}^{\eta_1}, \ g_{12}=a_{12}\mathrm{e}^{\xi_1},\\&g_{31}=\Delta_{11}\mathrm{e}^{\eta_1+\xi_1+\xi_1^*}, \ g_{32}=\Delta_{12}\mathrm{e}^{\eta_1+\xi_1+\eta_1^*},\\&f_2=\delta_1\mathrm{e}^{\eta_1+\eta_1^*}+\delta_2\mathrm{e}^{\xi_1+\xi_1^*}, \ f_4=\delta_{11}\mathrm{e}^{\eta_1+\eta_1^*+\xi_1+\xi_1^*},\\&\eta_1=k_1t+l_1y+w_1(z), \ \ \xi_1=p_1t+q_1y+r_1(z),\end{aligned} \quad (10)$$

with k_1,l_1,p_1 and q_1 being four complex parameters and the superscript $*$ denoting the complex conjugation. By substituting Eq. (10) into the bilinear forms (6) and collecting the coefficients of different powers of ε, we have

$$
\begin{aligned}
&w_1(z)=\frac{\mathrm{i}}{2}(k_1^2+l_1^2)\Big(\int\beta(z)\mathrm{d}z\Big),\ \ r_1(z)=\frac{\mathrm{i}}{2}(p_1^2+q_1^2)\Big(\int\beta(z)\mathrm{d}z\Big),\\
&\delta_1=\frac{|a_{11}|^2}{(k_1+k_1^*)^2+(l_1+l_1^*)^2},\quad \delta_2=\frac{|a_{12}|^2}{(p_1+p_1^*)^2+(q_1+q_1^*)^2},\\
&\Delta_{11}=\frac{a_{11}|a_{12}|^2[(k_1-p_1)(p_1+p_1^*)+(l_1-q_1)(q_1+q_1^*)]}{[(k_1+p_1^*)(p_1+p_1^*)+(l_1+q_1^*)(q_1+q_1^*)][(p_1+p_1^*)^2+(q_1+q_1^*)^2]},\\
&\Delta_{12}=-\frac{|a_{11}|^2a_{12}[(k_1-p_1)(k_1+k_1^*)+(l_1-q_1)(l_1+l_1^*)]}{[(k_1^*+p_1)(k_1+k_1^*)+(l_1^*+q_1)(l_1+l_1^*)][(k_1+k_1^*)^2+(l_1+l_1^*)^2]},\\
&\delta_{11}=\frac{a_{11}\Delta_{11}^*+a_{11}^*\Delta_{11}+a_{12}\Delta_{12}^*+a_{12}^*\Delta_{12}-[[(k_1+k_1^*)-(p_1+p_1^*)]^2+[(l_1+l_1^*)-(q_1+q_1^*)]^2]\delta_1\delta_2}{[(k_1+k_1^*)+(p_1+p_1^*)]^2+[(l_1+l_1^*)+(q_1+q_1^*)]^2},
\end{aligned}
\tag{11}
$$

and the constraint condition among k_1, q_1, p_1 and l_1, i.e.,

$$
q_1+q_1^*=\frac{(p_1+p_1^*)(l_1+l_1^*)}{k_1+k_1^*}. \tag{12}
$$

The inserting of $w_1(z)$ and $r_1(z)$ in Eq. (11) into the wave variables η_1 and ξ_1 yields

$$
\begin{aligned}
&\eta_1=k_1t+l_1y+\frac{\mathrm{i}}{2}(k_1^2+l_1^2)\Big(\int\beta(z)\mathrm{d}z\Big),\\
&\xi_1=p_1t+q_1y+\frac{\mathrm{i}}{2}(p_1^2+q_1^2)\Big(\int\beta(z)\mathrm{d}z\Big).
\end{aligned}
\tag{13}
$$

After setting the formal expansion parameter $\varepsilon=1$, the explicit forms of nondegenerate one-soliton solutions of vcC-NLS equations in the truncated series expansions form are

$$
\begin{aligned}
&u=\frac{(a_{11}\mathrm{e}^{\eta_1}+\Delta_{11}\mathrm{e}^{\eta_1+\xi_1+\xi_1^*})\mathrm{e}^{\int\rho(z)\mathrm{d}z}}{1+\delta_1\mathrm{e}^{\eta_1+\eta_1^*}+\delta_2\mathrm{e}^{\xi_1+\xi_1^*}+\delta_{11}\mathrm{e}^{\eta_1+\eta_1^*+\xi_1+\xi_1^*}},\\
&v=\frac{(a_{12}\mathrm{e}^{\xi_1}+\Delta_{12}\mathrm{e}^{\eta_1+\xi_1+\eta_1^*})\mathrm{e}^{\int\rho(z)\mathrm{d}z}}{1+\delta_1\mathrm{e}^{\eta_1+\eta_1^*}+\delta_2\mathrm{e}^{\xi_1+\xi_1^*}+\delta_{11}\mathrm{e}^{\eta_1+\eta_1^*+\xi_1+\xi_1^*}},
\end{aligned}
\tag{14}
$$

where Δ_{11}, Δ_{12}, δ_1, δ_2 and δ_{11} are shown as Eq. (11), and a_{11} and a_{12} are still two arbitrary constants. According to the expression (14), it is evident that the solitons propagating in the two components are represented by two arbitrary functions $\beta(z)$ and $R(z)$, which are associated with $\rho(z)$ by formula (5), and five arbitrary complex parameters a_{11}, a_{12} and k_1, l_1, p_1. These wave parameters of different values control the shapes, amplitudes, width and velocities of solitons in Kerr media.

3.2. Nondegenerate two-soliton solutions of Eq. (3)

For investigating the interaction between nondegenerate solitons, we consider constructing nondegenerate two-soliton solutions for Eq. (3), that is, truncating the expansions (8) to the following terms:

$$
\begin{aligned}
&g_1=\varepsilon g_{11}+\varepsilon^3g_{31}+\varepsilon^5g_{51}+\varepsilon^7g_{71},\\
&g_2=\varepsilon g_{12}+\varepsilon^3g_{32}+\varepsilon^5g_{52}+\varepsilon^7g_{72},\\
&f=1+\varepsilon^2f_2+\varepsilon^4f_4+\varepsilon^6f_6+\varepsilon^8f_8.
\end{aligned}
\tag{15}
$$

Now we take the seed solutions as $g_{11}=a_{11}\mathrm{e}^{\eta_1}+a_{21}\mathrm{e}^{\eta_2}$, $g_{12}=a_{12}\mathrm{e}^{\xi_1}+a_{22}\mathrm{e}^{\xi_2}$, where $\eta_s=k_st+l_sy+w_s(z)$, $\xi_s=p_st+q_sy+r_s(z)$ $(s=1,2)$, $\{k_s,\ l_s,\ p_s,\ q_s\ (s=1,2)\}$ are complex constants, and $w_s(z)$, $r_s(z)$ $(s=1,2)$ are four complex functions of z. Therefore, similar to the last subsection, by vanishing the coefficients of all different powers of ε and then letting $\varepsilon=1$, the nondegenerate two-soliton solutions of Eq. (3) are given as

$$
\begin{aligned}
&u=\frac{(g_{11}+g_{31}+g_{51}+g_{71})\,\mathrm{e}^{\int\rho(z)\mathrm{d}z}}{1+f_2+f_4+f_6+f_8},\\
&v=\frac{(g_{12}+g_{32}+g_{52}+g_{72})\,\mathrm{e}^{\int\rho(z)\mathrm{d}z}}{1+f_2+f_4+f_6+f_8},
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
g_{31}=&\,\mathrm{e}^{\eta_1}\sum_{s,j=1,2}\Delta_{sj}\mathrm{e}^{\xi_s+\xi_j^*}+\mathrm{e}^{\eta_2}\sum_{s,j=1,2}\Lambda_{sj}\mathrm{e}^{\xi_s+\xi_j^*}\\
&+\mathrm{e}^{\eta_1+\eta_2}\sum_{j=1,2}\Delta_{3j}\mathrm{e}^{\eta_j^*},\\
g_{32}=&\,\mathrm{e}^{\xi_1}\sum_{s,j=1,2}\rho_{sj}\mathrm{e}^{\eta_s+\eta_j^*}+\mathrm{e}^{\xi_2}\sum_{s,j=1,2}\mu_{sj}\mathrm{e}^{\eta_s+\eta_j^*}\\
&+\mathrm{e}^{\xi_1+\xi_2}\sum_{j=1,2}\rho_{3j}\mathrm{e}^{\xi_j^*},\\
g_{51}=&\,\mathrm{e}^{\eta_1+\eta_2+\eta_1^*}\sum_{s,j=1,2}\Theta_{sj}\mathrm{e}^{\xi_s+\xi_j^*}\\
&+\mathrm{e}^{\eta_1+\eta_2+\eta_2^*}\sum_{s,j=1,2}T_{sj}\mathrm{e}^{\xi_s+\xi_j^*}\\
&+\mathrm{e}^{\xi_1+\xi_1^*+\xi_2+\xi_2^*}\sum_{s=1,2}\Delta_{s4}\mathrm{e}^{\eta_s},\\
g_{52}=&\,\mathrm{e}^{\xi_1+\xi_2+\xi_1^*}\sum_{s,j=1,2}\nu_{sj}\mathrm{e}^{\eta_s+\eta_j^*}\\
&+\mathrm{e}^{\xi_1+\xi_2+\xi_2^*}\sum_{s,j=1,2}\phi_{sj}\mathrm{e}^{\eta_s+\eta_j^*}\\
&+\mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}\sum_{s=1,2}\rho_{s4}\mathrm{e}^{\xi_s},\\
g_{71}=&\,\mathrm{e}^{\xi_1+\xi_1^*+\xi_2+\xi_2^*+\eta_1+\eta_2}\sum_{j=1,2}\Delta_{5j}\mathrm{e}^{\eta_j^*},\\
g_{72}=&\,\mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\xi_1+\xi_2}\sum_{j=1,2}\rho_{5j}\mathrm{e}^{\xi_j^*},\\
f_2=&\sum_{s,j=1,2}\delta_{sj}\mathrm{e}^{\eta_s+\eta_j^*}+\sum_{s,j=1,2}\sigma_{sj}\mathrm{e}^{\xi_s+\xi_j^*},\\
f_8=&\,\delta_{18}\mathrm{e}^{\xi_1+\xi_1^*+\xi_2+\xi_2^*+\eta_1+\eta_1^*+\eta_2+\eta_2^*},\\
f_4=&\sum_{s,j=1,2}A_{sj}\mathrm{e}^{\eta_s+\eta_j^*}\sum_{s,j=1,2}B_{sj}\mathrm{e}^{\xi_s+\xi_j^*}+\delta_{17}\mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}
\end{aligned}
$$

