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# Team-based fixed-time containment control for multi-agent systems with disturbances

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We investigate the fixed-time containment control (FCC) problem of multi-agent systems (MASs) under discontinuous communication. A saturation function is used in the controller to achieve the containment control in MASs. One difference from using a symbolic function is that it avoids the differential calculation process for discontinuous functions, which further ensures the continuity of the control input. Considering the discontinuous communication, a dynamic variable is constructed, which is always non-negative between any two communications of the agent. Based on the designed variable, the dynamic event-triggered algorithm is proposed to achieve FCC, which can effectively reduce controller updating. In addition, we further design a new event-triggered algorithm to achieve FCC, called the team-trigger mechanism, which combines the self-triggering technique with the proposed dynamic event trigger mechanism. It has faster convergence than the proposed dynamic event triggering technique and achieves the tradeoff between communication cost, convergence time and number of triggers in MASs. Finally, Zeno behavior is excluded and the validity of the proposed theory is confirmed by simulation.

**Keywords:** fixed-time containment control, dynamic event-triggered strategy, team-based triggered strategy, multi-agent systems

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## 1. Introduction

The distributed cooperative control problem has attracted much attention due to its widely potential applications including smart grids,[1] deception attacks,[2] and so on. The issue of consensus control is an important part of cooperative control, and to enable the agent to follow the specified trajectory and to reach the predetermined goal, a leader-follower approach was introduced in Ref. [3], where the follower enters and follows the route designed by the leader through the information passed by the leader and eventually converges to the predetermined goal. The consensus control problem for multi-agent systems (MASs) with two or more leaders, also known as the containment control problem, has the property of guiding followers into the convex hull region formed by the leaders, which makes containment control of great interest and widely used in many real-life situations. For example, in order to avoid densely populated residential areas while removing hazardous chemicals, rescue robots must follow a designated route. The robots without sensors are steered into the convex hull that is spanned by the robots with sensors and they then advance in the convex hull region because only a few robots have sensors to monitor the surroundings. Therefore, it is of great interest to study the problem of containment control.

So far, many major and interesting findings on containment control of MASs have been found like Refs. [4–6]. Notably, most of the literature focuses on the asymptotic or exponential containment control problem, which implies that the agent cannot follow the desired trajectory and achieve the desired goal in finite time. This has led some researchers to further focus on studying the finite-time containment problem for MASs. Attitude containment control of a first-order MASs with stationary leaders was studied by proposing a model-independent strategy in Ref. [7]. In Ref. [8], the finite-time containment control problem for double-integral MASs with multiple dynamic leaders was discussed, and further finite-time containment control without velocity measurements was achieved in Ref. [9]. However, these derived upper bounds on the convergence time were predicated on the need to know information about the initial state of the agents, which is sometimes not easy to know. Based on this, the fixed-time consensus control algorithm was further proposed in Refs. [10,11], applying this principle to MASs with one leader and multiple leaders, and successfully implemented fixed-time tracking control and fixed-time containment control (FCC) in Refs. [12–14]. Considering the presence of unavoidable external disturbances in real systems, FCC with disturbances was further investigated in Refs. [15,16].

Note that all the above tasks are carried out with continuous communication. However, on the one hand, the failure of external devices may prevent communication and hinder task completion. On the other hand, continuous message monitoring inevitably consumes excessive resources. Therefore, it is important to study the containment control problem for MASs

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with intermittent communication. In Refs. [17,18], an event-triggered control (ETC) mechanism for consensus control was proposed by constructing combinations of measurement errors, which implies inter-agent communication only when a specific event is triggered. Dynamic event-triggered control (DETC) is a mechanism further develop by Girard *et al.* in Ref. [19]. It creates a new trigger function by including a dynamic variable, which has the advantage of lowering the number of controller updates. Correspondingly, ETC-based fixed-time containment control has been achieved in Refs. [20–22]. However, we notice that the problem of DETC-based containment control is rarely considered. In addition, inter-state combinations using measurement errors inevitably employ adjacent relative information, which increases the problem of continuous communication to some extent. For this reason, some scholars have tried to use self-triggering methods to determine the next trigger moment directly from the current moment in Refs. [23–25], while this has the disadvantage of increasing the number of triggers. Based on this, some scholars have tried to weigh the relationship among convergence speed, communication cost and the number of triggers, and have proposed the idea of team triggering to solve this difficulty discussed in Ref. [26], which has achieved initial success in consensus control but has received little attention in containment control.

The above discussion inspires the research in this paper, which further considers the use of dynamic event-triggered strategies and team-triggered techniques to achieve FCC for MASs with disturbances. First, FCC is achieved. Unlike the fixed-time controller created in Ref. [24], on the one hand, we have no requirement for the parity of the parameters, and on the other hand, the symbolic function is replaced by the saturation function, which avoids the chattering effect that may be caused by the discontinuity of the symbolic function and also better excludes the Zeno behavior, which is also effectively illustrated in the simulation section. Second, a dynamic event-triggered algorithm is proposed to achieve FCC, which can effectively reduce controller updates. Finally, by combining the self-triggering technique with the proposed dynamic event triggering mechanism, an algorithm called team-triggered mechanism is further designed, which does not have triggers as many as the self-triggering technique and has a faster convergence time than the proposed dynamic event triggering technique, enabling the system to make a trade-off between communication cost, convergence time and the number of triggers. Moreover, unlike the undirected graphs presented in Refs. [27–29], we discuss containment control based on directed graphs.

This paper is structured as follows. Section 2 introduces the preparatory knowledge including two parts of graph theory and problem description, the main results of this paper are presented in detail in Section 3 and the corresponding simulation results are given in Section 4, finally Section 5 is the conclusion.

## 2. Preliminaries and problem statement

### 2.1. Graph theory

Consider a directed graph $G = \{V, A, E\}$ with $m$ agents, where $V$, $E$, $A$ denote the set of nodes $V = \{1, \ldots, m\}$, the set of edges $E \subseteq V \times V$, and adjacency matrix $\mathrm{A} = (a_{ij})_{m\times m}$ with $a_{ij} > 0$, if agent $i$ can obtain the information from agent $j$; and $a_{ij} = 0$, otherwise. Let $d_i = \sum_{j=1, j\neq i}^{m} a_{ij}$, the degree matrix $D = \mathrm{diag}\{d_1, \ldots, d_m\}$ and the Laplacian matrix $L = D - A = (l_{ij})$. Furthermore, a directed graph is said to satisfy the detail balance condition if there exists some scalar $\mu_i > 0$ such that $\mu_i a_{ij} = \mu_j a_{ji}$. The containment control problem studied in this paper is based on this graph-theoretic knowledge.

### 2.2. Problems formulation

Consider the MASs with disturbances consisting of $m$ $(m = m_1 + m_2)$ agents. Let $\mathcal{F}_1 = \{1, 2, \ldots, m_1\}$ and $\mathcal{F}_2 = \{m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\}$ represent the set of followers and leaders, respectively. Their dynamics can be expressed by

$$\dot{x}_i(t) = u_i(t) + d_i(t), \quad i \in \mathcal{F}_1, \tag{1}$$

$$\dot{x}_i(t) = d_i(t), \quad i \in \mathcal{F}_2, \tag{2}$$

where $x_i(t)$, $u_i(t)$ and $d_i(t)$ denote the agent state, control input and external disturbance, respectively.

**Assumption 1**[30] Assumed that the communication graph $G_{\mathcal{F}_1}$ (communication topology consisting of agents in $\mathcal{F}_1$) is a connected directed graph and satisfies the detail balance criterion. In addition, every leader has at least one follower as a neighbor.

**Assumption 2** The external disturbance of each agent $i$ is bounded by a non-negative constant $\bar{d}$, i.e., for any $i \in \mathcal{F}_1 \cup \mathcal{F}_2$, there exists $\bar{d} \geqslant 0$ such that $|d_i(t)| \leq \bar{d}$.

