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Residual symmetry, CRE integrability and interaction solutions of two higher-dimensional shallow water wave equations

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Two $(3+1)$-dimensional shallow water wave equations are studied by using residual symmetry and the consistent Riccati expansion (CRE) method. Through localization of residual symmetries, symmetry reduction solutions of the two equations are obtained. The CRE method is applied to the two equations to obtain new Bäcklund transformations from which a type of interesting interaction solution between solitons and periodic waves is generated.

Keywords: $(3+1)$-dimensional shallow water wave equation, residual symmetry, consistent Riccati expansion

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1. Introduction

In the past few decades, various methods have been proposed and developed to obtain exact solutions to nonlinear partial differential equations (PDEs). These include the inverse scattering transform,[1] Hirota's bilinear method,[2] Darboux transformation,[3,4] Painlevé analysis,[5] the Riemann–Hilbert approach,[6,7] etc. It is well known that symmetry analysis can be used not only for simplifying PDEs but also for obtaining their exact solutions. Thanks to Lie's first theorem,[8] one can find the finite transformation corresponding to a Lie point symmetry group. There are standard methods to find the Lie point symmetry group; for example, one can use the classical Lie group approach or the nonclassical Lie group approach to obtain Lie point symmetry and similarity solutions of a nonlinear PDE.[9–11] On the other hand, although there is no unified way to obtain the nonlocal symmetries of nonlinear systems, we may obtain them through inverse recursion operators,[12] Lax pairs,[13] Bäcklund transformation,[14] conformal invariance,[15] potential systems,[16] etc.

In recent years, Lou found that, for many integrable systems, a nonlocal symmetry called residual symmetry can be obtained via truncated Painlevé expansion.[16] It was found that the residual symmetry of a system can be localized into a Lie point symmetry by prolonging the system by introducing new dependent variables. In this way, many nonlinear systems have been studied with residual symmetry and new symmetry reduction solutions have been obtained, such as the nonlinear Schrödinger equation,[17] the Kadomtsev–Petviashvili equation,[18] the Ablowitz-Kaup–Newell–Segur system[19] and the Gardner equation.[20]

The auxiliary equation method is a simple but effective way to get special solutions, including traveling solitary wave solutions of nonlinear equations. To obtain more generalized solutions, Lou[21] generalized the Riccati expansion method to the consistent Riccati expansion (CRE) method, through which a new integrable property called the CRE integrable is defined and new interaction solutions between solitons and periodic waves are generated for many nonlinear equations.[22–28] The observation of solitons interacting with periodic background waves has been documented in experimental studies.[29]

In this paper, using residual symmetry and the CRE method, we focus on two $(3+1)$-dimensional shallow water wave equations as follows:[30]

$$u_{yzt}+u_{xxxyz}-6u_xu_{xyz}-6u_{xz}u_{xy}=0, \tag{1}$$

$$u_{xzt}+u_{xxxyz}-2(u_{xx}u_{yz}+u_yu_{xxz})-4(u_xu_{xyz}+u_{xz}u_{xy})=0, \tag{2}$$

where $u=u(x,y,z,t)$ in Eqs. (1) and (2) is the free-surface displacement from the equilibrium level of an inviscid incompressible liquid and both of these equations can be reduced to the potential KdV equation by setting $x=y=z$. Both Eqs. (1) and (2) can be considered as natural expansions of low-dimensional shallow water wave equations, which have many important applications in physical situations such as fluid flows, marine environments, solid-state physics and plasma physics. In Ref. [31], new soliton-like solutions are obtained for the $(2+1)$-dimensional generalized shallow water wave equation. New analytic solutions, including periodic solitary wave, cross-kink soliton and doubly periodic breather-type solutions, were obtained for a type of $(3+1)$-dimensional shallow water wave equation by using an optimal system of Lie symmetry vectors.[32] In Ref. [33], Hirota's bilinear method was used to determine the multiple-soliton solutions of the $(3+1)$-dimensional shallow water wave equations (1) and (2), which also testified the complete integrability of the two equations. Despite these abundant findings on the $(3+1)$-dimensional shallow water wave equations, to our

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knowledge, interaction solutions between solitons and background periodic waves have not yet been obtained and are hard to obtain using traditional methods.

This paper is organized as follows. In Section 2, residual symmetries are obtained and localized in two new prolonged systems for Eqs. (1) and (2), respectively, and new Bäcklund transformations of these two equations are constructed using Lie's first theorem. In Section 3, using the standard Lie symmetry method we first get the Lie point symmetry group and then give corresponding symmetry reduction solutions for both of the two shallow water wave equations (1) and (2). In Section 5, using the CRE method, we give new Bäcklund transformations of the shallow water wave equations (1) and (2) from which interaction solutions between solitons and periodic waves are generated. The last section provides a short summary.