$$+\delta_{19}\mathrm{e}^{\xi_1+\xi_1^*+\xi_2+\xi_2^*},$$
$$f_6=\mathrm{e}^{\xi_1+\xi_1^*+\xi_2+\xi_2^*}\sum_{s,j=1,2}\Phi_{sj}\mathrm{e}^{\eta_s+\eta_j^*}+\mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}\sum_{s,j=1,2}\Psi_{sj}\mathrm{e}^{\xi_s+\xi_j^*},$$
$$w_s(z)=\frac{\mathrm{i}}{2}(k_s^2+l_s^2)\Big(\int\beta(z)\mathrm{d}z\Big),$$
$$r_s(z)=\frac{\mathrm{i}}{2}(p_s^2+q_s^2)\Big(\int\beta(z)\mathrm{d}z\Big),\quad(s=1,2).\tag{17}$$

Here k_j, p_j, q_j and l_j $(j=1,2)$ satisfy the constraint conditions

$$k_2=\frac{(q_2-q_1)k_2^*+2(p_1-p_2)l_2^*}{q_1-q_2},$$
$$l_1=\frac{(2k_1-k_2+k_2^*)l_2^*}{k_2+k_2^*},\quad l_2=l_2^*,$$
$$q_1=\frac{q_1^*(p_1+p_2^*)-q_2^*(p_1+p_1^*)}{p_1^*-p_2^*},$$
$$q_2=\frac{q_1^*(p_2+p_2^*)-q_2^*(p_2+p_1^*)}{p_1^*-p_2^*}.\tag{18}$$

Owing to the lengthiness of parameters δ_{17}, δ_{18}, δ_{19}, Δ_{sj}, Λ_{sj}, Δ_{3j}, ρ_{sj}, μ_{sj}, ρ_{3j}, Θ_{sj}, T_{sj}, Δ_{s4}, ν_{sj}, ϕ_{sj}, ρ_{s4}, Δ_{5j}, ρ_{5j}, δ_{sj}, σ_{sj}, A_{sj}, B_{sj}, Φ_{sj}, Ψ_{sj} $(s,j=1,2)$, the specific forms of parameters are shown in the Appendix. From Eqs. (16)–(18), it can be discovered that the complicated profiles of the nondegenerate two-soliton solutions are designed by nine arbitrary wave parameters $\beta(z)$, $R(z)$, k_1, p_j, a_{sj} $(s,j=1,2)$.

4. Properties of nondegenerate solitons

Based on the nondegenerate soliton solutions (14) and (16) with Eqs. (17) and (18), now we analyze the influences of different parameters, including the nonlinear effect $R(z)$, the diffraction $\beta(z)$ and the wave numbers, on the profiles and propagating behaviors of nondegenerate solitons in nonlinear optics.

4.1. Nondegenerate one-soliton solutions of Eq. (3)

First, we discuss how to design different profiles of nondegenerate one-soliton solutions for vcCNLS equations (3) by selecting different diffraction parameter $\beta(z)$ and nonlinear coefficient $R(z)$. It can be seen from the nondegenerate one-soliton solutions (14) with Eq. (5) that the function $\mathrm{e}^{\int\rho(z)\mathrm{d}z}$ modulating the amplitudes of the solitons that are present in two components u and v is determined by the diffraction and nonlinear coefficients. Bringing in $\rho(z)$ in different ways, a different dynamical phenomenon can be excited. For instance, when the periodic functions like $R(z)=\cos(z)$ and $\beta(z)=\cos^3(z)$ are settled on the diffraction and nonlinear coefficients, the profiles of the nondegenerate solitons in the two components $|u|$ and $|v|$ are modulated by the periodic function $|\cos(z)|$ at the same time, which is determined from the relation (5). At a certain transmission distance z, the wave packets still propagate in the t and y directions in the nondegenerate soliton forms. The corresponding evolution plots of nondegenerate solitons for $y=0$ and $z=0.5$ are shown in Figs. 1(a), 1(b) and 1(c), 1(d), respectively, whereas other parameters are $k_1=l_1=1$, $p_1=q_1=0.95$ and $a_{11}=2$, $a_{12}=4$. In addition, if the diffraction and nonlinear coefficients $\beta(z)$ and $R(z)$ are continued to be changed to $R(z)=\mathrm{sech}(z)$ and $\beta(z)=\mathrm{sech}^3(z)$, which makes $\mathrm{e}^{\int\rho(z)\mathrm{d}z}=\mathrm{sech}(z)$, the amplitudes of the nondegenerate solitons decay to zeros quickly along the z and t axis, as shown in Figs. 1(e) and 1(f). The solutions $|u|$ and $|v|$ are localized both in t and z directions. That is to say, under the action of the inhomogeneous coefficients, there appears remarkable difference between the solutions of vcCNLS equations and those of the constant coefficient CNLS equations.[11] In fact, due to the existence of the relation (5) between $\beta(z)$, $R(z)$ and $\rho(z)$, it is not difficult to verify that given different functions $\beta(z)$ and $R(z)$, the same amplitude modulation function $\rho(z)$ can be constructed, that is to say, different diffractions and nonlinear effects may produce the same nondegenerate soliton envelope. For instance, by taking $\{R(z)=\mathrm{sech}(z),\beta(z)=\mathrm{sech}^3(z)\}$, $\{R(z)=\cosh(z),\beta(z)=\mathrm{sech}(z)\}$, $\{R(z)=\sinh(z),\beta(z)=\tanh(z)\mathrm{sech}(z)\}$ and $\{R(z)=\tanh(z),\beta(z)=\tanh(z)\mathrm{sech}^2(z)\}$, one can receive the same amplitude modulation function $\mathrm{e}^{\int\rho(z)\mathrm{d}z}=\mathrm{sech}(z)$, but different wave variables η_1 and ξ_1. In contrast to the variable coefficient case, the nondegenerate one-soliton exhibits the shape-preserving propagation as soon as the diffraction parameter $\beta(z)$ and the nonlinear coefficient $R(z)$ are selected as some real constants.

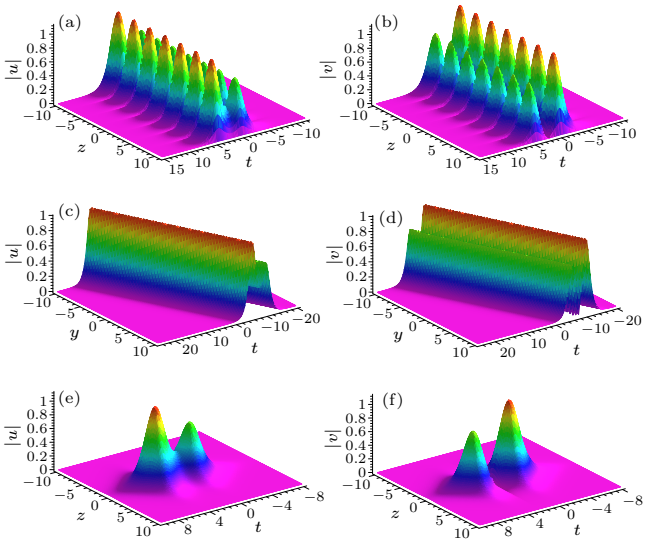


Fig. 1. Evolutions of the nondegenerate one-solitons with the parameters chosen as [(a), (b)] $R(z)=\cos(z)$, $\beta(z)=\cos^3(z)$ for $y=0$; [(c), (d)] $R(z)=\cos(z)$, $\beta(z)=\cos^3(z)$ for $z=0.5$; [(e), (f)] $R(z)=\mathrm{sech}(z)$, $\beta(z)=\mathrm{sech}^3(z)$ for $y=0$. Other parameters are $k_1=l_1=1$, $p_1=q_1=0.95$, $a_{11}=2$, and $a_{12}=4$.