**Remark 1** Herein, we further consider MASs with disturbances, which are unavoidable in practical engineering. Note that Assumption 2 has been widely used to deal with perturbation effects, and many types of nonlinear perturbations (e.g., sinusoidal perturbations) encountered in real engineering are consistent with this assumption. What's more, Assumption 1 is the basic condition for containment control, which ensures that each follower has direct or indirect access to leader information.

**Lemma 1**[13] For any $\xi_i > 0$, $i = 1, 2, \ldots, n$, suppose $0 \leq \delta \leq 1$, then $\sum_{i=0}^{n} \xi_i{}^{\delta} \geq (\sum_{i=0}^{n} \xi_i)^{\delta}$ holds, otherwise $\sum_{i=0}^{n} \xi_i{}^{\delta} \geq n^{1-\delta}(\sum_{i=0}^{n} \xi_i)^{\delta}$ holds.

**Lemma 2**[31] According to Assumption 1, the Laplacian matrix $L$ is denoted as $L = \begin{bmatrix} L_1 & L_2 \\ 0_{m_2\times m_1} & 0_{m_2\times m_2} \end{bmatrix}$, where $L_1 \in R^{m_1\times m_1}$, $L_2 \in R^{m_1\times m_2}$.

**Lemma 3**[31] For a given system $\dot{x}(t) = f(t,x)$, suppose that there exist parameters $\alpha, \beta > 0$, $0 < a < 1 < b$ such that the corresponding Lyapunov functions $\dot{V}(x) \leq -\alpha V^a(x) - \beta V^b(x)$. Then for any initial state of the agent, we can obtain the settling time $t \leq \frac{1}{\alpha(1-a)} + \frac{1}{\beta(b-1)}$, i.e., achieve the goal of fixed-time convergence.

**Lemma 4**[32] Let matrix $\Omega = -L_1^{-1}L_2 \in R^{m_1 \times m_2}$, then $\Omega_{ij} \geq 0$ holds, furthermore, one has $\sum_{j=1}^{m_2} \Omega_{ij} = 1$, that is, the sum of each row of matrix $\Omega$ is equal to 1.

**Lemma 5**[33] Assuming that matrix $\mu = \mathrm{diag}\{\mu_1, \ldots, \mu_{m_1}\}$, we can obtain $L_1\mu > 0$.

In what follows, the introduction of the convex hull is important for the study of containment control, $\mathrm{Co}(y)$ is said to be the convex hull of the set $y = \{y_1, \ldots, y_n\}$ if it satisfies $\mathrm{Co}(y) = \{\sum_{i=1}^n \nu_i y_i \mid y_i \in Y, \nu_i \geq 0, \sum_{i=1}^n \nu_i = 1\}$. Let $x_F(t) = (x_1(t), \ldots, x_{m_1}(t))^{\mathrm{T}}$, $x_L(t) = (x_{m_1+1}(t), \ldots, x_{m_1+m_2}(t))^{\mathrm{T}}$, according to Lemma 2, we have $x_p(t) = (x_{p1}(t), \ldots, x_{pm_1}(t))^{\mathrm{T}}$ inside the convex hull spanned by the leader's state, where $x_p(t) = \Omega x_L(t) \in R^{m_1 \times 1}$. Define the containment control error as follows: $\Delta_x = x_F(t) - x_p(t) \in R^{m_1 \times 1}$, i.e., $\Delta_{xi}(t) = x_i(t) - x_{pi}(t)$, $i \in \mathcal{F}_1$. Our goal is to design appropriate event-triggered control protocol such that all follower agents enter and remain within the convex hull generated by the leader within a fixed time.

**Definition 1** The fixed-time containment control of MASs can be achieved if there exists a constant $T$ for any initial states satisfying $\lim_{t \to T} |\Delta_{xi}(t)| = 0$ and $|\Delta_{xi}(t)| = 0$ for any $t \geq T$, $i \in \mathcal{F}_1$.

## 3. Main results

In this section, we propose two algorithms to achieve FCC. The first algorithm reduces the number of controller updated by designing a dynamic variable, inspired by the application of dynamic event triggering techniques in consensus control. Then, the team-triggered algorithm is further designed to implement containment control, which is a combination of a self-triggering strategy and the proposed dynamic triggering mechanism.

The fixed-time containment control protocol for each follower $i$ via an event-triggered strategy is designed as follows:

$$u_i(t) = -\kappa_1 \mathrm{sig}^{\alpha}(\zeta_i(t_k^i)) - \kappa_2 \zeta_i(t_k^i) - \kappa_3 \mathrm{sat}(\zeta_i(t_k^i)), \quad t \in [t_k^i, t_{k+1}^i), \tag{3}$$

where $\kappa_1$, $\kappa_2$, $\kappa_3$ are positive constants, $\alpha > 1$, and $t_k^i$ is the latest event instant of follower $i$; $\mathrm{sig}^{\alpha}(\zeta) = \mathrm{sign}(\zeta)|\zeta|^{\alpha}$ and $\mathrm{sat}(\cdot)$ is called saturation function, which satisfies

$$\mathrm{sat}(\zeta_i(t)) = \begin{cases} \frac{\zeta_i(t)}{\delta}, & |\zeta_i(t)| \leq \delta, \\ \mathrm{sign}(\zeta_i(t)), & |\zeta_i(t)| > \delta, \end{cases}$$

where $\mathrm{sign}(\cdot)$ denotes symbolic function. Auxiliary variable $\zeta_i(t)$ is

$$\zeta_i(t) = \sum_{i=1}^{m} a_{ij}(x_i(t) - x_j(t)), \quad i \in \mathcal{F}_1. \tag{4}$$

Let $\zeta_F(t) = (\zeta_1(t), \ldots, \zeta_{m_1}(t))^{\mathrm{T}}$, according to Lemma 2 one has

$$\begin{aligned} \zeta_F &= L_1 x_F + L_2 x_L = L_1(x_F - (\Omega x_L)) \\ &= L_1(x_F - x_p) = L_1 \Delta_x. \end{aligned} \tag{5}$$

The measurement error of $i$ is designed as

$$\begin{aligned} \varepsilon_i(t) = {} & \kappa_1 \mathrm{sig}^{\alpha}(\zeta_i(t_k^i)) + \kappa_2 \zeta_i(t_k^i) + \kappa_3 \mathrm{sat}(\zeta_i(t_k^i)) \\ & - \kappa_1 \mathrm{sig}^{\alpha}(\zeta_i(t)) - \kappa_2 \zeta_i(t) - \kappa_3 \mathrm{sat}(\zeta_i(t)). \end{aligned} \tag{6}$$

The dynamic event-triggered function of each follower agent $i$ reads

$$\phi_i(t) = \vartheta_i(|\varepsilon_i(t)| - \sigma\varphi_i(t)) - \Gamma_i(t), \tag{7}$$

where $\sigma > 0$ and $\vartheta_i > 0$ means trigger adjustable parameter, $\varphi_i(t)$ can be written as

$$\varphi_i(t) = \frac{\mu_m}{\mu_{\mathrm{M}}}[\kappa_1 |\zeta_i(t)|^{\alpha} + \kappa_2 |\zeta_i(t)| + \kappa_3], \tag{8}$$

where $\mu_m = \min_{i \in \mathcal{F}_1} \mu_i$, and $\mu_{\mathrm{M}} = \max_{i \in \mathcal{F}_1} \mu_i$. $\Gamma_i(t)$ is a dynamic variable satisfying the following differential equation:

$$\begin{aligned} \dot{\Gamma}_i(t) = {} & \frac{s}{\mu_m}|\zeta_i(t)|(\sigma\varphi_i(t) - |\varepsilon_i(t)|) - \lambda_1 \Gamma_i^{\frac{1+\alpha}{2}}(t) \\ & - \lambda_2 \Gamma_i^2(t) - \lambda_3 \Gamma_i^{1/2}(t), \quad t \in [t_k^i, t_{k+1}^i), \end{aligned} \tag{9}$$

with $s > 0$. In addition, $|\varepsilon_i(t)| - \sigma\varphi_i(t)$ is noted as being a function triggered by a static event, marked as $\psi_i(t) \triangleq |\varepsilon_i(t)| - \sigma\varphi_i(t)$. Thus, the next trigger moment is

$$t_{k+1}^i = \inf\{t_k^i > t \mid \phi_i(t) \geq 0\}. \tag{10}$$

**Remark 2** Unlike the fixed-time controller created in Ref. [24], on the one hand, we have no requirement for the parity of the parameters, and on the other hand, the symbolic function is replaced by the saturation function, which avoids the chattering effect that may be caused by the discontinuity of the symbolic function, which is also effectively illustrated in the simulation section.