2. Residual symmetries and related Bäcklund transformations for Eqs. (1) and (2)

For Eqs. (1) and (2), we take truncated the Painlevé expansion as

$$u = \frac{u_0}{\phi} + u_1, \tag{3}$$

where ϕ is a singular manifold and u_0, u_1 are functions of x, y, z, t. By substituting Eq. (3) into Eqs. (1) and (2) and vanishing all different powers of $1/\phi$, we have

$$u_0 = -2\phi_x, \tag{4}$$

$$u_1 = \frac{\phi_{xx}}{2\phi_x} + \frac{1}{4}\int \frac{\phi_{xx}^2}{\phi_x^2}\,\mathrm{d}x, \tag{5}$$

for both Eqs. (1) and (2). Then, the $(3+1)$-dimensional shallow water wave equations can be reduced to their Schwarzian form

$$K + C = 0 \tag{6}$$

for Eq. (1) and

$$KP + P_{xx} + C + \int P_x K\,\mathrm{d}x = 0 \tag{7}$$

for Eq. (2) with Schwarzian variables $K = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2}$, $C = \frac{\phi_t}{\phi_x}$ and $P = \frac{\phi_y}{\phi_x}$.

Substituting Eqs. (4) and (5) into Eq. (3), we get a new Bäcklund transformation for Eq. (1) (or Eq. (2)).

Theorem 1 If ϕ is a solution of the Schwartzian equation (6) (or Eq. (7)), then

$$u = -\frac{2\phi_x}{\phi} + \frac{\phi_{xx}}{2\phi_x} + \frac{1}{4}\int \frac{\phi_{xx}^2}{\phi_x^2}\,\mathrm{d}x \tag{8}$$

is a solution of Eq. (1) (or Eq. (2)).

As we know, any Schwarzian-form equations are invariant under the Möbious transformation

$$\phi \longrightarrow \frac{a\phi + b}{c\phi + d}, \quad ac \neq bd, \tag{9}$$

which means that Schwarzian equations have symmetries $\sigma_\phi = d_1$, $\sigma_\phi = d_2\phi$ and

$$\sigma_\phi = d_3\phi^2, \tag{10}$$

with arbitrary constants d_1, d_2 and d_3. It can be verified that u_0 in Eq. (4) satisfies the linearized equations of Eqs. (1) and (2) so we get a nonlocal symmetry for both of the two equations, which is called residual symmetry.

As nonlocal symmetry cannot be used to construct its finite transformation, for the residual symmetry of Eq. (4) we localize it in a new prolonged system by introducing a new variable

$$g \equiv \phi_x. \tag{11}$$

It can be verified that the symmetries of

$$\sigma_u = 2g, \tag{12a}$$

$$\sigma_g = 2g\phi, \tag{12b}$$

$$\sigma_\phi = \phi^2 \tag{12c}$$

satisfy the linearized system of Eqs. (1), (6), (11) and the linearized system of Eqs. (2), (7) and (11), which means that the residual symmetry can be localized in the two prolonged systems. Equivalently, the symmetry of Eq. (12) can be expressed in a vector form as

$$V = 2g\partial_u + 2g\phi\partial_g + \phi^2\partial_\phi \tag{13}$$

for both of the two prolonged systems.

By using Lie's first theorem and solving the initial value problem of Eq. (12)

$$\frac{\mathrm{d}\hat{u}(\varepsilon)}{\mathrm{d}\varepsilon} = 2\hat{g}(\varepsilon), \quad \hat{u}(0) = u, \tag{14}$$

$$\frac{\mathrm{d}\hat{g}(\varepsilon)}{\mathrm{d}\varepsilon} = 2\hat{\phi}(\varepsilon)\hat{g}(\varepsilon), \quad \hat{g}(0) = g, \tag{15}$$

$$\frac{\mathrm{d}\hat{\phi}(\varepsilon)}{\mathrm{d}\varepsilon} = \hat{\phi}(\varepsilon)^2, \quad \hat{\phi}(0) = \phi, \tag{16}$$

we get the following theorem.

Theorem 2 If $\{u, g, \phi\}$ is a solution of the prolonged system (1), (6), (11) (or (2), (7) and (11)), then so is $\{\hat{u}, \hat{g}, \hat{\phi}\}$ with

$$\hat{u} = u - \frac{2\varepsilon g}{\varepsilon\phi - 1}, \tag{17a}$$

$$\hat{g} = \frac{g}{(\varepsilon\phi - 1)^2}, \tag{17b}$$

$$\hat{\phi} = -\frac{\phi}{\varepsilon\phi - 1}, \tag{17c}$$

where ε is an arbitrary group parameter.

3. New symmetry reduction solutions of Eqs. (1) and (2)

For the prolonged shallow water system (1), (6), (11) the Lie point symmetry can be written in vector form as

$$V = X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z}+T\frac{\partial}{\partial t} +U\frac{\partial}{\partial u}+G\frac{\partial}{\partial g}+\Phi\frac{\partial}{\partial \phi}, \tag{18}$$

or, equivalently, written as a function form

$$\sigma_u = Xu_x+Yu_y+Zu_z+Tu_t-U, \tag{19a}$$
$$\sigma_g = Xg_x+Yg_y+Zu_z+Tg_t-G, \tag{19b}$$
$$\sigma_\phi = X\phi_x+Y\phi_y+Zu_z+T\phi_t-\Phi, \tag{19c}$$

which means that the prolonged shallow water system (1), (6), (11) is invariant under the transformation

$$\{x,y,z,t,u,g,\phi\} \to \{x+\varepsilon X, y+\varepsilon Y, z+\varepsilon Z, t+\varepsilon T, u+\varepsilon U, g+\varepsilon g, \phi+\varepsilon\Phi\}. \tag{20}$$