Moreover, according to the explicit solutions (14) with Eqs. (11), (12) and (13), by selecting different wave parameters k_1, p_1, l_1, a_{11} and a_{12}, the nondegenerate one-solitons with different intensities, propagation velocities and phases can be designed. Particularly, under the special conditions $k_1=p_1$

and $l_1 = q_1$, the nondegenerate one-soliton solutions (14) degenerate into the modulated bright soliton forms

$$u = \frac{a_{11}\,\mathrm{e}^{\eta_1}\mathrm{e}^{\int \rho(z)\,\mathrm{d}z}}{1+\mathrm{e}^{\eta_1+\eta_1^*+\delta}},\quad v = \frac{a_{12}\,\mathrm{e}^{\eta_1}\mathrm{e}^{\int \rho(z)\,\mathrm{d}z}}{1+\mathrm{e}^{\eta_1+\eta_1^*+\delta}}, \tag{19}$$

which can be rewritten as

$$\begin{aligned} u =& 4(k_{1\mathrm{R}}^2 + l_{1\mathrm{R}}^2)A_1 \mathrm{e}^{\mathrm{i}\eta_{1\mathrm{I}}}\mathrm{sech}\left(\eta_{1\mathrm{R}} + \frac{\delta}{2}\right)\mathrm{e}^{\int \rho(z)\,\mathrm{d}z},\\ v =& 4(k_{1\mathrm{R}}^2 + l_{1\mathrm{R}}^2)A_2 \mathrm{e}^{\mathrm{i}\eta_{1\mathrm{I}}}\mathrm{sech}\left(\eta_{1\mathrm{R}} + \frac{\delta}{2}\right)\mathrm{e}^{\int \rho(z)\,\mathrm{d}z}, \end{aligned} \tag{20}$$

where $k_{1\mathrm{R}}$, $k_{1\mathrm{I}}$, $l_{1\mathrm{R}}$, $l_{1\mathrm{I}}$, $\eta_{1\mathrm{R}}$ and $\eta_{1\mathrm{I}}$ represent the real and imaginary parts of k_1, l_1 and η_1, respectively; $\eta_{1\mathrm{R}} = k_{1\mathrm{R}}t + l_{1\mathrm{R}}y - (k_{1\mathrm{R}}k_{1\mathrm{I}} + l_{1\mathrm{R}}l_{1\mathrm{I}})\int \beta(z)\,\mathrm{d}z$, $\eta_{1\mathrm{I}} = k_{1\mathrm{I}}t + l_{1\mathrm{I}}y + \frac{1}{2}(k_{1\mathrm{R}}^2 - k_{1\mathrm{I}}^2 + l_{1\mathrm{R}}^2 - l_{1\mathrm{I}}^2)\int \beta(z)\,\mathrm{d}z$, $A_s = \frac{a_{1s}}{\sqrt{|a_{11}|^2+|a_{12}|^2}}$ $(s = 1,2)$ and $\mathrm{e}^{\delta} = \frac{|a_{11}|^2+|a_{12}|^2}{(k_1+k_1^*)^2+(l_1+l_1^*)^2}$.

4.2. Nondegenerate two-soliton solutions of Eq. (3)

Now we continue to take into account the effects of varying diffraction and nonlinear parameters on the shapes of the nondegenerate two-soliton solutions. Figure 2 shows the evolutions of two solitons with different $\beta(z)$ and $R(z)$. It can be seen from Figs. 2(a) and 2(b) that nondegenerate two-solitons propagate as a tanh function due to the variable coefficients are chosen as $\beta(z) = \tanh(z)$ and $R(z) = \sinh(z)\cosh(z)(\sinh(z) - \cosh(z))^2$, which leads to $\mathrm{e}^{\int \rho(z)\,\mathrm{d}z} = \tanh(z) + 1$. Other wave parameters are $l_1 = -2$, $l_2 = 2.2$, $p_1 = -1.2$, $p_2 = 1.25$, $q_2 = 2.5$ and $a_{11} = 5$, $a_{12} = 3$, $a_{21} = a_{22} = 2$. Since the diffraction and nonlinear coefficients are given by $\beta(z) = \sin(z)$ and $R(z) = \tan(z)\sec(z)$, the wave peaks of two solitons increase and decrease with z at a certain frequency, and the envelopes of the nondegenerate two-solitons vary periodically as $\mathrm{e}^{\int \rho(z)\,\mathrm{d}z} = |\cos(z)|$, which are depicted in Figs. 2(c) and 2(d).

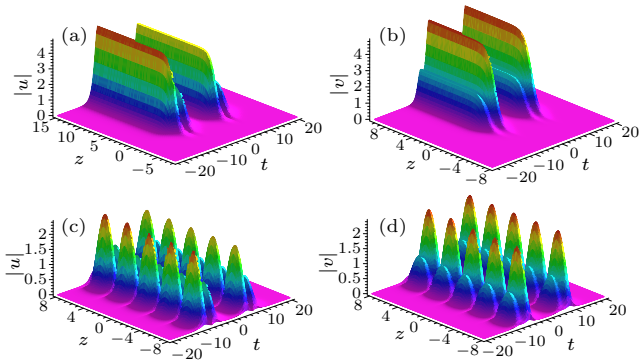


Fig. 2. Evolutions of the nondegenerate two-solitons with $y = 0$ and the parameters chosen as [(a), (b)] $\beta(z) = \tanh(z)$, $R(z) = \sinh(z)\cosh(z)(\sinh(z) - \cosh(z))^2$; [(c), (d)] $\beta(z) = \sin(z)$, $R(z) = \tan(z)\sec(z)$.

In particular, if the variable coefficients are taken as some constants, which makes the gain (loss) parameter $\rho(z) = 0$, the nondegenerate two solitons will travel with some specific amplitudes. For example, when $\beta(z) = 2$ and $R(z) = 1$ are selected, the vcCNLS equations (3) have two stable double-hump soliton solutions, which can be found in Figs. 3(a) and 3(b), where other parameters are chosen as $l_1 = 0.65 + 1.45\mathrm{i}$, $l_2 = 0.45$, $p_1 = 0.5 + 0.6\mathrm{i}$, $p_2 = 0.405 - 0.55\mathrm{i}$, $q_2 = 0.48\mathrm{i}$, $k_2 = 0.38 - 0.55\mathrm{i}$, $a_{11} = a_{22} = 1 + 0.8\mathrm{i}$, $a_{12} = a_{21} = 1.2 + \mathrm{i}$. The corresponding two-dimensional evolution plots for $y = 0$ and $z = \pm 8$ are depicted in Figs. 3(c) and 3(d), where one may observe that the nondegenerate two-solitons maintain the double-hump structures before and after the collisions, but the energies are redistributed after the inelastic collisions. Moreover, via computing the maximum values of the nondegenerate two-solitons, it can be proved that the total energies of the nondegenerate two-solitons before and after collision remain unchanged, which is just one of the prominent characteristics of the nondegenerate solitons. Sequentially, we find that when the imaginary parts of all wave numbers are taken as the same values, which makes the nondegenerate two solitons of both components propagate at the same speed, the nondegenerate two solitons in this case are bounded into double-hump two-soliton molecules,[39] which can be seen from Fig. 4, where we have set the wave parameters as $l_1 = -2.34$, $l_2 = 2.2$, $p_1 = -1.2$, $p_2 = 1.25$, $q_2 = 2.5$, $a_{11} = a_{12} = a_{21} = a_{22} = 2$, $\beta(z) = 1$ and $R(z) = 2$.

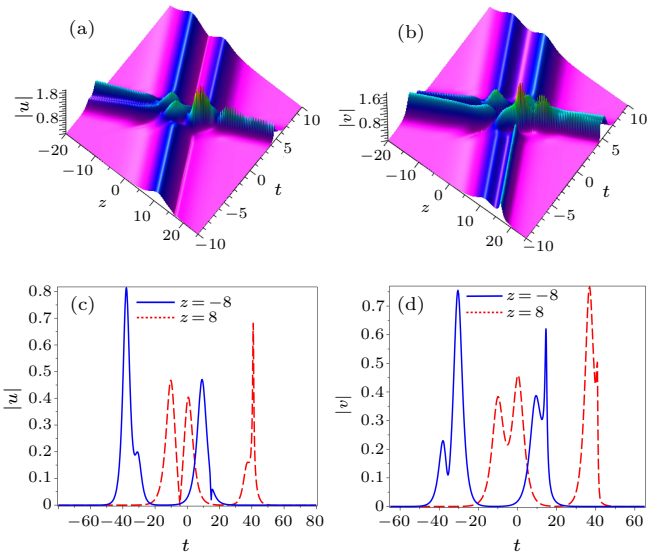


Fig. 3. The inelastic collisions between nondegenerate two-solitons for $y = 0$. [(a), (b)] Dynamical evolution diagrams of nondegenerate soliton collisions. [(c), (d)] The corresponding strength plots for both components $|u|$ and $|v|$ before ($z = -8$) and after ($z = 8$) collision.

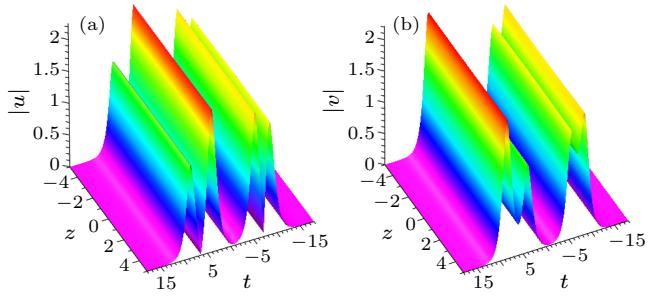


Fig. 4. Stably propagating nondegenerate two-soliton molecules. The wave parameters are chosen as $l_1 = -2.34$, $l_2 = 2.2$, $p_1 = -1.2$, $p_2 = 1.25$, $q_2 = 2.5$, $a_{11} = a_{12} = a_{21} = a_{22} = 2$, $\beta(z) = 1$, and $R(z) = 2$.