**Lemma 6** Construct the Lyapunov candidate function as

$$V = \frac{1}{2}\zeta_F^{\mathrm{T}}(L_1\mu)^{-1}\zeta_F + \sum_{i \in \mathcal{F}_1} \Gamma_i, \tag{11}$$

then $V$ is a validate Lyapunov function.

**Proof** Without loss of generality, let $V_1 = \frac{1}{2}\zeta_F^{\mathrm{T}}(L_1\mu)^{-1}\zeta_F$ and $V_2 = \sum_{i \in \mathcal{F}_1} \Gamma_i$. According to Lemma 5, it is obvious that

$$0 \leq V_1 \leq \frac{1}{2}\lambda_{\mathrm{M}}((L_1\mu)^{-1}) \sum_{i \in \mathcal{F}_1} |\zeta_i|^2, \tag{12}$$

where $\lambda_{\mathrm{M}}(L)$ denote maximum eigenvalues of matrix $L$. Thus to guarantee that $V$ is valid, $\Gamma_i \geq 0$ must be met. In fact, for any $t \in [t_k^i, t_{k+1}^i)$, using Eqs. (7) and (10) yields

$$|\varepsilon_i(t)| \leqslant \sigma\varphi_i(t) + \frac{\Gamma_i(t)}{\vartheta_i}. \tag{13}$$

It can be further deduced that

$$\dot{\Gamma}_i(t) \geqslant -\frac{s}{\mu_m}|\zeta_i(t)|\frac{\Gamma_i(t)}{\vartheta_i} - \lambda_1\Gamma_i^{\frac{1+\alpha}{2}}(t) - \lambda_2\Gamma_i^2(t) - \lambda_3\Gamma_i^{1/2}(t), \tag{14}$$

and we have

$$\Gamma_i(t) \geqslant \varepsilon^{\int_0^t Q_i(s)\,\mathrm{d}s}\Gamma_i(0) > 0, \tag{15}$$

where $Q_i(t) = -\frac{s}{\mu_m\vartheta_i}|\zeta_i(t)| - \lambda_1\Gamma_i^{\frac{1+\alpha}{2}}(t) - \lambda_2\Gamma_i^2(t) - \lambda_3\Gamma_i^{1/2}(t)$. Therefore, $V$ is a validate Lyapunov function.

**Remark 3** Lemma 6 has shown that the constructed Lyapunov function is reasonable, and in combination with Eq. (15) we find that the dynamic variables satisfying Eq. (9) are always non-negative between any two communications of the agents, which further suggests that the average inter-event time of the dynamic event trigger Eq. (10) can be significantly increased compared to the traditional ETC mechanism.

**Theorem 1** Under Assumptions 1 and 2, if $|\zeta_i(t)| > \delta$, then saturation function $\mathrm{sat}(\cdot) = \mathrm{sign}(\cdot)$ and the MASs consisting of Eqs. (1) and (2) can achieve fixed-time containment control within $T_1$ under controller (3) and triggered condition (10), where the fixed-time $T_1 \leq \frac{2}{\rho_2} + \frac{2}{\rho_1 m_1^{\frac{1-\alpha}{2}}(\alpha-1)}$ with $\rho_1$ and $\rho_2$ satisfies, respectively,

$$\rho_1 = \min\left\{\frac{\kappa_1(1-\sigma)m_1^{\frac{1-\alpha}{2}}2^{\frac{1+\alpha}{2}}}{[\lambda_{\mathrm{M}}((L_1\mu)^{-1})]^{\frac{1+\alpha}{2}}\mu_{\mathrm{M}}}, \lambda_1 m_1^{\frac{1-\alpha}{2}}\right\},$$
$$\rho_2 = \min\left\{\frac{\kappa_3(1-\sigma)}{\sqrt{2}[\lambda_{\mathrm{M}}((L_1\mu)^{-1})]^{1/2}\mu_{\mathrm{M}}}, \lambda_3\right\}.$$

**Proof** Choose the above Lyapunov candidate function $V$. Before deriving the derivative for $V$, we analyze $\zeta_F$ and using Eq. (5) to derive the derivative of $\zeta_F(t)$ yields

$$\begin{aligned}\dot{\zeta}_F =& L_1(-\varepsilon_F - \kappa_1\,\mathrm{sig}^{\alpha}(\zeta_F) - k_2\zeta_F \\ &- \kappa_3\,\mathrm{sign}(\zeta_F) + d_F + L_1^{-1}L_2 d_L),\end{aligned} \tag{16}$$

where $\varepsilon_F$, $d_F$, and $d_L$ is column stack vector of $\varepsilon_i$, $d_i$ ($i \in \mathcal{F}_1$) and $d_i$ ($i \in \mathcal{F}_2$), respectively. One can further calculate $\dot{V}$, i.e.,

$$\begin{aligned}\dot{V} =& \zeta_F^{\mathrm{T}}(L_1\mu)^{-1}\dot{\zeta}_F + \sum_{i\in\mathcal{F}_1}\dot{\Gamma}_i \\ \leq& \frac{1}{\mu_{\mathrm{M}}}\sum_{i\in\mathcal{F}_1}(-\kappa_1|\zeta_i|^{1+\alpha} - \kappa_2|\zeta_i|^2 - \kappa_3|\zeta_i|) \\ &+ \frac{1}{\mu_m}\|\zeta_F\|\|d_F + L_1^{-1}L_2 d_L\| + \sum_{i\in\mathcal{F}_1}\Big(\frac{s}{\mu_m}|\zeta_i|(\sigma\varphi_i - |\varepsilon_i|) \\ &- \lambda_1\Gamma_i^{\frac{1+\alpha}{2}} - \lambda_2\Gamma_i^2 - \lambda_3\Gamma_i^{1/2}\Big) + \frac{1}{\mu_m}\sum_{i\in\mathcal{F}_1}|\zeta_i||\varepsilon_i| \\ \leq& \frac{1-s}{\mu_m}\sum_{i\in\mathcal{F}_1}|\zeta_i||\varepsilon_i| - \frac{\kappa_1(1-\sigma s)}{\mu_{\mathrm{M}}}\sum_{i\in\mathcal{F}_1}|\zeta_i|^{1+\alpha} \\ &- \frac{\kappa_2(1-\sigma s)}{\mu_{\mathrm{M}}}\sum_{i\in\mathcal{F}_1}|\zeta_i|^2 - \frac{\kappa_3(1-\sigma s)}{\mu_{\mathrm{M}}}\sum_{i\in\mathcal{F}_1}|\zeta_i| \\ &+ \frac{1}{\mu_m}\|\zeta_F\|\|d_F + L_1^{-1}L_2 d_L\| \\ &- \sum_{i\in\mathcal{F}_1}\Big(\lambda_1\Gamma_i^{\frac{1+\alpha}{2}} + \lambda_2\Gamma_i^2 + \lambda_3\Gamma_i^{1/2}\Big).\end{aligned} \tag{17}$$