By substituting Eq. (19) into the linearized equations of the system (1), (6), (11) and vanishing all coefficients of different derivatives of the independent variables of u, g and ϕ, we get over-determined linear equations for the infinitesimals X,Y,Z,T,U,G,Φ, calculated by computer algebra. The final result is

$$\begin{aligned} &X = \frac{c_1x}{3}+f_7(y,z),\ \ Y = f_2(y),\ \ Z = f_1(z),\ \ T = c_1t+c_2,\\ &U = -\frac{c_1u}{3}+f_3(y,z)g+f_6(y,z)+f_5(y,t)+f_4(z,t),\\ &G = \frac{1}{3}g\left[3f_3(y,z)\phi+3f_8(y,z)-c_1\right],\\ &\Phi = \frac{1}{2}f_3(y,z)\phi^2+f_8(y,z)\phi+f_9(y,z), \end{aligned} \tag{21}$$

where f_i $(i=1,2,\ldots,9)$ are arbitrary functions of indicated variables and c_1, c_2 are arbitrary constants. By setting $f_3(y,z)=2$ and f_i $(i=1,2,4,5,6,7,8,9)=c_1=c_2=0$ the symmetry of Eq. (21) is degenerated into the special case of Eq. (13).

Substituting Eq. (21) into Eq. (19) we get

$$\begin{aligned} \sigma_u &= \left(\frac{c_1x}{3}+f_7(y,z)\right)u_x+f_2(y)u_y+(c_1t+c_2)u_t\\ &\quad+\frac{c_1u}{3}-f_3(y,z)g-f_6(y,z)-f_5(y,t)-f_4(z,t),\\ \sigma_g &= \left(\frac{c_1x}{3}+f_7(y,z)\right)g_x+f_2(y)g_y+(c_1t+c_2)g_t\\ &\quad-\frac{1}{3}g\left[3f_3(y,z)\phi+3f_8(y,z)-c_1\right],\\ \sigma_\phi &= \left(\frac{c_1x}{3}+f_7(y,z)\right)\phi_x+f_2(y)\phi_y+(c_1t+c_2)\phi_t\\ &\quad-\frac{1}{2}f_3(y,z)\phi^2-f_8(y,z)\phi-f_9(y,z). \end{aligned} \tag{22}$$

The group invariant solutions of the prolonged system (1), (6), (11) can be obtained by solving the equations in Eq. (22) under symmetry constraints $\sigma_u=\sigma_g=\sigma_\phi=0$. Without loss of generality, considering the case of $f_7(y,z)=c_7, f_1(z)=c_3z+c_4, f_2(y)=c_5y+c_6, f_3=1, f_8(y,z)=c_{11}, f_4(z,t)=f_5(y,t)=f_6(y,z)=0$, we get

$$\begin{aligned} \phi &= -\tanh\left[\frac{\delta(c_1\Phi+\ln(c_1t+c_2))}{2c_1}\right]\delta-c_{10},\\ \delta &= \sqrt{c_{10}^2-2c_{11}}, \end{aligned} \tag{23}$$

$$g = -\frac{2G}{(c_1t+c_2)^{\frac{1}{3}}\left[\frac{\cosh[\delta(c_1\Phi+\ln(c_1t+c_2))]}{c_1}+1\right]}, \tag{24}$$

$$u = \frac{U}{(c_1t+c_2)^{\frac{1}{3}}}+\frac{4G}{\delta(c_1t+c_2)^{\frac{1}{3}}\left[\mathrm{e}^{\delta\Phi}(c_1t+c_2)^{\frac{\delta}{c_1}}+1\right]}, \tag{25}$$

where U, G and Φ are group invariant functions of new group invariant variables of

$$\xi = \frac{c_1x+3c_9}{(c_1t+c_2)^{\frac{1}{3}}c_1},\ \ \eta = \frac{c_5y+c_6}{(c_1t+c_2)^{\frac{c_5}{c_1}}c_5},\ \ \tau = \frac{c_3z+c_4}{(c_1t+c_2)^{\frac{c_3}{c_1}}c_3}.$$

The corresponding symmetry reduction equations of the prolonged system (1), Eqs. (6) and (11) can be found by substituting Eq. (23) into this system and the results are

$$G-\frac{1}{2}\delta^2\Phi_\xi = 0, \tag{26}$$

$$\begin{aligned} &6c_5\Phi_\eta\Phi_\xi\eta+6c_3\Phi_\tau\Phi_\xi\tau+3\delta^2\Phi_\xi^4+2c_1\Phi_\xi^2\xi\\ &-6(\Phi_{\xi\xi\xi}+1)\Phi_\xi+9\Phi_{\xi\xi}^2 = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} &3c_5\Phi_\eta\Phi_\xi\eta+3c_3\Phi_\tau\Phi_\xi\tau+3\delta^2\Phi_\xi^4\\ &+(c_1\xi-6\Phi_{\xi\xi}\delta-6U_\xi)\Phi_\xi^2-3\Phi_\xi+3\Phi_{\xi\xi}^2 = 0. \end{aligned} \tag{28}$$