5. Conclusions

In summary, we have investigated the nondegenerate one-soliton and two-soliton solutions relating to the Hirota bilinear form for the $(2+1)$-dimensional CNLS equations (3) with variable coefficients. The results show that the shapes of nondegenerate solitons are controllable by choosing different wave parameters, especially the diffraction coefficient $\beta(z)$ and the nonlinearity $R(z)$. Concretely, we have demonstrated how to design the poses and the propagation velocities of nondegenerate solitons by choosing different diffraction coefficients, nonlinear effects and wave numbers, and also analyzed the dynamic behaviors of these nondegenerate solitons based on the analytical expressions (14) with Eqs. (11)–(13), and expressions (16) with Eqs. (17) and (18). For the illustrative purpose, we respectively demonstrate the profiles of nondegenerate one-solitons of cosine function and hyperbolic secant function forms and nondegenerate two-solitons of hyperbolic tangent function and sine function forms. For the nondegenerate two-soliton solutions, it is found that despite the collisions between two double-hump solitons are inelastic and the energies are redistributed after the inelastic collisions, the total energies of the nondegenerate two-solitons remain the same before and after collision. Moreover, as soon as the velocity of one double-hump soliton resonant with that of the other one, the double-hump two-soliton molecule can be constructed, which is shown in Fig. 4. Most importantly, we hope that the results obtained in this paper may raise the possibility of relative experiments and potential applications.

Acknowledgements

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Appendix A

Choosing the formal expansion parameter $\varepsilon = 1$, we can obtain the nondegenerate two-soliton solutions

$$u = \frac{g_{11}+g_{31}+g_{51}+g_{71}}{1+f_2+f_4+f_6+f_8}, \quad v = \frac{g_{12}+g_{32}+g_{52}+g_{72}}{1+f_2+f_4+f_6+f_8},$$

where

$$\begin{aligned}
g_{11} &= a_{11}\mathrm{e}^{\eta_1} + a_{21}\mathrm{e}^{\eta_2},\\
g_{31} &= \mathrm{e}^{\eta_1}\sum_{s,j=1,2}\Delta_{sj}\mathrm{e}^{\xi_s+\xi_j^*} + \mathrm{e}^{\eta_2}\sum_{s,j=1,2}\Lambda_{sj}\mathrm{e}^{\xi_s+\xi_j^*}\\
&\quad + \mathrm{e}^{\eta_1+\eta_2}\sum_{j=1,2}\Delta_{3j}\mathrm{e}^{\eta_j^*},\\
g_{12} &= a_{12}\mathrm{e}^{\xi_1} + a_{22}\mathrm{e}^{\xi_2},\\
g_{32} &= \mathrm{e}^{\xi_1}\sum_{s,j=1,2}\rho_{sj}\mathrm{e}^{\eta_s+\eta_j^*} + \mathrm{e}^{\xi_2}\sum_{s,j=1,2}\mu_{sj}\mathrm{e}^{\eta_s+\eta_j^*}\\
&\quad + \mathrm{e}^{\xi_1+\xi_2}\sum_{j=1,2}\rho_{3j}\mathrm{e}^{\xi_j^*},\\
g_{51} &= \mathrm{e}^{\eta_1+\eta_2+\eta_1^*}\sum_{s,j=1,2}\Theta_{sj}\mathrm{e}^{\xi_s+\xi_j^*}\\
&\quad + \mathrm{e}^{\eta_1+\eta_2+\eta_2^*}\sum_{s,j=1,2}T_{sj}\mathrm{e}^{\xi_s+\xi_j^*}\\
&\quad + \mathrm{e}^{\xi_1+\xi_1^*+\xi_2+\xi_2^*}\sum_{s=1,2}\Delta_{s4}\mathrm{e}^{\eta_s},\\
g_{52} &= \mathrm{e}^{\xi_1+\xi_2+\xi_1^*}\sum_{s,j=1,2}\nu_{sj}\mathrm{e}^{\eta_s+\eta_j^*}\\
&\quad + \mathrm{e}^{\xi_1+\xi_2+\xi_2^*}\sum_{s,j=1,2}\phi_{sj}\mathrm{e}^{\eta_s+\eta_j^*}\\
&\quad + \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}\sum_{s=1,2}\rho_{s4}\mathrm{e}^{\xi_s},\\
g_{71} &= \mathrm{e}^{\xi_1+\xi_1^*+\xi_2+\xi_2^*+\eta_1+\eta_2}\sum_{j=1,2}\Delta_{5j}\mathrm{e}^{\eta_j^*},\\
g_{72} &= \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\xi_1+\xi_2}\sum_{j=1,2}\rho_{5j}\mathrm{e}^{\xi_j^*},\\
f_2 &= \sum_{s,j=1,2}\delta_{sj}\mathrm{e}^{\eta_s+\eta_j^*} + \sum_{s,j=1,2}\sigma_{sj}\mathrm{e}^{\xi_s+\xi_j^*},\\
f_8 &= \delta_{18}\mathrm{e}^{\xi_1+\xi_1^*+\xi_2+\xi_2^*+\eta_1+\eta_1^*+\eta_2+\eta_2^*},\\
f_4 &= \sum_{s,j=1,2}A_{sj}\mathrm{e}^{\eta_s+\eta_j^*}\sum_{s,j=1,2}B_{sj}\mathrm{e}^{\xi_s+\xi_j^*}\\
&\quad + \delta_{17}\mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*} + \delta_{19}\mathrm{e}^{\xi_1+\xi_1^*+\xi_2+\xi_2^*},\\
f_6 &= \mathrm{e}^{\xi_1+\xi_1^*+\xi_2+\xi_2^*}\sum_{s,j=1,2}\Phi_{sj}\mathrm{e}^{\eta_s+\eta_j^*}\\
&\quad + \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}\sum_{s,j=1,2}\Psi_{sj}\mathrm{e}^{\xi_s+\xi_j^*}, \quad (s,j=1,2).
\end{aligned}$$

Let us define $(p_1 \pm p_2) = P^{\pm}$, $(p_1 \pm p_1^*) = P_1^{\pm}$, $(p_1 \pm p_2^*) = P_2^{\pm}$, $(p_2 \pm p_1^*) = P_3^{\pm}$, $(p_2 \pm p_2^*) = P_4^{\pm}$, $(q_1 \pm q_2) = Q^{\pm}$, $(q_1 \pm q_1^*) = Q_1^{\pm}$, $(q_1 \pm q_2^*) = Q_2^{\pm}$, $(q_2 \pm q_1^*) = Q_3^{\pm}$, $(q_2 \pm q_2^*) = Q_4^{\pm}$, $(k_1 \pm k_2) = K^{\pm}$, $(k_1 \pm k_1^*) = K_1^{\pm}$, $(k_1 \pm k_2^*) = K_2^{\pm}$, $(k_2 \pm k_1^*) = K_3^{\pm}$, $(k_2 \pm k_2^*) = K_4^{\pm}$, $(l_1 \pm l_2) = L^{\pm}$, $(l_1 \pm l_1^*) = L_1^{\pm}$, $(l_1 \pm l_2^*) = L_2^{\pm}$, $(l_2 \pm l_1^*) = L_3^{\pm}$, $(l_2 \pm l_2^*) = L_4^{\pm}$, $(k_1 \pm p_1) = U_1^{\pm}$, $(k_1 \pm p_2) = U_2^{\pm}$, $(k_2 \pm p_1) = U_3^{\pm}$, $(k_2 \pm p_2) = U_4^{\pm}$, $(k_1 \pm p_1^*) = U_5^{\pm}$, $(k_1 \pm p_2^*) = U_6^{\pm}$, $(k_2 \pm p_1^*) = U_7^{\pm}$, $(k_2 \pm p_2^*) = U_8^{\pm}$, $(l_1 \pm q_1) = V_1^{\pm}$, $(l_1 \pm q_2) = V_2^{\pm}$, $(l_2 \pm q_1) = V_3^{\pm}$, $(l_2 \pm q_2) = V_4^{\pm}$, $(l_1 \pm q_1^*) = V_5^{\pm}$, $(l_1 \pm q_2^*) = V_6^{\pm}$, $(l_2 \pm q_1^*) = V_7^{\pm}$, $(l_2 \pm q_2^*) = V_8^{\pm}$, and the wave parameters are

$$\begin{aligned}
\delta_{sj} &= \frac{a_{s1}a_{j1}^*}{(k_s+k_j^*)^2+(l_s+l_j^*)^2},\\
\Delta_{3j} &= \frac{a_{11}\delta_{2j}(K^-K_{j+2}^+ + L^-L_{j+2}^+) - a_{21}\delta_{1j}(K^-K_j^+ + L^-L_j^+)}{K_j^+K_{j+2}^+ + L_j^+L_{j+2}^+},\\
\sigma_{sj} &= \frac{a_{s2}a_{j2}^*}{(p_s+p_j^*)^2+(q_s+q_j^*)^2},\\
\rho_{3j} &= \frac{a_{12}\sigma_{2j}(P^-P_{j+2}^+ + Q^-Q_{j+2}^+) - a_{22}\sigma_{1j}(P^-P_j^+ + Q^-Q_j^+)}{P_j^+P_{j+2}^+ + Q_j^+Q_{j+2}^+},
\end{aligned}$$

$$\Delta_{sj}=\frac{a_{11}\sigma_{sj}[U_s^-(p_s+p_j^*)+V_s^-(q_s+q_j^*)]}{[(k_1+p_j^*)(p_s+p_j^*)+(l_1+q_j^*)(q_s+q_j^*)]},\ \rho_{sj}=\frac{a_{12}\delta_{sj}[(p_1-k_s)(k_s+k_j^*)+(q_1-l_s)(l_s+l_j^*)]}{[(p_1+k_j^*)(k_s+k_j^*)+(q_1+l_j^*)(l_s+l_j^*)]},$$