From Eq. (13), we further obtain

$$\begin{aligned}\frac{1-s}{\mu_m}\sum_{i\in\mathcal{F}_1}|\zeta_i||\varepsilon_i| \leq& \frac{(1-s)\sigma\kappa_1}{\mu_{\mathrm{M}}}\sum_{i\in\mathcal{F}_1}|\zeta_i|^{\alpha+1} \\ &+ \frac{(1-s)\sigma\kappa_2}{\mu_{\mathrm{M}}}\sum_{i\in\mathcal{F}_1}|\zeta_i|^2 + \frac{(1-s)\sigma\kappa_3}{\mu_{\mathrm{M}}}\sum_{i\in\mathcal{F}_1}|\zeta_i| \\ &+ \frac{1-s}{\mu_m\vartheta_i}\sum_{i\in\mathcal{F}_1}|\zeta_i||\Gamma_i|,\end{aligned} \tag{18}$$

and by Assumption 2, it is inferred that

$$\begin{aligned}&\|d_F + L_1^{-1}L_2 d_L\| \leq \|d_F\| + \sigma_{\mathrm{M}}(L_1^{-1}L_2)\|d_L\| \\ &\leq \Big(\sum_{i\in\mathcal{F}_1}|d_i|^2\Big)^{1/2} + \sigma_{\mathrm{M}}(L_1^{-1}L_2)\Big(\sum_{i\in\mathcal{F}_2}|d_i|^2\Big)^{1/2} \\ &\leq (m_1 + m_2\sigma_{\mathrm{M}}(L_1^{-1}L_2))\bar{d},\end{aligned} \tag{19}$$

where $\sigma_{\mathrm{M}}(A)$ represents the maximum singular value of the matrix $A$. On the basis of Eqs. (18) and (19), one can further calculate $\dot{V}$ to obtain

$$\begin{aligned}\dot{V} \leq& -\frac{\kappa_1(1-\sigma)}{\mu_{\mathrm{M}}}\sum_{i\in\mathcal{F}_1}|\zeta_i|^{1+\alpha} - \sum_{i\in\mathcal{F}_1}\lambda_1\Gamma_i^{\frac{1+\alpha}{2}} \\ &- \frac{\kappa_2(1-\sigma)}{\mu_{\mathrm{M}}}\sum_{i\in\mathcal{F}_1}|\zeta_i|^2 - \sum_{i\in\mathcal{F}_1}\lambda_2\Gamma_i^2 + \frac{1-s}{\mu_m\vartheta_i}\sum_{i\in\mathcal{F}_1}|\zeta_i||\chi_i| \\ &- \frac{\kappa_3(1-\sigma)}{\mu_{\mathrm{M}}}\sum_{i\in\mathcal{F}_1}|\zeta_i| + \frac{(m_1 + m_2\sigma_{\mathrm{M}}(L_1^{-1}L_2))\bar{d}}{\mu_m}\sum_{i\in\mathcal{F}_1}|\zeta_i| \\ &- \sum_{i\in\mathcal{F}_1}\lambda_3\Gamma_i^{1/2}.\end{aligned} \tag{20}$$

Assume that $\sqrt{\lambda_1\kappa_2(1-\sigma)} = \frac{(1-s)\sqrt{\mu_{\mathrm{M}}}}{2\mu_m\vartheta_i}$ and $\kappa_3 \geq \frac{(m_1+m_2\sigma_{\mathrm{M}}(L_1^{-1}L_2))2\mu_{\mathrm{M}}\bar{d}}{(1-\sigma)\mu_m}$, combine with inequalities

$$\sum_{i\in\mathcal{F}_1}|\zeta_i|^{1+\alpha} = \sum_{i\in\mathcal{F}_1}\zeta_i^{2\left(\frac{1+\alpha}{2}\right)} \geq m_1^{\frac{1-\alpha}{2}}\Big(\sum_{i\in\mathcal{F}_1}|\zeta_i|^2\Big)^{\frac{1+\alpha}{2}}, \tag{21}$$

$$\sum_{i\in\mathcal{F}_1}|\zeta_i| = \sum_{i\in\mathcal{F}_1}\zeta_i^{2(\frac{1}{2})} \geq \Big(\sum_{i\in\mathcal{F}_1}|\zeta_i|^2\Big)^{1/2}, \tag{22}$$

we can further calculate $\dot{V}$ as

$$\begin{aligned}\dot{V} \leq& -\frac{\kappa_1(1-\sigma)m_1^{\frac{1-\alpha}{2}}}{\mu_{\mathrm{M}}}\Big(\sum_{i\in\mathcal{F}_1}|\zeta_i|^2\Big)^{\frac{1+\alpha}{2}} \\ &- \lambda_1 m_1^{\frac{1-\alpha}{2}}\Big(\sum_{i\in\mathcal{F}_1}\Gamma_i\Big)^{\frac{1+\alpha}{2}} - \frac{\kappa_3(1-\sigma)}{2\mu_{\mathrm{M}}}\Big(\sum_{i\in\mathcal{F}_1}|\zeta_i|^2\Big)^{1/2} \\ &- \lambda_3\Big(\sum_{i\in\mathcal{F}_1}\Gamma_i\Big)^{1/2}.\end{aligned} \tag{23}$$

Then with Eq. (12), we can reach

$$\dot{V} \leq -\frac{\kappa_1(1-\sigma)m_1^{\frac{1-\alpha}{2}}2^{\frac{1+\alpha}{2}}}{\Big[\lambda_{\mathrm{M}}\Big((L_1\mu)^{-1}\Big)\Big]^{\frac{1+\alpha}{2}}\mu_{\mathrm{M}}}V_1^{\frac{1+\alpha}{2}} - \lambda_1 m_1^{\frac{1-\alpha}{2}}V_2^{\frac{1+\alpha}{2}}$$

$$-\frac{\kappa_3(1-\sigma)}{\sqrt{2}\left[\lambda_{\mathrm{M}}\left((L_1\mu)^{-1}\right)\right]^{1/2}\mu_{\mathrm{M}}}V_1^{1/2}-\lambda_3 V_2^{1/2}$$
$$\leq -\rho_1 m_1^{\frac{1-\alpha}{2}}V^{\frac{1+\alpha}{2}}-\rho_2 V^{1/2}. \tag{24}$$

Moreover, according to Lemma 3 one has $V\to 0$ within $T_1$, where the setting time $T_1$ satisfies $T_1\leq \frac{2}{\rho_2}+\frac{2}{\rho_1 m_1^{\frac{1-\alpha}{2}}(\alpha-1)}$. Combining Eqs. (5) and (11) clearly gives $\Delta_{xi}\to 0$ within $T_1$.

**Remark 4** Unlike the controller used in Refs. [24–26], the first term of the control protocol proposed in this paper is employed to replace $\zeta_i^{\alpha}(t_k^i)$ with $\mathrm{sig}^{\alpha}(\zeta_i(t_k^i))$. Assume that the controller is used $\zeta_i^{\alpha}(t_k^i)$, then an analysis similar to Eq. (17) yields

$$\begin{aligned}\dot{V}\leq&\frac{1}{\mu_{\mathrm{M}}}\sum_{i\in F_1}(-\kappa_1\zeta_i^{1+\alpha}-\kappa_2\zeta_i^2-\kappa_3|\zeta_i|)\\&+\frac{1}{\mu_m}\|\zeta_F\|\|d_F+L_1^{-1}L_2d_L\|\\&+\sum_{i\in F_1}\Big(\frac{s}{\mu_m}|\zeta_i|(\sigma\varphi_i-|\varepsilon_i|)-\lambda_1\Gamma_i^{\frac{1+\alpha}{2}}\\&-\lambda_2\Gamma_i^2-\lambda_3\Gamma_i^{1/2}\Big)+\frac{1}{\mu_m}\sum_{i\in F_1}|\zeta_i||\varepsilon_i|\,.\end{aligned}$$

If the parity of the parameter $\alpha$ is not specified, then

$$\sum_{i\in F_1}\zeta_i^{1+\alpha}=\sum_{i\in F_1}\zeta_i^{2\left(\frac{1+\alpha}{2}\right)}\geq m_1^{\frac{1-\alpha}{2}}\Big(\sum_{i\in F_1}|\zeta_i|^2\Big)^{\frac{1+\alpha}{2}}$$

will hold only when $\zeta_i(t)\geq 0$, which is practically difficult to achieve, so specifying the parameter $\alpha$ as odd is the only way to achieve FCC in this scheme. This further shows that one of the advantages of the controller proposed in this paper is the relaxation of the requirement for $\alpha$. Furthermore, it is clear that the first term of the control protocol is necessary to achieve FCC, while $\kappa_2$ is used to weigh the dynamic variables.