By solving Eq. (28) for Φ, we can get solutions for G and U by Eqs. (26) and (27), and then new solutions of the shallow water equation (1) by Eq. (25). To give a concrete example, by taking a special solution of Eq. (28) as

$$\Phi = -ck_2\xi+l_1\eta+\omega_1\tau+cE_{\mathrm{F}}(\mathrm{sn}(l_2\eta+k_2\xi+\omega_2\tau,m),m), \tag{29}$$

where E_{F} is an incomplete elliptic integral of the first kind and $c, l_1, \omega_1, l_2, \omega_2, k_2, m, n$ are all arbitrary constants, we get a new solution of Eq. (1)

$$u = \frac{2\delta ck_2\left[\mathrm{cn}(X,m)\mathrm{dn}(X,m)-S(X,m)\right]}{S(X,m)(c_1t+c_2)^{\frac{1}{3}}\left\{\exp\{\delta[-ck_2\xi+l_1\eta+\omega_1\tau+cE_{\mathrm{F}}(\mathrm{sn}(X,m),m)]\}(c_1t+c_2)^{\frac{\delta}{c_1}}+1\right\}},$$

$$S(X,m)=\sqrt{1-\mathrm{sn}(X,m)^2}\sqrt{1-\mathrm{sn}(X,m)^2m^2},\ \ X=l_2\eta+k_2\xi+\omega_2\tau, \tag{30}$$

which can be verified by substituting it into Eq. (1).

For the prolonged system (2), (7), (11), using a similar procedure as in the case of the prolonged system (1), (6), (11), we can get the Lie point symmetry as

$$\begin{aligned}
&X=-\frac{xf_4(z)}{2}+\frac{xf_1(z)}{2}+f_6(z,t),\ \ Y=f_4(z)y+f_5(z),\\
&Z=f_3(z),\ \ T=f_1(z)t+f_2(z),\\
&U=\frac{uf_4(z)}{2}-\frac{uf_1(z)}{2}+f_7(y,z)g-\frac{f_{6,t}(z,t)y}{2}+f_8(z,t),\\
&G=\frac{1}{2}g[2f_7(y,z)\phi-f_1(z)+f_4(z)+2f_9(y,z)],\\
&\Phi=\frac{1}{2}f_7(y,z)\phi^2+f_9(y,z)\phi+f_{10}(y,z).
\end{aligned}\tag{31}$$

where f_i $(i=1,2,\ldots,10)$ are arbitrary functions of indicated variables. When taking $f_7(y,z)=2$, $f_i=0$ $(i\neq 7)$ in Eq. (31), the symmetry degenerates into the special case of Eq. (13).

By setting $f_1(z)=c_1$, $f_2(z)=c_2$, $f_3(z)=c_3z+c_4$, $f_4(z)=c_5$, $f_5(z)=c_6$, $f_6(z,t)=c_7$, $f_7(y,z)=1$, $f_8(z,t)=0$, $f_9(y,z)=c_8$, $f_{10}(y,z)=c_9$ with c_i $(i=1,2,\ldots,9)$ being arbitrary constants in Eq. (31), we get symmetry reduction solutions of the prolonged system (2), (7), (11)

$$\begin{aligned}
&\phi=-\tanh\left[\frac{\delta_1(c_1\Phi+\ln(c_1t+c_2))}{2c_1}\right]\delta_1-c_8,\\
&\delta_1=\sqrt{c_8^2-2c_9},
\end{aligned}\tag{32}$$

$$g=-\frac{2(c_1t+c_2)^{\frac{c_5-c_1}{2c_1}}G}{\left[\frac{\cosh[\delta_1(c_1\Phi+\ln(c_1t+c_2))]}{c_1}+1\right]}, \tag{33}$$

$$u=(c_1t+c_2)^{\frac{c_5-c_1}{2c_1}}U+\frac{4(c_1t+c_2)^{\frac{c_5-c_1}{2c_1}}G}{\delta_1\left[\mathrm{e}^{\delta_1\Phi}(c_1t+c_2)^{\frac{\delta_1}{c_1}}+1\right]}, \tag{34}$$

where U, G, Φ are group invariant functions of new group invariant variables

$$\begin{aligned}
&\xi=\frac{(c_1-c_5)x+2c_7}{(c_1t+c_2)^{\frac{c_1-c_5}{2c_1}}(c_1-c_5)},\ \ \eta=\frac{c_5y+c_6}{(c_1t+c_2)^{\frac{c_5}{c_1}}c_5},\\
&\tau=\frac{c_3z+c_4}{(c_1t+c_2)^{\frac{c_3}{c_1}}c_3}.
\end{aligned}$$