$$\Lambda_{sj}=\frac{a_{21}\sigma_{sj}[U_{s+2}^-(p_s+p_j^*)+V_{s+2}^-(q_s+q_j^*)]}{[(k_2+p_s^*)(p_s+p_j^*)+(l_2+q_s^*)(q_s+q_j^*)]},\ \mu_{sj}=\frac{a_{22}\delta_{sj}[(p_2-k_s)(k_s+k_j^*)+(q_2-l_s)(l_s+l_j^*)]}{[(p_2+k_j^*)(k_s+k_j^*)+(q_2+l_j^*)(l_s+l_j^*)]}.$$

$$A_{sj}B_{mn}=\frac{a_{s1}M_{nm}^*+a_{j1}^*N_{mn}+a_{m2}\alpha_{js}^*+a_{n2}^*\beta_{sj}-[[(k_s+k_j^*)-(p_m+p_n^*)]^2+[(l_s+l_j^*)-(q_m+q_n^*)]^2]\delta_{sj}\sigma_{mn}}{[(k_s+k_j^*)+(p_m+p_n^*)]^2+[(l_s+l_j^*)+(q_m+q_n^*)]^2},$$

with $s=1,\ N_{mn}=\Delta_{mn},\ s=2,\ N_{mn}=\Lambda_{mn};\ j=1,M_{nm}^*=\Delta_{nm}^*,\ j=2,M_{nm}^*=\Lambda_{nm}^*;\ m=1,\ \beta_{sj}=\rho_{sj},\ m=2,\ \beta_{sj}=\mu_{sj};\ n=1,\ \alpha_{js}^*=\rho_{js}^*,\ n=2,\ \alpha_{js}^*=\mu_{js}^*$.

$$\begin{aligned}\delta_{17}=&\{a_{11}\Delta_{32}^*+a_{11}^*\Delta_{32}+a_{21}\Delta_{31}^*+a_{21}^*\Delta_{31}-[(K_1^+-K_4^+)^2+(L_1^+-L_4^+)]^2\delta_{11}\delta_{22}-[(K_1^--K_4^-)^2\\&+(L_1^--L_4^-)^2]\delta_{12}\delta_{21}\}\{[(K_1^+-K_4^+)^2+(L_1^+-L_4^+)]^2\}^{-1},\\ \delta_{19}=&\{a_{12}\rho_{32}^*+a_{12}^*\rho_{32}+a_{22}\rho_{31}^*+a_{22}^*\rho_{31}-[(P_1^+-P_4^+)^2+(Q_1^+-Q_4^+)^2]\sigma_{11}\sigma_{22}-[(P_1^--P_4^-)^2\\&+(Q_1^--Q_4^-)^2]\sigma_{12}\sigma_{21}\}\{(P_1^++P_4^+)^2+(Q_1^++Q_4^+)^2\}^{-1}.\\ \boldsymbol{V}_{sj}=&\frac{C_{sj}^m\delta_{2m}\Delta_{sj}-D_{sj}^m\delta_{1m}\Lambda_{sj}-E_{sj}^m\sigma_{sj}\Delta_{3m}-F_{sj}^m a_{21}A_{1m}B_{sj}+G_{sj}^m a_{11}A_{2m}B_{sj}}{K_m^+(K_{m+2}^++p_s+p_j^*)+(p_s+p_j^*)(k_2+p_j^*)+L_m^+(L_{m+2}^++q_s+q_j^*)+(q_s+q_j^*)(l_2+q_j^*)},\end{aligned}$$

with $m=1,\ \boldsymbol{V}_{sj}=\Theta_{sj},\ m=2,\boldsymbol{V}_{sj}=T_{sj}$, and $C_{sj}^m=K_{m+2}^+[K^-+(p_s+p_j^*)]-(p_s+p_j^*)(k_1+p_j^*)+L_{m+2}^+[L^-+(q_s+q_j^*)]-(q_s+q_j^*)(l_1+q_j^*)$, $D_{sj}^m=K_m^+[K^--(p_s+p_j^*)]+(p_s+p_j^*)(k_2+p_j^*)+L_m^+[L^-+(q_s+q_j^*)]+(q_s+q_j^*)(l_2+q_j^*)$, $E_{sj}^m=K_m^+[K_{m+2}^+-(p_s+p_j^*)]-(p_s+p_j^*)(k_2-p_s)+L_m^+[L_{m+2}^+-(q_s+q_j^*)]-(q_s+q_j^*)(l_2-q_s)$, $F_{sj}^m=K_m^+[K^-+(p_s+p_j^*)]-(p_s+p_j^*)(k_2-p_j)+L_m^+[L^-+(q_s+q_j^*)]+(q_s+q_j^*)(l_1-q_j)$, $G_{sj}^m=K_{m+2}^+[K^--(p_s+p_j^*)]+(p_s+p_j^*)(k_1-p_j)+L_{m+2}^+[L^--(q_s+q_j^*)]+(q_s+q_j^*)(l_1-q_j)$.

$$\Delta_{s4}=\frac{C_{s4}\sigma_{11}W_{22}+D_{s4}\sigma_{12}W_{21}-E_{s4}\sigma_{21}W_{12}-F_{s4}\sigma_{22}W_{11}+G_{s4}a_{s1}\delta_{19}}{(k_s+p_1^*)(P_1^++P_4^+)+P_2^+P_4^++(l_s+q_1^*)(Q_1^++Q_4^+)+Q_2^+Q_4^+},$$

with $s=1,\ W=\Delta;\ s=2,\ W=\Lambda$, and $C_{s4}=(k_s+p_2^*)(P_1^+-p_4^+)-P^-P_1^++(l_s+q_2^*)(Q_1^+-Q_4^+)-Q^-Q_1^+$, $D_{s4}=(k_s+p_1^*)(P_1^--P_4^-)-P^-P_2^++(l_s+q_1^*)(Q_1^--Q_4^-)-Q^-Q_2^+$, $E_{s4}=(k_s+p_2^*)(P_1^--P_4^-)-P^-P_3^++(l_s+q_2^*)(Q_1^--Q_4^-)-Q^-Q_3^+$, $F_{s4}=(k_s+p_1^*)(P_1^+-P_4^+)-P^-P_4^++(l_s+q_1^*)(Q_1^+-Q_4^+)-Q^-Q_4^+$, $G_{s4}=(k_s-p_1)(P_1^++P_4^+)-P_3^+P_4^++(l_s-q_1)(Q_1^++Q_4^+)-Q_3^+Q_4^+$.

$$\boldsymbol{v}_{sj}=\frac{-c_{sj}^m\sigma_{1m}\mu_{sj}-d_{sj}^m\sigma_{2m}\rho_{sj}-e_{sj}^m\delta_{sj}\rho_{3m}-f_{sj}^m a_{22}A_{sj}B_{1m}-g_{sj}^m a_{12}A_{sj}B_{2m}}{(k_s+k_j^*)(k_j^*+p_2+P_m^+)+P_m^+P_{m+2}^++(l_s+l_j^*)(l_j^*+q_2+Q_m^+)+Q_m^+Q_{m+2}^+},$$

with $m=1,\ \boldsymbol{v}_{sj}=\nu_{sj},\ m=2,\ \boldsymbol{v}_{sj}=\phi_{sj}$, and $c_{sj}^m=(k_s+k_j^*)[(k_j^*+p_2)-P_m^+]+P_m^+P^-+(l_s+l_j^*)[(l_j^*+q_2)-Q_m^+]+Q_m^+Q^-$, $d_{sj}^m=(k_s+k_j^*)[(k_j^*+p_1)-P_{m+2}^+]-P_{m+2}^+P^-+(l_s+l_j^*)[(l_j^*+q_1)-Q_{m+2}^+]-Q_{m+2}^+Q^-$, $e_{sj}^m=(k_s+k_j^*)[(k_s-p_2)-P_m^+]+P_m^+P_{m+2}^++(l_s+l_j^*)[(l_s-q_2)-Q_m^+]+Q_m^+Q_{m+2}^+$, $f_{sj}^m=(k_s+k_j^*)[(k_s-p_2)+P_m^+]+P_m^+P^-+(l_s+l_j^*)[(l_s-q_2)+Q_m^+]+Q_m^+Q^-$, $g_{sj}^m=(k_s+k_j^*)[(k_s-p_1)+P_{m+2}^+]-P_{m+2}^+P^-+(l_s+l_j^*)[(l_s-q_1)+Q_{m+2}^+]-Q_{m+2}^+Q^-$.

$$\rho_{s4}=\frac{c_{s4}\delta_{11}w_{22}+d_{s4}\delta_{12}w_{21}-e_{s4}\delta_{21}w_{12}-f_{s4}\delta_{22}w_{11}-g_{s4}a_{s2}\delta_{17}}{(p_s+k_1^*)(K_1^++K_4^+)+K_2^+K_4^++(q_s+l_1^*)(L_1^++L_4^+)+L_2^+L_4^+},$$

with $s=1,\ w=\rho;\ s=2,\ w=\nu$, and $c_{s4}=(p_s+k_2^*)(K_1^+-K_4^+)-K^-K_1^++(q_s+l_2^*)(L_1^+-L_4^+)-L^-L_1^+$, $d_{s4}=(p_s+k_1^*)(K_1^--K_4^-)-K^-K_2^++(q_s+l_1^*)(L_1^--L_4^-)-L^-L_2^+$, $e_{s4}=(p_s+k_2^*)(K_1^--K_4^-)-K^-K_3^++(q_s+l_2^*)(L_1^--L_4^-)-L^-L_3^+$, $f_{s4}=(p_s+k_1^*)(K_1^+-K_4^+)-K^-K_4^++(q_s+l_1^*)(L_1^+-L_4^+)-L^-L_4^+$, $g_{s4}=(p_sK_4+)(K_1^++K_4+)-K_3^+K_4^++(q_s-l_1)(L_1^++L_4^+)-L_3^+L_4^+$.