In what follows, assume that $|\zeta_i(t)|\leq\delta$, saturation function $\mathrm{sat}(\zeta_i(t))=\zeta_i(t)/\delta$. To distinguish the sign from that in Theorem 1, mark $\breve{\varepsilon}_i(t)$ as the new combined error, with

$$\begin{aligned}\breve{\varepsilon}_i(t)=&\kappa_1\,\mathrm{sig}^{\alpha}(\zeta_i(t_k^i))+\kappa_2\zeta_i(t_k^i)+\kappa_3\frac{\zeta_i(t_k^i)}{\delta}\\&-\kappa_1\,\mathrm{sig}^{\alpha}(\zeta_i(t))-\kappa_2\zeta_i(t)-\kappa_3\frac{\zeta_i(t)}{\delta}.\end{aligned}\tag{25}$$

Correspondingly,

$$\breve{\phi}_i(t)=\vartheta_i(|\breve{\varepsilon}_i(t)|-\sigma\varphi_i(t))-\breve{\Gamma}_i(t),\tag{26}$$

$$\begin{aligned}\dot{\breve{\Gamma}}_i(t)=&\frac{s}{\mu_m}|\zeta_i(t)|(\sigma\varphi_i(t)-|\breve{\varepsilon}_i(t)|)-\lambda_1\breve{\Gamma}_i^{\frac{1+\alpha}{2}}(t)-\lambda_2\breve{\Gamma}_i^2(t)\\&-\lambda_3\breve{\Gamma}_i^{1/2}(t),\ \ t\in[t_k^i,t_{k+1}^i).\end{aligned}\tag{27}$$

Similar to Theorem 1, the main conclusion of this scenario is given in Theorem 2.

**Theorem 2** Under Assumptions 1 and 2, when $|\zeta_i(t)|\leq\delta$, the fixed-time containment control of MASs consisting of Eqs. (1) and (2) can also be achieved under controller (3) and triggered condition (10) with the settling time $T_2=2T_1$.

**Proof** Select the same Lyapunov candidate function $V$, and deriving it by

$$\begin{aligned}\dot{\zeta}_F=&L_1\Big(-\breve{\varepsilon}_F-\kappa_1\,\mathrm{sig}^{\alpha}(\zeta_F)-k_2\zeta_F\\&-\kappa_3\frac{\zeta_F}{\delta}+d_F+L_1^{-1}L_2d_L\Big)\end{aligned}\tag{28}$$

for $V$ gives

$$\begin{aligned}\dot{V}\leq&\frac{1-s}{\mu_m}\sum_{i\in F_1}|\zeta_i||\breve{\varepsilon}_i|-\frac{\kappa_1(1-\sigma s)}{\mu_{\mathrm{M}}}\sum_{i\in F_1}|\zeta_i|^{1+\alpha}\\&-\frac{\kappa_2(1-\sigma s)}{\mu_{\mathrm{M}}}\sum_{i\in F_1}|\zeta_i|^2-\frac{\kappa_3}{\delta\mu_{\mathrm{M}}}\sum_{i\in F_1}|\zeta_i|^2\\&+\frac{\kappa_3\sigma s}{\mu_{\mathrm{M}}}\sum_{i\in F_1}|\zeta_i|+\frac{1}{\mu_m}\|\zeta_F\|\|d_F+L_1^{-1}L_2d_L\|\\&-\sum_{i\in F_1}(\lambda_1\breve{\Gamma}_i^{\frac{1+\alpha}{2}}+\lambda_2\breve{\Gamma}_i^2+\lambda_3\breve{\Gamma}_i^{1/2}).\end{aligned}\tag{29}$$

Combining Eqs. (13) and (19), further derivation of $V$ yields

$$\begin{aligned}\dot{V}\leq&-\frac{\kappa_1(1-\sigma)}{\mu_{\mathrm{M}}}\sum_{i\in F_1}|\zeta_i|^{1+\alpha}-\sum_{i\in F_1}\lambda_1\breve{\Gamma}_i^{\frac{1+\alpha}{2}}\\&-\frac{\kappa_3}{\delta\mu_{\mathrm{M}}}\sum_{i\in F_1}|\zeta_i|^2-\sum_{i\in F_1}\lambda_3\breve{\Gamma}_i^{1/2}-\frac{\kappa_2(1-\sigma)}{\mu_{\mathrm{M}}}\sum_{i\in F_1}|\zeta_i|^2\\&-\sum_{i\in F_1}\lambda_2\breve{\Gamma}_i^2+\frac{1-s}{\mu_m\vartheta_i}\sum_{i\in F_1}|\zeta_i||\breve{\Gamma}_i|+\frac{\kappa_3\sigma}{\mu_{\mathrm{M}}}\sum_{i\in F_1}|\zeta_i|\\&+\frac{(m_1+m_2\sigma_{\mathrm{M}}(L_1^{-1}L_2))\bar{d}}{\mu_m}\sum_{i\in F_1}|\zeta_i|.\end{aligned}\tag{30}$$

Assume that the parameter $\kappa_2$ satisfies the same assumptions as Theorem 1 then we have

$$\begin{aligned}\dot{V}\leq&-\frac{\kappa_1(1-\sigma)}{\mu_{\mathrm{M}}}\sum_{i\in F_1}|\zeta_i|^{1+\alpha}-\sum_{i\in F_1}\lambda_1\breve{\Gamma}_i^{\frac{1+\alpha}{2}}\\&-\frac{\kappa_3(1-\sigma)}{\mu_{\mathrm{M}}}\sum_{i\in F_1}|\zeta_i|^2-\sum_{i\in F_1}\lambda_3\breve{\Gamma}_i^{1/2}\\&+\Big(\frac{\kappa_3}{\mu_{\mathrm{M}}}+\frac{(m_1+m_2\sigma_{\mathrm{M}}(L_1^{-1}L_2))\bar{d}}{\mu_m}\Big)\sum_{i\in F_1}|\zeta_i|\\&-\frac{\kappa_3}{\delta\mu_{\mathrm{M}}}\sum_{i\in F_1}|\zeta_i|^2,\end{aligned}\tag{31}$$

and for convenience, let $\tilde{\rho}_1=\frac{\kappa_3}{\mu_{\mathrm{M}}}+\frac{(m_1+m_2\sigma_{\mathrm{M}}(L_1^{-1}L_2))\bar{d}}{\mu_m}$ and $\tilde{\rho}_2=\frac{\kappa_3}{\delta\mu_{\mathrm{M}}}$, the inequalities

$$\begin{aligned}\sum_{i\in F_1}(\tilde{\rho}_1|\zeta_i|-\tilde{\rho}_2|\zeta_i|^2)=&\sum_{i\in F_1}\Big[-\tilde{\rho}_2\Big(|\zeta_i|-\frac{\tilde{\rho}_1}{2\tilde{\rho}_2}\Big)^2+\frac{\tilde{\rho}_1^2}{4\tilde{\rho}_2}\Big]\\\leq&\sum_{i\in F_1}\frac{\tilde{\rho}_1^2}{4\tilde{\rho}_2}\leq\frac{m_1\tilde{\rho}_1^2}{4\tilde{\rho}_2}\doteq\tilde{\rho}_3\end{aligned}\tag{32}$$

will be used, and we have

$$\dot{V}\leq-\rho_1 m_1^{\frac{1-\alpha}{2}}V^{\frac{1+\alpha}{2}}-\rho_2V^{1/2}+\tilde{\rho}_3.\tag{33}$$