By substituting Eq. (32) into the prolonged system (2), (7), (11) we get the corresponding symmetry reduction equations

$$G-\frac{1}{2}\delta_1^2\Phi_\xi=0, \tag{35}$$

$$\begin{aligned}
&2\delta_1^2\Phi_\eta\Phi_\xi^4+[(c_5-c_1)\xi-4\Phi_{\xi\eta}\delta_1-4U_\eta]\Phi_\xi^3\\
&-2\{[4\Phi_{\xi\xi}\delta_1+c_5\eta+4U_\xi]\Phi_\eta+c_3\Phi_\tau\tau-2\Phi_{\xi\xi\eta}-1\}\Phi_\xi^2\\
&+4(\Phi_\eta\Phi_{\xi\xi\xi}-\Phi_{\xi\eta}\Phi_{\xi\xi})\Phi_\xi-2\Phi_\eta\Phi_{\xi\xi}^2=0,
\end{aligned}\tag{36}$$

$$\begin{aligned}
&2\delta_1^2\Phi_{\xi\eta}\Phi_\xi^4+(c_1-c_5)\Phi_\xi^3+(2c_3\Phi_{\xi\tau}\tau+2c_5\Phi_{\xi\eta}\eta\\
&-2\Phi_{\xi\xi\xi\eta})\Phi_\xi^2+[2(3\Phi_{\xi\xi\eta}+1-c_3\Phi_\tau\tau-c_5\Phi_\eta\eta)\Phi_{\xi\xi}\\
&+2\Phi_{\xi\eta}\Phi_{\xi\xi\xi}]\Phi_\xi-6\Phi_{\xi\eta}\Phi_{\xi\xi}^2=0,
\end{aligned}\tag{37}$$

Solving Φ by Eq. (37), we can obtain new exact solutions of Eq. (2) by using Eqs. (36), (35) and (34). In particular, when solutions of Eq. (37) are taken as periodic functions, the solutions of (34) are interaction solutions between solitons and background periodic waves.

4. CRE integrability and new interaction solutions of Eqs. (1) and (2)

Using the CRE method, the generalized Riccati expansions of Eqs. (1) and (2), which can be determined by balancing the highest nonlinear term and the dispersion term, take the form

$$u=v_0+v_1R(w),\ \ (w=w(x,y,t)), \tag{38}$$

where v_0,v_1 are arbitrary functions of (x,y,z,t) and $R(w)$ is a solution of the Riccati equation

$$R_w=a_0+a_1R+a_2R^2. \tag{39}$$

Substituting Eq. (38) with Eq. (39) into Eqs. (1) and (2), respectively, and vanishing different powers of $R(w)$, we obtain the same expressions of

$$v_1=2a_2w_x, \tag{40}$$

$$v_{0,x}=-a_0a_2w_x^2+\frac{1}{4}a_1^2w_x^2+a_1w_{xx}+\frac{w_{xxx}}{2w_x}-\frac{w_{xx}^2}{4w_x^2}, \tag{41}$$

for both Eqs. (1) and (2). It can be checked that all equations obtained by vanishing different powers of $R(w)$ for Eqs. (1) and (2) are consistent with each other, which proves that both of the shallow water equations are CRE integrable.[21] Finally, we obtain

$$\begin{aligned}
&2w_x^2(QPC_x+QPK_x+QC_y+QK_y+PC_z+PK_z)\\
&+4QP_xw_x^2[\delta_2w_x^2+K+C]+w_x^2[\delta_2w_x^2+2(K+C)]P_z\\
&+w_x^2[5P\delta_2w_x^2+6P(K+C)]Q_x+12PQw_xw_{xx}[\delta_2w_x^2+(K+C)]\\
&+6PQa_1\delta_2w_x^5+12PQa_1(K+C)w_x^3=0
\end{aligned}\tag{42}$$

for Eq. (1) and

$$C_x+K_y+\delta_2P_xw_x^2+P\delta_2w_{xx}w_x=0 \tag{43}$$

for Eq. (2), where $\delta_2=4a_0a_2-a_1^2$ and the Schwarzian variables are $K=\frac{w_{xxx}}{w_x}-\frac{3}{2}\frac{w_{xx}^2}{w_x^2}$, $C=\frac{w_t}{w_x}$, $P=\frac{w_y}{w_x}$ and $Q=\frac{w_z}{w_x}$.

From the above discussions we get a Bäcklund transformation for Eqs. (1) and (2), which can be summarized as follows.

Theorem 3 If w is a solution of Eq. (42) (or (43)), then

$$u = 2a_2 w_x R(w) - a_0 a_2 w_x^2 + \frac{1}{4}a_1^2 w_x^2 + a_1 w_{xx} + \frac{w_{xxx}}{2w_x} - \frac{w_{xx}^2}{4w_x^2} \tag{44}$$

is a solution of Eq. (1) (or Eq. (2)), where $R = R(w)$ is a solution of the Riccati equation (39).