$$\begin{aligned}\Psi_{sj}=&\{H_{sj}-C_{sj}\delta_{11}A_{22}B_{sj}-D_{sj}\delta_{12}A_{21}B_{sj}-E_{sj}\delta_{21}A_{12}B_{sj}\\&-F_{sj}\delta_{22}A_{11}B_{sj}-G_{sj}\sigma_{sj}\delta_{17}\}\{K_1^+[K_1^++2K_4^+\\&+2(p_s+p_j^*)]+(K_4^++p_s+p_j^*)^2+L_1^+[L_1^++2L_4^+\\&+2(q_s+q_j^*)]+(L_4^++q_s+q_j^*)^2\}^{-1},\end{aligned}$$

with $s=1,\ \omega_{11}=\rho_{11},\ \tau_{22}=\rho_{22},\ \omega_{12}=\rho_{12},\ \omega_{21}=\rho_{21},\ s=2,\ \omega_{11}=\mu_{11},\ \tau_{22}=\mu_{22},\ \omega_{12}=\mu_{12},\ \omega_{21}=\mu_{21};\ j=1,\ \tau_{22}^*=\rho_{22}^*,\ \omega_{11}^*=\rho_{11}^*,\ \tau_{12}^*=\rho_{12}^*,\ \tau_{21}^*=\rho_{21}^*,\ j=2,\ \tau_{22}^*=\mu_{22}^*,\ \omega_{11}^*=\mu_{11}^*,\ \tau_{12}^*=\mu_{12}^*,\ \tau_{21}^*=\mu_{21}^*$, and $C_{sj}=K_1^+[K_1^+-2K_4^+-2(p_s+p_j^*)]+(K_4^++p_s+p_j^*)^2+L_1^+[L_1^+-2L_4^+-2(q_s+q_j^*)]+(L_4^++q_s+q_j^*)^2$, $D_{sj}=K_1^-[K_1^--2K_4^--2(p_s+p_j^*)]+(K_4^-+p_s+p_j^*)^2+L_1^-[L_1^--2L_4^--2(q_s+q_j^*)]+(L_4^-+q_s+q_j^*)^2$, $E_{sj}=K_1^-[K_1^--2K_4^-+2(p_s+p_j^*)]+(K_4^--p_s-p_j^*)^2+L_1^-[L_1^--2L_4^-+2(q_s+q_j^*)]+(L_4^--q_s-q_j^*)^2$, $F_{sj}=K_1^+[K_1^+-2K_4^++2(p_s+p_j^*)]+(K_4^+-p_s-p_j^*)^2+L_1^+[L_1^+-2L_4^++2(q_s+q_j^*)]+(L_4^+-q_s-q_j^*)^2$, $G_{sj}=K_1^+[K_1^++2K_4^+-2(p_s+p_j^*)]+(K_4^+-p_s-p_j^*)^2+L_1^+[L_1^++2L_4^+-2(q_s+q_j^*)]+(L_4^+-q_s-$

$q_j^*)^2$, $H_{sj}=\Delta_{sj}\Delta_{32}^*+\Delta_{31}\Lambda_{js}^*+\Delta_{32}\Delta_{js}^*+\Lambda_{sj}\Delta_{31}^*+a_{11}^*T_{sj}+a_{11}T_{js}^*+a_{s2}\rho_{j4}^*+a_{j2}^*\rho_{s4}+a_{21}\Theta_{js}^*+a_{21}^*\Theta_{sj}+\omega_{11}\tau_{22}^*+\omega_{11}^*\tau_{22}+\omega_{12}\tau_{12}^*+\omega_{21}\tau_{21}^*$.

$$\begin{aligned}\Phi_{sj}=&\{H'_{sj}-C'_{sj}\sigma_{11}A_{sj}B_{22}-D'_{sj}\sigma_{12}\nu_{sj}B_{21}-E'_{sj}\sigma_{21}A_{sj}B_{12}\\&-F'_{sj}\sigma_{22}A_{sj}B_{11}-G'_{11}\delta_{sj}\delta_{19}\}\{(k_s+k_j^*)[k_s+k_j^*\\&+2(P_1^++P_4^+)]+(P_1^++P_4^+)^2+(l_s+l_j^*)[l_s+l_j^*\\&+2(Q_1^++Q_4^+)]+(Q_1^++Q_4^+)^2\}^{-1},\end{aligned}$$

with $s=1$, $\Omega_{11}=\Delta_{11}$, $\Pi_{22}=\Delta_{22}$, $\Omega_{12}=\Delta_{12}$, $\Omega_{21}=\Delta_{21}$, $s=2$, $\Omega_{11}=\Lambda_{11}$, $\Pi_{22}=\Lambda_{22}$, $\Omega_{12}=\Lambda_{12}$, $\Omega_{21}=\Lambda_{21}$; $j=1$, $\Pi_{22}^*=\Delta_{22}^*$, $\Omega_{11}^*=\Delta_{11}^*$, $\Pi_{12}^*=\Delta_{12}^*$, $\Pi_{21}^*=\Delta_{21}^*$, $j=2$, $\Pi_{22}^*=\Lambda_{22}^*$, $\Omega_{11}^*=\Lambda_{11}^*$, $\Pi_{12}^*=\Lambda_{12}^*$, $\Pi_{21}^*=\Lambda_{21}^*$, and $C'_{sj}=(k_s+k_j^*)[k_s+k_j^*-2(P_1^+-P_4^+)]+(P_1^+-P_4^+)^2+(l_s+l_j^*)[l_s+l_j^*-2(Q_1^+-Q_4^+)]+(Q_1^+-Q_4^+)^2$, $D'_{sj}=(k_s+k_j^*)[k_s+k_j^*-2(P_1^--P_4^-)]+(P_1^--P_4^-)^2+(l_s+l_j^*)[l_s+l_j^*-2(Q_1^--Q_4^-)]+(Q_1^--Q_4^-)^2$, $E'_{sj}=(k_s+k_j^*)[k_s+k_j^*+2(P_1^--P_4^-)]+(P_1^--P_4^-)^2+(l_s+l_j^*)[l_s+l_j^*+2(Q_1^--Q_4^-)]+(Q_1^--Q_4^-)^2$, $F'_{sj}=(k_s+k_j^*)[k_s+k_j^*+2(P_1^+-P_4^+)]+(P_1^+-P_4^+)^2+(l_s+l_j^*)[l_s+l_j^*+2(Q_1^+-Q_4^+)]+(Q_1^+-Q_4^+)^2$, $G'_{sj}=(k_s+k_j^*)[k_s+k_j^*-2(P_1^++P_4^+)]+(P_1^++P_4^+)^2+(l_s+l_j^*)[l_s+l_j^*-2(Q_1^++Q_4^+)]+(Q_1^++Q_4^+)^2$, $H'_{sj}=\mu_{sj}\rho_{31}^*+\rho_{31}\mu_{js}^*+\rho_{sj}\rho_{32}^*+\rho_{32}\rho_{js}^*+a_{s1}\Delta_{j4}^*+a_{j1}^*\Delta_{s4}+a_{12}\phi_{js}^*+a_{12}^*\phi_{sj}+a_{22}\nu_{js}^*+a_{22}^*\nu_{sj}+\Omega_{11}\Pi_{22}^*+\Omega_{11}^*\Pi_{22}+\Omega_{12}\Pi_{12}^*+\Omega_{21}\Pi_{21}^*$.

$$\begin{aligned}\Delta_{5j}=&\{(M_j^1\Delta_{11}A_{2j}B_{22}+M_j^2\Delta_{12}A_{2j}B_{21}+M_j^3\Delta_{21}A_{2j}B_{12}\\&+M_j^4\Delta_{22}A_{2j}B_{11}-M_j^5\Lambda_{11}A_{1j}B_{22}-M_j^6\Lambda_{12}A_{1j}B_{21}\\&-M_j^7\Lambda_{21}A_{1j}B_{12}-M_j^8\Lambda_{22}A_{1j}B_{11}-M_j^9\delta_{1j}\Delta_{24}\\&+M_j^{10}\delta_{2j}\Delta_{14}-M_j^{11}\delta_{19}\Delta_{3j}-M_j^{12}a_{21}\Phi_{1j}+M_j^{13}a_{11}\Phi_{2j}\\&-M_j^{14}\sigma_{11}O_{22}-M_j^{15}\sigma_{12}O_{21}-M_j^{16}\sigma_{21}O_{12}\\&-M_j^{17}\sigma_{22}O_{11})\}\{K_j^+(K_{j+2}^++P_1^++P_4^+)+P_1^+(U_7^++P_4^+)\\&+U_8^+P_4^++L_j^+(L_{j+2}^++Q_1^++Q_4^+)+Q_1^+(V_7^++Q_4^+)\\&+V_8^+Q_4^+\}^{-1},\end{aligned}$$