Since the parameter $\delta$ is adjustable, choosing an appropriate $\delta$ can make $\tilde{\rho}_3 \leq \frac{\rho_1 m_1^{\frac{1-\alpha}{2}}}{2} V^{\frac{1+\alpha}{2}} + \frac{\rho_2}{2} V^{1/2}$, namely,

$$\dot{V} \leq -\frac{\rho_1 m_1^{\frac{1-\alpha}{2}}}{2} V^{\frac{1+\alpha}{2}} - \frac{\rho_2}{2} V^{1/2}. \tag{34}$$

Therefore, it can be deduced from Lemma 3 that at a fixed time $V \to 0$, it shows $\Delta_{xi} \to 0$ within $T_2 = 2T_1$. This concludes the proof of Theorem 2.

**Theorem 3** Under Assumptions 1 and 2, the fixed-time containment control of MASs consisting of Eqs. (1) and (2) can be achieved under controller (3) and triggered condition (10), and the settling time $T = 3T_1$.

**Proof** From Theorem 1, it is clear that each agent will enter the $|\zeta_i(t)| \leq \delta$ zone within $T_1$, and Theorem 2 proves that the agent can achieve fixed-time containment control within $2T_1$ when $|\zeta_i(t)| \leq \delta$. Therefore, combining Theorems 1 and 2 yields this theorem immediately.

**Corollary 1** If the next trigger moment of each agent satisfies

$$t_{k+1}^i = \inf\{t_k^i > t \mid \psi_i(t) \geq 0\}, \tag{35}$$

the systems (1) and (2) can also achieve fixed-time containment control under controller (3) if Assumptions 1 and 2 are satisfied.

**Proof** Choosing the Lyapunov candidate function as $V_1$. On the one hand, when we consider the case of $|\zeta_i(t)| \leq \delta$, a calculation similar to Eq. (17) yields

$$\begin{aligned}\dot{V}_1 \leq & \frac{1}{\mu_{\mathrm{M}}} \sum_{i \in F_1} (-\kappa_1 |\zeta_i|^{1+\alpha} - \kappa_2 |\zeta_i|^2 - \kappa_3 |\zeta_i|) \\ & + \frac{1}{\mu_m} (\|\zeta_F\| \|d_F + L_1^{-1} L_2 d_L\| + \sum_{i \in F_1} |\zeta_i| |\varepsilon_i|).\end{aligned}$$

When $\psi_i(t) \leq 0$ gets $|\varepsilon_i(t)| \leqslant \sigma \varphi_i(t)$. Further calculation of $\dot{V}_1$ in combination with the proof of Theorem 1 yields

$$\begin{aligned}\dot{V}_1 \leq & -\frac{\kappa_1 (1-\sigma) m_1^{\frac{1-\alpha}{2}} 2^{\frac{1+\alpha}{2}}}{\left[\lambda_{\mathrm{M}}\left((L_1 \mu)^{-1}\right)\right]^{\frac{1+\alpha}{2}} \mu_{\mathrm{M}}} V_1^{\frac{1+\alpha}{2}} \\ & - \frac{\kappa_3 (1-\sigma)}{\sqrt{2}\left[\lambda_{\mathrm{M}}\left((L_1 \mu)^{-1}\right)\right]^{1/2} \mu_{\mathrm{M}}} V_1^{1/2}.\end{aligned}$$

In virtue to Lemma 3, the fixed-time containment control can be achieved. On the other hand, when $|\zeta_i(t)| > \delta$ is satisfied, the same conclusion can be obtained in a similar way to Theorem 2, and the proof is omitted here. The proof of this inference is completed.

**Remark 5** In general, Eq. (35) is referred to as a static trigger condition, characterized by the fact that no dynamic variables are involved, which is obviously a special case of condition (10) when $\Gamma_i(t) = 0$. It can also be seen that one of the advantages of a dynamic variable-based triggering mechanism is that the triggering parameters can be dynamically adjusted to achieve the desired system performance.

**Theorem 4** Under Assumptions 1 and 2, multi-agent systems (1) and (2) based on controller (3) and dynamic event triggering function (7) do not show Zeno behavior.

**Proof** First, the analysis of the Lyapunov candidate function in conjunction with Theorem 1 shows that $|\zeta_i(t)| \leq \|\zeta_F(t)\| \leq \sqrt{\frac{2V_1(0)}{\lambda_m[(L_1\mu)^{-1}]}} \triangleq \gamma_1$. Next, let $\gamma_2 = \kappa_1 \alpha \gamma_1^{\alpha-1} + \kappa_2 + \frac{\kappa_3}{\delta}$, $\gamma_3(t_{k'(t)}^j) = \kappa_1 |\zeta_j(t_{k'(t)}^j)|^\alpha + \kappa_2 |\zeta_j(t_{k'(t)}^j)|$, and $\gamma_4 = \kappa_3 + [m_1 + m_2 \sigma_{\mathrm{M}}(L_1^{-1} L_2)]\bar{d}$, differentiation of Eq. (6) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\varepsilon_i(t)| \leq & |\dot{\varepsilon}_i(t)| \leq |\kappa_1 \alpha |\zeta_i(t)|^{\alpha-1} + \kappa_2 + \frac{\kappa_3}{\delta}| |\dot{\zeta}_i(t)| \\ & \leq \gamma_2 \sum_{j \in N_i} a_{ij}(u_j(t) + \|d_F(t) + L_1^{-1} L_2 d_L(t)\|) \\ & \leq \gamma_2 \bar{a}(\gamma_3(t_{k'(t)}^j) + \gamma_4),\end{aligned} \tag{36}$$

where $\bar{a} \triangleq \max_{i \in F_1}\{\sum_{j \in N_i} |a_{ij}|\}$. Then, integration of Eq. (33) yields

$$\begin{aligned}|\varepsilon_i(t)| \leq & \int_{t_k^i}^t \gamma_2 \bar{a}(\gamma_3(t_{k'(t)}^j) + \gamma_4)\,\mathrm{d}s + |\varepsilon_i(t_k^i)| \\ = & \int_{t_k^i}^t \gamma_2 \bar{a}(\gamma_3(t_{k'(t)}^j) + \gamma_4)\,\mathrm{d}s.\end{aligned} \tag{37}$$

Recall that the trigger condition (10) shows

$$\begin{aligned}& |\varepsilon_i(t_{k+1}^i)| \\ = & \frac{\Gamma_i(t_{k+1}^i)}{\vartheta_i} + \frac{\sigma \mu_m}{\mu_{\mathrm{M}}}\left[\kappa_1 |\zeta_i(t_{k+1}^i)|^\alpha + \kappa_2 |\zeta_i(t_{k+1}^i)| + \frac{\kappa_3}{\delta}\right] \\ \leq & \int_{t_k^i}^{t_{k+1}^i} \gamma_2 \bar{a}(\gamma_3(t_{k'(t)}^j) + \gamma_4)\,\mathrm{d}s \leq \int_{t_k^i}^{t_{k+1}^i} \gamma_2 \bar{a} \gamma_5\,\mathrm{d}s,\end{aligned} \tag{38}$$

where $\gamma_5 = \kappa_1 \gamma_1^\alpha + \kappa_2 \gamma_1 + \gamma_4$. Therefore, the time interval between the two events is

$$\tau_i = t_{k+1}^i - t_k^i \geq \frac{\kappa_3 \sigma \mu_m}{\gamma_2 \bar{a} \gamma_5 \mu_{\mathrm{M}}} > 0, \tag{39}$$

which means that no Zeno phenomenon will occur.