As a special case of the CRE method we consider the application of the consistent tanh expansion (CTE) (where $R(w) = \tanh(w)$ in Eq. (39)) on Eqs. (1) and (2). To this end, we take

$$u = u_1 \tanh(w) + u_0, \tag{45}$$

and substitute it into Eqs. (1) and (2). Similar to the CRE method, by vanishing different powers of $\tanh(w)$, Eq. (1) is changed to

$$\begin{aligned}
&16PQw_x^9 - \{[4(2C+7K)P+6P_{xx}]Q + 32Q_xP_x + 14PQ_{xx} \\
&+8P_{xz}\}w_x^7 - (56P_xQ + 70Q_xP + 14P_z)w_{xx}w_x^6 - \{84PQw_{xx}^2 \\
&-[4(CK+K^2)P + P_{xx}C + (C_x+K_x)P_x + KP_{xx} + C_{xy} + K_{xy}]Q \\
&-[3Q_{xx}C + 2(C_x+K_x)Q_x + 3KQ_{xx} + C_{xz} + K_{xz}]P \\
&-2(2Q_xP_x + P_{xz})C - 2(2KP_x + C_y + K_y)Q_x - 2(C_z+K_z)P_x \\
&-(C_x+K_x)P_z - 2P_{xz}K - C_{yz} - K_{yz}\}w_x^5 + \{[3(C_x+K_x)P \\
&+6P_xC + 6KP_x + 3C_y + 3K_y]Q + 3(3Q_xC + 3KQ_x + C_z + K_z)P \\
&+3P_zC + 3KP_z\}w_{xx}w_x^4 + 9(K+C)PQw_{xx}^2w_x^3 = 0,
\end{aligned} \tag{46}$$

and Eq. (2) is changed to

$$\begin{aligned}
&[-4Q_xw_{xx}w_x + 4Qw_x^4 - 2(QK+Q_{xx})w_x^2 - 3Qw_{xx}^2](C_x+K_y) \\
&+(80Q_xw_x^3w_{xx} - 16Qw_x^6 + 20(QK+Q_{xx})w_x^4 + 70Qw_x^2w_{xx}^2)P_x \\
&+(4w_x^4K + 18w_x^2w_{xx}^2)P_z + [12(PK+P_{xx})w_x^4 + (90Pw_{xx}^2 \\
&-C_{xx} - K_{xy})w_x^2]Q_x - 16QPw_x^5w_{xx} + 4(K_zP + P_{xxz})w_x^4 \\
&+(48QKP + 12P_{xx}Q + 36Q_{xx}P + 20P_{xz})w_{xx}w_x^3 \\
&-(C_{xxz} + K_{xyz})w_x^2 + [72PQw_{xx}^3 - (C_{xx}Q + K_{xy}Q \\
&+2C_{xz} + 2K_{yz})w_{xx}]w_x = 0.
\end{aligned} \tag{47}$$

Like Theorem 3, we get a new Bäcklund transformation for Eqs. (1) and (2).

Theorem 4 If w satisfies Eq. (46) (or Eq. (47)), then

$$u = -2w_x \tanh(w) + \int w_x^2 \mathrm{d}x + \frac{w_{xx}}{2w_x} + \frac{1}{4}\int \frac{w_{xx}^2}{w_x^2}\mathrm{d}x \tag{48}$$

is a new solution of Eq. (1) (or Eq. (2)).

Below, we give some new solutions of the shallow water equations (1) or (2) by using Theorem 4. We assume that the solution of Eq. (46) or (47) has the form of

$$w = kx + ly + mz + \omega t + g, \tag{49}$$

where k, l, m, ω are arbitrary constants and g is an arbitrary function of x, y, z and t. Some nontrivial solutions of the shallow water equation (1) (or Eq. (2)) can be generated by trivial solutions of Eq. (46) (or Eq. (47)).

We consider two special cases.

Case 1 For Eq. (1), it can be easily checked that

$$w = k_2x + l_2y + m_2z + d, \tag{50}$$

with k_2, l_2, m_2, d being arbitrary constants is a trivial solution of Eq. (46), then we get a nontrivial kink soliton solution

$$u = 2k_2 \tanh(k_2x + l_2y + m_2z + d) - k_2^2x, \tag{51}$$

for Eq. (1). As for Eq. (2), after taking similar operations we get a special solution

$$u = -2k \tanh(k_2x + l_2y + \omega_2t + d_2) + k_2^2x, \tag{52}$$

with arbitrary constants k_2, l_2, m_2, d_2, which coincide with the result of Eq. (3.17) in Ref. [34] under the condition of $x = z = 0$ and some special restrictions on a function therein.

Case 2 We consider a special solution of Eq. (46) (or Eq. (47)) as

$$\begin{aligned}
w &= k_1x + l_1y + m_1z + \omega_1t + W(X), \\
X &= k_2x + l_2y + m_2z + \omega_2t,
\end{aligned} \tag{53}$$

with arbitrary constants $k_1, l_1, m_1, \omega_1, k_2, l_2, m_2, \omega_2$, which leads to a new kind of solution for Eqs. (1) and (2).