with $j=1$, $O=\Theta$; $j=2$, $O=T$, and $M_j^1=K_{j+2}^+(K^-+P_1^+-P_4^+)-P_1^+(U_5^+-P_4^+)+U_2^-P_4^++L_{j+2}^+(L^-+Q_1^+-Q_4^+)-Q_1^+(V_5^+-Q_4^+)+V_2^-Q_4^+$, $M_j^2=K_{j+2}^+(K^-+P_1^--P_4^-)-P_2^+(U_5^--P_4^-)+U_2^-P_3^++L_{j+2}^+(L^-+Q_1^--Q_4^-)-Q_2^+(V_5^--Q_4^-)+V_2^-Q_3^+$, $M_j^3=K_{j+2}^+(K^--P_1^-+P_4^-)+P_2^+(U_1^-+P_3^+)-U_5^+P_3^++L_{j+2}^+(L^--Q_1^-+Q_4^-)+Q_2^+(V_1^-+Q_3^+)-V_5^+Q_3^+$, $M_j^4=K_{j+2}^+(K^--P_1^++P_4^+)+P_1^+(U_1^-+P_4^+)-U_6^+P_4^++L_{j+2}^+(L^--Q_1^++Q_4^+)+Q_1^+(V_1^-+Q_4^+)-V_6^+Q_4^+$, $M_j^5=K_j^+(K^--P_1^++P_4^+)+P_1^+(U_7^+-P_4^+)-U_4^-P_4^++L_j^+(L^--Q_1^++Q_4^+)+Q_1^+(V_7^+-Q_4^+)-V_4^-Q_4^+$, $M_j^6=K_j^+(K^--P_1^-+P_4^-)+P_2^+(U_7^--P_4^-)-U_4^-P_3^++L_j^+(L^--Q_1^-+Q_4^-)+Q_2^+(V_7^--Q_4^-)-V_4^-Q_3^+$, $M_j^7=K_j^+(K^-+P_1^--P_4^-)-P_2^+(U_3^-+P_3^+)+U_7^+P_3^++L_j^+(L^-+Q_1^--Q_4^-)-Q_2^+(V_3^-+Q_3^+)+V_7^+Q_3^+$, $M_j^8=K_j^+(K^-+P_1^+-P_4^+)-P_1^+(U_3^-+P_4^+)+U_8^+P_4^++L_j^+(L^-+Q_1^+-Q_4^+)-Q_1^+(V_3^-+Q_4^+)+V_8^+Q_4^+$, $M_j^9=K_j^+(K^--P_1^+-P_4^+)+P_1^+(U_7^++P_4^+)+U_8^+P_4^++L_j^+(L^--Q_1^+-Q_4^+)+Q_1^+(V_7^++Q_4^+)+V_8^+Q_4^+$, $M_j^{10}=K_{j+2}^+(K^-+P_1^+-P_4^+)-P_1^+(U_5^++P_4^+)-U_6^+P_4^++L_{j+2}^+(L^-+Q_1^+-Q_4^+)-Q_1^+(V_5^++Q_4^+)-V_6^+Q_4^+$, $M_j^{11}=K_{j+2}^+(K_j^+-P_1^+-P_4^+)-P_1^+(U_1^--P_4^+)-U_2^-P_4^++L_{j+2}^+(L_j^+-Q_1^+-Q_4^+)-Q_1^+(V_1^--Q_4^+)-V_2^-Q_4^+$, $M_j^{12}=K_j^+(K^-+P_1^+-P_4^+)-P_1^+(U_3^--P_4^+)-U_4^-P_4^++L_j^+(L^-+Q_1^+-Q_4^+)-Q_1^+(V_3^--Q_4^+)-V_4^-Q_4^+$, $M_j^{13}=K_{j+2}^+(K^--P_1^+-P_4^+)+P_1^+(U_1^--P_4^+)+U_2^-P_4^++L_{j+2}^+(L^--Q_1^+-Q_4^+)+Q_1^+(V_1^--Q_4^+)+V_2^-Q_4^+$, $M_j^{14}=K_j^+(K_{j+2}^+-P_1^++P_4^+)-P_1^+(U_3^-+P_4^+)+U_8^+P_4^++L_j^+(L_{j+2}^+-Q_1^++Q_4^+)-Q_1^+(V_3^-+Q_4^+)+V_8^+Q_4^+$, $M_j^{15}=K_j^+(K_{j+2}^+-P_1^-+P_4^-)-P_2^+(U_3^-+P_3^+)+U_7^+P_3^++L_j^+(L_{j+2}^+-Q_1^-+Q_4^-)-Q_2^+(V_3^-+Q_3^+)+V_7^+Q_3^+$, $M_j^{16}=K_j^+(K_{j+2}^++P_1^--P_4^-)+P_2^+(U_7^--P_4^-)-U_4^-P_3^++L_j^+(L_{j+2}^++Q_1^--Q_4^-)+Q_2^+(V_7^--Q_4^-)-V_4^-Q_3^+$, $M_j^{17}=K_j^+(K_{j+2}^++P_1^+-P_4^+)+P_1^+(U_7^+-P_4^+)-U_4^-P_4^++L_j^+(L_{j+2}^++Q_1^+-Q_4^+)+Q_1^+(V_7^+-Q_4^+)-V_4^-Q_4^+$.

$$\begin{aligned}\rho_{5j}=&\{(M'_{1j}\rho_{11}A_{22}B_{2j}+M'_{2j}\rho_{12}A_{21}B_{2j}+M'_{3j}\rho_{21}A_{12}B_{2j}\\&-M'_{4j}\rho_{22}A_{11}B_{2j}+M'_{5j}\mu_{11}A_{22}B_{1j}+M'_{6j}\mu_{12}A_{21}B_{1j}\\&+M'_{7j}\mu_{21}A_{12}B_{1j}-M'_{8j}\mu_{22}A_{11}B_{1j}-M'_{9j}\sigma_{1j}\rho_{24}\\&-M'_{10,j}\sigma_{2j}\rho_{14}-M'_{11,j}\delta_{17}\rho_{3j}-M'_{12,j}a_{22}\Psi_{1j}\\&-M'_{13,j}a_{12}\Psi_{2j}-M'_{14,j}\delta_{11}o_{22}-M'_{15,j}\delta_{12}o_{21}\\&-M'_{16,j}\delta_{21}o_{12}-M'_{17,j}\delta_{22}o_{11})\}\{K_1^+(K_4^++U_5^{+*}+P_{j+2}^+)\\&+P_j^+(K_4^++P_{j+2}^+)+K_4^+U_8^{+*}+L_1^+(L_4^++V_5^{+*}+Q_{j+2}^+)\\&+Q_j^+(L_4^++Q_{j+2}^+)+L_4^+V_8^{+*}\}^{-1},\end{aligned}$$

with $j=1$, $o=\nu$; $j=2$, $o=\phi$, and $M'_{1j}=K_1^+(K_4^+-U_6^{-*}-P_j^-)-P_{j+2}^+(K_4^+-P^-)-U_3^-K_4^++L_1^+(L_4^+-V_6^{-*}-Q_j^-)-Q_{j+2}^+(L_4^+-Q^-)-V_3^-L_4^+$, $M'_{2j}=K_3^+(K^-+U_8^{-*}+P_j^-)+P_{j+2}^+(K_2^++P^-)-U_7^{+*}K_2^++L_3^+(L^-+V_8^{-*}+Q_j^-)+Q_{j+2}^+(L_2^++Q^-)-V_7^{+*}L_2^+$, $M'_{3j}=K_3^+(K_1^-+U_8^{+*}-P_j^-)-P_{j+2}^+(K_2^+-P^-)-U_1^-K_2^++L_3^+(L_1^-+V_8^{+*}-Q_j^-)-Q_{j+2}^+(L_2^+-Q^-)-V_1^-L_2^+$, $M'_{4j}=K_1^+(K^--U_8^{-*}-P_j^-)-P_{j+2}^+(K_4^++P^-)+U_7^{+*}K_4^++L_1^+(L^--V_8^{-*}-Q_j^-)-Q_{j+2}^+(L_4^++Q^-)+V_7^{+*}L_4^+$, $M'_{5j}=K_1^+(K_4^+-U_6^{+*}+P_j^+)-P_j^+(K_4^++P^-)-U_4^-K_4^++L_1^+(L_4^+-V_6^{+*}+Q_j^+)-Q_j^+(L_4^++Q^-)-V_4^-L_4^+$, $M'_{6j}=K_3^+(K^-+U_8^{+*}-P_j^+)+P_j^+(K_2^+-P^-)-U_8^{+*}K_2^++L_3^+(L^-+V_8^{+*}-Q_j^+)+Q_j^+(L_2^+-Q^-)-V_8^{+*}L_2^+$, $M'_{7j}=K_3^+(K_1^-+U_8^{-*}+P_j^+)-P_j^+(K_2^++P^-)-U_2^-K_2^++L_3^+(L_1^-+V_8^{-*}+Q_j^+)-Q_j^+(L_2^++Q^-)-V_2^-L_2^+$, $M'_{8j}=K_1^+(K^--U_8^{+*}+P_j^+)-P_j^+(K_4^+-P^-)+U_8^{+*}K_4^++L_1^+(L^--V_8^{+*}+Q_j^+)-Q_j^+(L_4^+-Q^-)+V_8^{+*}L_4^+$, $M'_{9j}=K_1^+(K_4^++U_6^{+*}-P_j^+)-P_j^+(K_4^+-P^-)+U_8^{+*}K_4^++L_1^+(L_4^++V_6^{+*}-Q_j^+)-Q_j^+(L_4^+-Q^-)+V_8^{+*}L_4^+$, $M'_{10,j}=K_1^+(K_4^++U_6^{-*}+P_j^+)-P_{j+2}^+(K_4^++P^-)+U_7^{+*}K_4^++L_1^+(L_4^++V_6^{-*}+Q_j^+)-Q_{j+2}^+(L_4^++Q^-)+V_7^{+*}L_4^+$, $M'_{11,j}=K_1^+(K^++U_8^{-*}-P_j^+)-P_{j+2}^+(K_4^+-P_j^+)+U_3^-K_4^++L_1^+(L^++V_8^{-*}-Q_j^+)-Q_{j+2}^+(L_4^+-Q_j^+)+V_3^-L_4^+$, $M'_{12,j}=K_1^+(K^++U_8^{-*}+P_j^+)+P_j^+(K_4^+-P^-)+U_4^-K_4^++L_1^+(L^++V_8^{-*}+Q_j^+)+Q_j^+(L_4^+-Q^-)+V_4^-L_4^+$, $M'_{13,j}=K_1^+(K^++U_8^{+*}-P_j^+)+P_{j+2}^+(K_4^+-P^-)+U_3^-K_4^++L_1^+(L^++V_8^{+*}-Q_j^+)+Q_{j+2}^+(L_4^+-Q^-)+V_3^-L_4^+$, $M'_{14,j}=K_1^+(K^--U_8^{+*}-P_j^+)+P_j^+(U_7^++U_8^{+*})+$