**Remark 6** Note that the parameter $\kappa_3$ plays an important role in this paper. On the one hand, it is clear from the proofs of Theorems 1 and 2 that $\kappa_3$ is not only effective in suppressing external perturbations but also key to achieving FCC. On the other hand, combined with Eq. (19), $\kappa_3$ ensures a positive lower bound on the event interval for each follower, thus effectively excluding Zeno behavior.

**Remark 7** When executing a dynamic time trigger, it is obvious that if $\phi_i(t) > 0$, there must be $\psi_i(t) > 0$. However, when executing a static trigger, $\psi_i(t) > 0$ but $\phi_i(t) > 0$ unnecessarily holds, which means that Eq. (10) is a sufficient non-essential condition for Eq. (35). This illustrates that using a dynamic trigger strategy reduces the frequency of controller updates.

**Remark 8** Both the event-triggering strategies (7) and (35) share the drawback that the event-triggered functions

$\phi_i(t)$ and $\psi_i(t)$ involve continuous-time adjacency information, which suggests that implementing the triggering mechanism requires continuous communication, which is preferably reduced in practical applications.

Based on the above discussion, we further designed the team trigger mechanism to trade-off convergence speed, communication cost and trigger time. Before introducing the team triggers, define a new function $\pi_i(t)$, which satisfies

$$\pi_i(t) = |\varepsilon_i(t)| + \frac{\sigma\mu_m}{\sigma\mu_m + \mu_{\mathrm{M}}}|\xi_i(t)|, \tag{40}$$

where $\xi_i(t) = \kappa_3 \operatorname{sat}(\zeta_i(t))$. Imitating the differentiation of Eq. (36) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\pi_i(t)| &\le |\dot{\pi}_i(t)| \le |\dot{\varepsilon}_i(t)| + \frac{\sigma\mu_m}{\sigma\mu_m + \mu_{\mathrm{M}}}|\dot{\zeta}_i(t)| \\ &\le \Big(\gamma_2 + \frac{\sigma\mu_m}{\sigma\mu_m + \mu_{\mathrm{M}}}\Big)\bar{a}\Big(\gamma_3(t^j_{k'(t)}) + \gamma_4\Big) \\ &\doteq \Upsilon\Big(t^j_{k'(t)}\Big).\end{aligned} \tag{41}$$

The team-based trigger function is designed as

$$\Pi_i(t) = \begin{cases} \phi_i(t), & \text{if } \frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\mathrm{M}}}|\xi_i(t^i_k)| + \eta \ge \Xi_i(t^i_k), \\ \int_{t^i_k}^{t} \Upsilon(t^j_{k'(t)})\mathrm{d}s + \frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\mathrm{M}}}|\xi_i(t^i_k)| - \Xi_i(t^i_k), & \text{otherwise}, \end{cases} \tag{42}$$

where $\eta$ is a positive adjustable parameter with a small value, and $\Xi_i(t^i_k)$ is designed as

$$\begin{aligned}\Xi_i(t^i_k) = \frac{\sigma\mu_m}{\sigma\mu_m + \mu_{\mathrm{M}}}|&\kappa_1 \operatorname{sig}^{\alpha}(\zeta_i(t^i_k)) \\ &+ \kappa_2\zeta_i(t^i_k) + \kappa_3 \operatorname{sat}(\zeta_i(t^i_k))|.\end{aligned} \tag{43}$$

**Remark 9** We provide Algorithm 1 to describe this process in order to better highlight the team-triggered functionality of the above architecture. After calculating and judging the sizes of $\frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\mathrm{M}}}|\xi_i(t^i_k)| + \eta$ and $\Xi_i(t^i_k)$, steps 3–8 in the algorithm are applied to dynamic variable-based processes, while steps 12–17 are self-triggering processes. Obviously, simple modifications can also be used for the previously designed dynamic event triggering mechanism, so only one algorithm process is given here. As before, more details are given through Theorem 5.

**Algorithm 1** Team-based event-triggered communication and control law.

**Require:**

At $t^i_0 = 0$, initialize $u_i(0) = -\kappa_1 \operatorname{sig}^{\alpha}(\zeta_i(0)) - \kappa_2\zeta_i(0) - \kappa_3 \operatorname{sat}(\zeta_i(0))$, $i \in \mathcal{F}_1$.

**Ensure:**

1: For any $t \in [t^i_k,\ t^i_{k+1})$, compute $|\xi_i(t^i_k)|$ and $\Xi_i(t^i_k)$ with (43) ;
2: **if** $\frac{\sigma\mu_{\mathrm{m}}}{\sigma\mu_{\mathrm{m}}+\mu_{\mathrm{M}}}\ |\xi_i(t^i_k)| + \eta \ge \Xi_i(t^i_k)$ **then**
3: compute $\Gamma_i(t)$ as the solution to (9)
4: **if** DET function $\phi_i(t)$ exceeds zeros at triggering moments $t = t^i_k$ **then**
5: measurement error of agent $i$ resets to zeros at $t = t^i_k$;
6: perform $x_i(t^i_k) \leftarrows x_j(t^i_k)$;
7: compute auxiliary variable $\zeta_i(t^i_k)$ with (4)
8: compute control $u_i(t)$ with (3)
9: **end if**
10: **end if**
11: **if** $\frac{\sigma\mu_{\mathrm{m}}}{\sigma\mu_{\mathrm{m}}+\mu_{\mathrm{M}}}\ |\xi_i(t^i_k)| + \eta < \Xi_i(t^i_k)$ **then**
12: compute $\Upsilon(t^j_{k'(t)})$ with (41);
13: **if** $\int_{t^i_k}^{t} \Upsilon(t^j_{k'(t)})\mathrm{d}s + \frac{\sigma\mu_{\mathrm{m}}}{\sigma\mu_{\mathrm{m}}+\mu_{\mathrm{M}}}\ |\xi_i(t^i_k)| - \Xi_i(t^i_k) \ge 0$ at $t = t^i_k$ **then**
14: $\varepsilon_i(t)$ resets to zeros at $t = t^i_k$;
15: perform $x_i(t^i_k) \leftarrows x_j(t^i_k)$;
16: compute auxiliary variable $\zeta_i(t^i_k)$ with (4)
17: compute control $u_i(t)$ with (3)
18: **end if**
19: **end if**

**Theorem 5** Under Assumptions 1 and 2, if the next trigger moment of each agent satisfies

$$t^i_{k+1} = \inf\{t^i_k > t \mid \Pi_i(t) \ge 0\}, \tag{44}$$

the fixed-time containment control of MASs consisting of Eqs. (1) and (2) can be achieved under controller (3). In addition, Zeno behavior can be avoided.

**Proof** In the case of $\frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\mathrm{M}}}|\xi_i(t^i_k)| + \eta \ge \Xi_i(t^i_k)$, which is the dynamic trigger condition described above, Theorems 3 and 5 have been shown to achieve fixed containment control and to avoid Zeno behavior. Next, we further analyze the case $\frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\mathrm{M}}}|\xi_i(t^i_k)| + \eta < \Xi_i(t^i_k)$, which is the self-triggering con-

dition.