$$\begin{aligned}
u_x = {}& 2(W_1k_2 + k_1)^2 \tanh(k_1x + l_1y + m_1z + \omega_1t + W)^2 \\
&- 2W_{1,X}k_2^2 \tanh(k_1x + l_1y + m_1z + \omega_1t + W) \\
&- \frac{1}{4(W_1k_2+k_1)^2}(4W_1^4k_2^4 + 16W_1^3k_1k_2^3 + 24W_1^2k_1^2k_2^2 \\
&- 2W_1W_{1,XX}k_2^4 + W_{1,X}^2k_2^4 + 16W_1k_1^3k_2 \\
&- 2W_{1,XX}k_1k_2^3 + 4k_1^4),
\end{aligned} \tag{54}$$

where $W_1(X) \equiv W(X)_X$. For Eq. (1), by substituting Eq. (53) into Eq. (46), we find that $W_1(X)$ satisfies the following elliptic function equation:

$$W_{1X}^2 = C_0 + C_1W_1 + C_2W_1^2 + C_3W_1^3 + C_4W_1^4, \tag{55}$$

with

$$\begin{aligned}
C_0 &= \frac{k_1[-12k_1^3k_2 + (C_2k_2^3 + \omega_2)k_1 + 3k_2\omega_1]}{3k_2^5}, \\
C_1 &= \frac{-8k_1^3k_2 + (C_2k_2^3 + \omega_2)k_1 + k_2\omega_1}{k_2^4}, \\
C_3 &= \frac{C_2k_2^3 + 24k_1^2k_2 - 2\omega_2}{3k_2^2k_1},
\end{aligned}$$

$$C_4 = 4, \tag{56}$$

and C_2 is an arbitrary constant. Equation (54) describes the interaction behavior between a kink soliton and elliptic waves. To illustrate this, we take the cnoidal solution of Eq. (55) to be

$$W_1 = \mu_0 + \mu_1 \mathrm{sn}(MX, N), \tag{57}$$

with arbitrary constants μ_0, μ_1, M and N. By substituting Eq. (57) into Eq. (55) with Eq. (56) and setting all coefficients of $\mathrm{sn}(MX, N)$ to zero, we have

$$\begin{aligned}
C_2 &= -\frac{4[(N^2+1)k_1^2 + 2\mu_0(N^2+1)k_2k_1 + \mu_0^2(N^2-5)k_2^2]}{k_2^2}, \\
M &= \frac{2(k_2\mu_0 + k_1)}{k_2}, \\
\omega_1 &= (6N^2+2)k_1^3 + (20N^2-4)\mu_0 k_2 k_1^2 \\
&\quad + (22N^2-14)\mu_0^2 k_2^2 k_1 + (8N^2-8)\mu_0^3 k_2^3, \\
\omega_2 &= -2(N^2-5)[(k_1^2 + 2\mu_0 k_2^2 k_1 + \mu_0^2 k_2^3], \\
\mu_1 &= \frac{MN}{2}.
\end{aligned} \tag{58}$$

It can be seen from the expressions of Eq. (54) with Eqs. (55)–(58) that the free parameters therein can be chosen as any real number with $k_1 k_2 \neq 0$ to avoid singularities. Figure 1 gives a picture of the interaction solution of Eq. (54) with Eqs. (55)–(58) by setting $y = t = 0$, the other parameters are fixed by

$$\begin{aligned}
&\mu_0 = -\frac{11}{4}, \ k_2 = 2, \ k_1 = 6, \ N = M = \frac{1}{2}, \ C_2 = \frac{2899}{16}, \\
&\omega_2 = \frac{19}{4}, \ l_1 = l_2 = m_1 = m_2 = 1, \ \omega_1 = \frac{27}{2}, \\
&\mu_1 = \frac{1}{8}, \ C_0 = \frac{7245}{32}, \ C_1 = \frac{10593}{32}, \ C_3 = 44.
\end{aligned} \tag{59}$$

As for Eq. (2), by substituting Eq. (53) into Eq. (47), we find that $W_1(X)$ satisfies Eq. (55) with

$$\begin{aligned}
C_0 &= \frac{k_1[(2C_1k_2^3 - C_2k_1k_2^2 + 4k_1^3)l_2 - k_1\omega_2 + k_2\omega_1]}{3k_2^4 l_2}, \\
C_3 &= -\frac{(C_1k_2^3 - 2C_2k_1k_2^2 - 16k_1^3)l_2 + k_1\omega_2 - k_2\omega_1}{3k_1^2k_2l_2}, \\
C_4 &= 4,
\end{aligned} \tag{60}$$

and C_1 and C_2 are arbitrary constants. After substituting Eq. (57) into Eq. (55) with Eq. (60) we have

$$\begin{aligned}
C_1 &= -16\mu_0^3 + (2N^2+2)M^2\mu_0, \\
C_2 &= (-N^2-1)M^2 + 24\mu_0^2, \ k_1 = \left(\frac{1}{2}M - \mu_0\right)k_2, \\
\omega_1 &= (N^2-1)M^3 l_2 k_2^2 + \left(\frac{1}{2}M - \mu_0\right)\omega_2, \ \mu_1 = \frac{MN}{2}.
\end{aligned} \tag{61}$$