$U_8^{+*}K_4^+ + L_1^+(L^- - V_8^{+*} - Q_j^+) + Q_j^+(V_7^+ + V_8^{+*}) + V_8^{+*}L_4^+$, $M'_{15,j} = K_3^+(K_1^- + U_8^{-*} - P_j^+) + P_j^+(U_5^- + U_8^{-*}) - U_2^-K_2^+ + L_3^+(L_1^- + V_8^{-*} - Q_j^+) + Q_j^+(V_5^- + V_8^{-*}) - V_2^-L_2^+$, $M'_{16,j} = K_3^+(K^- + U_8^{+*} + P_j^+) - P_j^+(K_2^+ + P_{j+2}^+) - U_8^{+*}K_2^+ + L_3^+(L^- + V_8^{+*} + Q_j^+) - Q_j^+(L_2^+ + Q_{j+2}^+) - V_8^{+*}L_2^+$, $M'_{17,j} = K_1^+(U_6^{+*} - K_4^+ + P_j^+) - P_j^+(K_4^+ - P_{j+2}^+) + U_4^-K_4^+ + L_1^+(V_6^{+*} - L_4^+ + Q_j^+) - Q_j^+(L_4^+ - Q_{j+2}^+) + V_4^-L_4^+$.

$$\begin{aligned}\delta_{18} = \{&N_1 - N_2\delta_{11}\Phi_{22} - N_3\delta_{12}\Phi_{21} - N_4\delta_{21}\Phi_{12} - N_5\delta_{22}\Phi_{11}\\ &- N_6\sigma_{11}\Psi_{22} - N_7\sigma_{12}\Psi_{21} - N_8\sigma_{21}\Psi_{12} - N_9\sigma_{22}\Psi_{11}\\ &- N_{10}\delta_{17}\delta_{19} - N_{11}A_{11}B_{11} - N_{12}A_{11}B_{12} - N_{13}A_{11}B_{21}\\ &- N_{14}A_{11}B_{22} - N_{15}A_{21}B_{11} - N_{16}A_{21}B_{12} - N_{17}A_{21}B_{21}\\ &- N_{18}A_{21}B_{22}\}\{K_+^+(K_+^+ + 2P_+^+) + (P_+^+)^2\\ &+ L_+^+(L_+^+ + 2Q_+^+) + (Q_+^+)^2\}^{-1}.\end{aligned}$$

with $K_\pm^+ = (K_1^+ \pm K_4^+)$, $K_-^- = (K_1^- - K_4^-)$, $L_\pm^+ = (L_1^+ \pm L_4^+)$, $L_-^- = (L_1^- - L_4^-)$, $P_\pm^+ = (P_1^+ \pm P_4^+)$, $P_-^- = (P_1^- - P_4^-)$, $Q_\pm^+ = (Q_1^+ \pm Q_4^+)$, $Q_-^- = (Q_1^- - Q_4^-)$, and $N_2 = K_-^+(K_-^+ - 2P_+^+) + (P_+^+)^2 + L_-^+(L_-^+ - 2Q_+^+) + (Q_+^+)^2$, $N_3 = K_-^-(K_-^- - 2P_+^+) + (P_+^+)^2 + L_-^-(L_-^- - 2Q_+^+) + (Q_+^+)^2$, $N_4 = K_-^-(K_-^- + 2P_+^+) + (P_+^+)^2 + L_-^-(L_-^- + 2Q_+^+) + (Q_+^+)^2$, $N_5 = K_-^+(K_-^+ + 2P_+^+) + (P_+^+)^2 + L_-^+(L_-^+ + 2Q_+^+) + (Q_+^+)^2$, $N_6 = K_+^+(K_+^+ - 2P_-^+) + (P_-^+)^2 + L_+^+(L_+^+ - 2Q_-^+) + (Q_-^+)^2$, $N_7 = K_+^+(K_+^+ - 2P_-^-) + (P_-^-)^2 + L_+^+(L_+^+ - 2Q_-^-) + (Q_-^-)^2$, $N_8 = K_+^+(K_+^+ + 2P_-^-) + (P_-^-)^2 + L_+^+(L_+^+ + 2Q_-^-) + (Q_-^-)^2$, $N_9 = K_+^+(K_+^+ + 2P_-^+) + (P_-^+)^2 + L_+^+(L_+^+ + 2Q_-^+) + (Q_-^+)^2$, $N_{10} = K_+^+(K_+^+ - 2P_+^+) + (P_+^+)^2 + L_+^+(L_+^+ - 2Q_+^+) + (Q_+^+)^2$, $N_{11} = K_-^+(K_-^+ + 2P_-^+) + (P_-^+)^2 + L_-^+(L_-^+ + 2Q_-^+) + (Q_-^+)^2$, $N_{12} = K_-^+(K_-^+ + 2P_-^-) + (P_-^-)^2 + L_-^+(L_-^+ + 2Q_-^-) + (Q_-^-)^2$, $N_{13} = K_-^+(K_-^+ - 2P_-^-) + (P_-^-)^2 + L_-^+(L_-^+ - 2Q_-^-) + (Q_-^-)^2$, $N_{14} = K_-^+(K_-^+ - 2P_-^+) + (P_-^+)^2 + L_-^+(L_-^+ - 2Q_-^+) + (Q_-^+)^2$, $N_{15} = K_-^-(K_-^- - 2P_-^+) + (P_-^+)^2 + L_-^-(L_-^- - 2Q_-^+) + (Q_-^+)^2$, $N_{16} = K_-^-(K_-^- - 2P_-^-) + (P_-^-)^2 + L_-^-(L_-^- - 2Q_-^-) + (Q_-^-)^2$, $N_{17} = K_-^-(K_-^- + 2P_-^-) + (P_-^-)^2 + L_-^-(L_-^- + 2Q_-^-) + (Q_-^-)^2$, $N_{18} = K_-^-(K_-^- + 2P_-^+) + (P_-^+)^2 + L_-^-(L_-^- + 2Q_-^+) + (Q_-^+)^2$, $N_{11} = a_{11}\Delta_{52}^* + a_{11}^*\Delta_{52} + a_{12}\rho_{52}^* + a_{12}^*\rho_{52} + a_{21}\Delta_{51}^* + a_{21}^*\Delta_{51} + a_{22}\rho_{51}^* + a_{22}^*\rho_{51} + \Delta_{14}\Delta_{32}^* + \Delta_{14}^*\Delta_{32} + \Delta_{24}\Delta_{31}^* + \Delta_{24}^*\Delta_{31} + \rho_{14}\rho_{32}^* + \rho_{14}^*\rho_{32} + \rho_{24}\rho_{31}^* + \rho_{24}^*\rho_{31} + \Delta_{11}T_{22}^* + \Delta_{11}^*T_{22} + \Delta_{12}T_{12}^* + \Delta_{12}^*T_{12} + \Delta_{21}T_{21}^* + \Delta_{21}^*T_{21} + \Delta_{22}T_{11}^* + \Delta_{22}^*T_{11} + \Theta_{11}\Lambda_{22}^* + \Theta_{11}^*\Lambda_{22} + \Theta_{12}\Lambda_{12}^* + \Theta_{12}^*\Lambda_{12} + \Theta_{21}\Lambda_{21}^* + \Theta_{21}^*\Lambda_{21} + \Theta_{22}\Lambda_{11}^* + \Theta_{22}^*\Lambda_{11} + \phi_{11}\rho_{22}^* + \phi_{11}^*\rho_{22} + \phi_{12}\rho_{12}^* + \phi_{12}^*\rho_{12} + \phi_{21}\rho_{21}^* + \phi_{21}^*\rho_{21} + \phi_{22}\rho_{11}^* + \phi_{22}^*\rho_{11} + \mu_{11}\nu_{22}^* + \mu_{11}^*\nu_{22} + \mu_{12}\nu_{12}^* + \mu_{12}^*\nu_{12} + \mu_{21}\nu_{21}^* + \mu_{21}^*\nu_{21} + \mu_{22}\nu_{11}^* + \mu_{22}^*\nu_{11}$.

All s and j not specified in this paper are $s, j = 1, 2$.

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