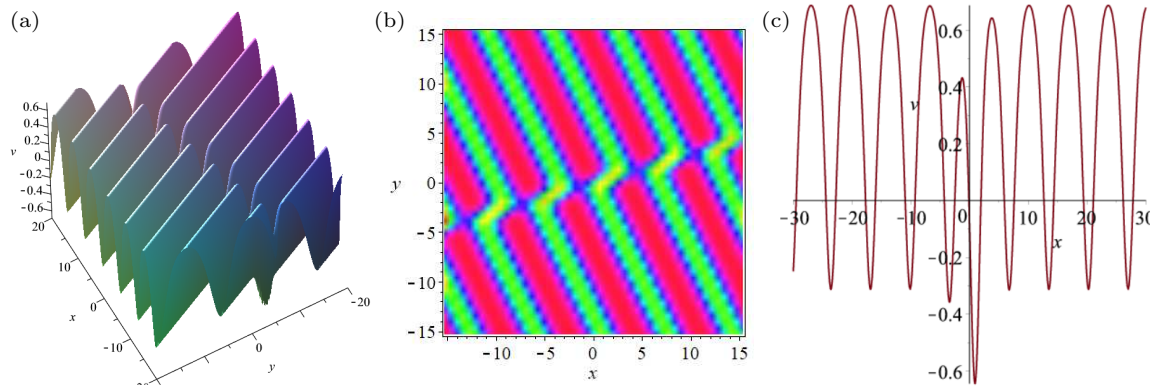


Fig. 1. The interaction solution (54) with Eqs. (55)–(58) where the parameters are fixed by Eq. (59) and $v = u_x$: (a) the three-dimensional plot with $z = t = 0$; (b) the density plot with $z = t = 0$; (c) the one-dimensional plot with $y = z = t = 0$.

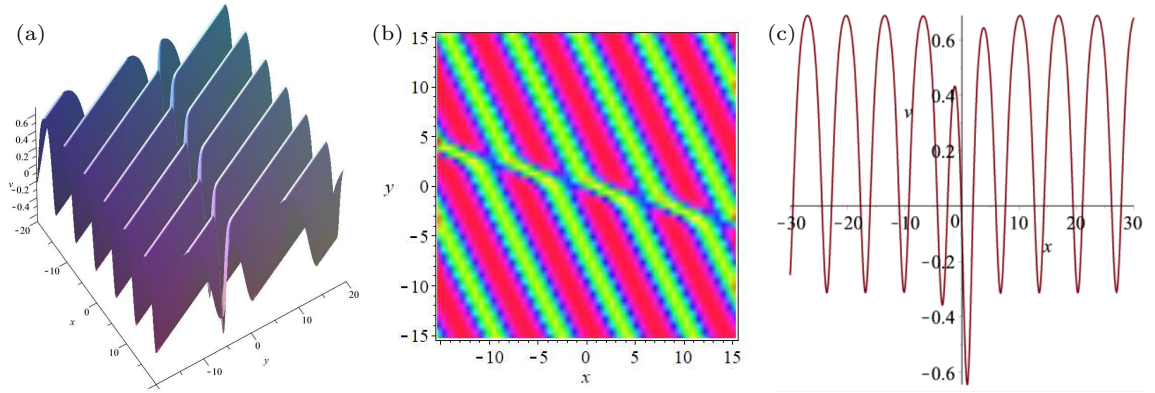


Fig. 2. The interaction solution (54) with Eqs. (55), (60), (57) and (61), where the parameters are fixed by Eq. (62) and $v = u_x$: (a) the three-dimensional plot with $z = t = 0$; (b) the density plot with $z = t = 0$; (c) the one-dimensional plot with $y = z = t = 0$.

Similarly, it can be seen from expressions of Eq. (54) with Eqs. (60) and (61) that the free parameters therein can be chosen any real number with $k_1k_2l_2 \neq 0$ to avoid singularities. Figure 2 displays the interaction solution (54) with Eqs. (55), (60), (57) and (61) where the parameters are fixed by

$$\begin{aligned} &M = N = \frac{1}{2},\ \ k_2 = \omega_2 = 2,\ \ l_1 = l_2 = m_1 = m_2 = \mu_0 = 1, \\ &C_0 = \frac{945}{256},\ \ C_1 = -\frac{123}{8},\ \ C_2 = \frac{379}{16},\ \ C_3 = -16, \\ &k_1 = -\frac{3}{2},\ \ \omega_1 = -\frac{15}{8},\ \ \mu_1 = \frac{1}{8}. \end{aligned} \tag{62}$$

It can be seen from Figs. 1 and 2 that the shallow water equations of Eqs. (1) and (2) have similar interaction behaviors between soliton and background periodic waves. but also have different interaction features, which are shown Figs. 1(b) and 2(b) as interactions along straight lines with different slopes or, in other words, the positions of interaction between soliton and periodic waves between Eqs. (1) and (2) are different.

5. Conclusion

In summary, two high-order shallow water wave equations are studied by using the standard Lie group method and the CRE method, respectively. After localization of the residual symmetries, new symmetry reduction solutions for the two shallow water wave equations are obtained, from which abundant new exact solutions can be generated. The two high-order shallow water wave equations are proved to be integrable in the sense of having consistent Riccati expansions. Some non-auto Bäcklund transformations for the two high-order shallow water wave equations are derived by using the CRE and CTE methods. Using the non-auto Bäcklund transformation theorems, a new type of interesting interaction solutions between solitons and cnoidal periodic waves is obtained and their detailed interaction behaviors are revealed by their plots and analysis. All the exact solutions in this paper are verified to be correct by substituting them into the $(3+1)$-dimensional shallow water wave equations (1) and (2).

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