Assuming that $\frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\rm M}}|\xi_i(t_k^i)|+\eta<\Xi_i(t_k^i)$, when $t\in[t_k^i,t_{k+1}^i)$ by the definition of $\Pi_i(t)$ we have

$$\int_{t_k^i}^{t}\Upsilon(t_{k'(t)}^j)\mathrm{d}s+\frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\rm M}}|\xi_i(t_k^i)|\le\Xi_i(t_k^i). \tag{45}$$

On the one hand, combining Eqs. (36), (37), (40), and (41) yields

$$\begin{aligned}\pi_i(t)&=|\varepsilon_i(t)|+\frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\rm M}}|\xi_i(t)|\\&\le\int_{t_k^i}^{t}\Upsilon(t_{k'(t)}^j)\mathrm{d}s+\frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\rm M}}|\xi_i(t_k^i)|\le\Xi_i(t_k^i).\end{aligned} \tag{46}$$

On the other hand,

$$\begin{aligned}&|\varepsilon_i(t)|+|\Phi_i(t)|+|\xi_i(t)|\\&\ge|\kappa_1\,\mathrm{sig}^{\alpha}(\zeta_i(t_k^i))+\kappa_2\zeta_i(t_k^i)+\kappa_3\,\mathrm{sat}(\zeta_i(t_k^i))|\\&=\frac{\sigma\mu_m+\mu_{\rm M}}{\sigma\mu_m}|\Xi_i(t_k^i)|\ge\frac{\sigma\mu_m+\mu_{\rm M}}{\sigma\mu_m}|\varepsilon_i(t)|,\end{aligned} \tag{47}$$

where $|\Phi_i(t)|=|\kappa_1\,\mathrm{sig}^{\alpha}(\zeta_i(t))+\kappa_2\zeta_i(t)|$. Further calculation of the above equation obviously yields

$$\begin{aligned}|\varepsilon_i(t)|&\le\frac{\sigma\mu_m}{\mu_{\rm M}}(|\Phi_i(t)|+|\xi_i(t)|)\\&\le\frac{\sigma\mu_m}{\mu_{\rm M}}(\kappa_1|\,\mathrm{sig}^{\alpha}(\zeta_i(t))|+\kappa_2|\zeta_i(t)|+\kappa_3|\,\mathrm{sat}(\zeta_i(t))|\\&\le\frac{\sigma\mu_m}{\mu_{\rm M}}(\kappa_1|\zeta_i(t)|^{\alpha}+\kappa_2|\zeta_i(t)|+\kappa_3)\doteq\sigma\varphi_i(t).\end{aligned} \tag{48}$$

That is, $|\varepsilon_i(t)|\le\sigma\varphi_i(t)+\frac{\Gamma_i(t)}{\vartheta_i}$, which implies that triggered condition based on $\Pi_i(t)$ is a sufficient condition for $\phi_i(t)\le0$ when $\frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\rm M}}|\xi_i(t_k^i)|+\eta\ge\Xi_i(t_k^i)$. When the next trigger moment occurs, we can obtain

$$\begin{aligned}\tau_i\doteq t_{k+1}^i-t_k^i&=\frac{\Xi_i(t_k^i)-\frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\rm M}}|\xi_i(t_k^i)|}{\Upsilon\left(t_{k'(t)}^j\right)}\\&\ge\frac{\eta}{\Upsilon\left(t_{k'(t)}^j\right)}>0.\end{aligned} \tag{49}$$

This means that Zeno behavior is ruled out. Furthermore, for any $t\in[t_k^i,t_k^i+\tau_i)$, if one of its neighbors triggers an event, it updates $\Upsilon(t_{k'(t)}^j)$ immediately, otherwise agent $i$ waits until $t_{k+1}^i$, which can also show that the team-based event-triggered condition has the same effect as the dynamic event-triggered condition, since for any $t\in[t_k^i,t_{k+1}^i)$, $\phi_i(t)\le0$ holds. Thereby FCC can be achieved and the setting time $T=3T_1$ according to the discussion of Theorem 3. The proof of this theorem is completed.

**Remark 10** For $\frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\rm M}}|\xi_i(t_k^i)|+\eta<\Xi_i(t_k^i)$, there exists $\pi_i(t)=\int_{t_k^i}^{t_{k+1}^i}\Upsilon(t_{k'(t)}^j)\mathrm{d}s+\frac{\sigma\mu_m}{\sigma\mu_m+\mu_{\rm M}}|\xi_i(t_k^i)|-\Xi_i(t_k^i)$. This is known as the self-triggering condition, which is characterized by the fact that only $\int_{t_k^i}^{t_{k+1}^i}\Upsilon(t_{k'(t)}^j)\mathrm{d}s$, $|\xi_i(t_k^i)|$ and $\Xi_i(t_k^i)$ are needed to determine the next trigger time directly, which means that there is no need for continuous communication between agents, an effect expected by the actual system, thus the use of team-triggered control can reduce the loss caused by communication to some extent.

**Remark 11** If only the symbolic function $\mathrm{sign}(\cdot)$ is used in Eq. (3), on the one hand, there will be unnecessary chattering due to the discontinuity in the control input that may result from the application of discontinuity function, which can be avoided by applying the saturation function, and on the other hand, the use of $\mathrm{sat}(\cdot)$ will eliminate the need to calculate the derivative of $\mathrm{sign}(\cdot)$, which will further help us to rule out Zeno behavior.

## 4. Simulation results

Consider the first-order multi-agent systems consisting of five agents with disturbance. As shown in Fig. 1, assuming that 1–3 are follower agents and 4–5 represent leader agents, we can immediately obtain

$$L_1=\begin{bmatrix}0.7&-0.4&0\\-0.44&1.04&-0.2\\0&-0.22&0.47\end{bmatrix},\ L_2=\begin{bmatrix}-0.2&-0.1\\0&0.4\\0.25&0\end{bmatrix},$$

and further calculations yield $\sigma_{\rm M}(L_1^{-1}L_2)=1.2247$. Their dynamics are that $\dot{x}_i(t)=u_i(t)+0.01\cos(1.5t)$, $i=1,2,3$, and $\dot{x}_i(t)=0.01\cos(2.5t)$, $i=4,5$, which obviously has $\bar{d}=0.01$. Under the controller (3), we can choose $\kappa_1=20$, $\kappa_2=28.74$, $\kappa_3=0.42$, and $\alpha=1.5$. For $\Gamma_i(t)$ that satisfies Eq. (9), we set $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=1$, and $s=0.05$. In addition, let $\sigma=0.05$, $\vartheta=0.1$, $\mu_1=1$, $\mu_2=1.1$, and $\mu_3=1.21$. It can be confirmed that $\kappa_2$, $\kappa_3$ satisfy $\sqrt{\lambda_1\kappa_2(1-\sigma)}=\frac{(1-s)\sqrt{\mu_{\rm M}}}{2\mu_m\vartheta_i}$ and $\kappa_3\ge\frac{(m_1+m_2\sigma_{\rm M}(L_1^{-1}L_2))2\mu_{\rm M}\bar{d}}{(1-\sigma)\mu_m}=0.1388$, respectively. According to Theorem 1, we calculate $\rho_1=0.7589$ and $\rho_2=0.2820$, which yield $T_1=14.0207$. Further combining Theorem 3 yields $T=3T_1=42.0621$.

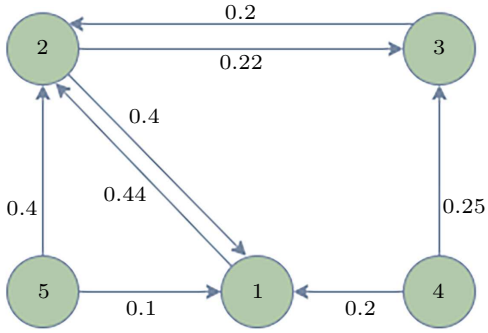



**Fig. 1.** Communication graph.

**Example 1** Containment control under DETC mechanism.

Assume that $x(0)=[9;-1.5;-12;1.5;-1.5]$ and $\Gamma(0)=[10;2;3]$. Under the DETC condition (10), Fig. 2 shows the

state trajectories of all agents, and Fig. 3 shows the plot of the containment error formed between the leader and the follower, which indicates that containment control is achieved at 1.190 s. This convergence time is much less than the calculated upper time limit of $T$.

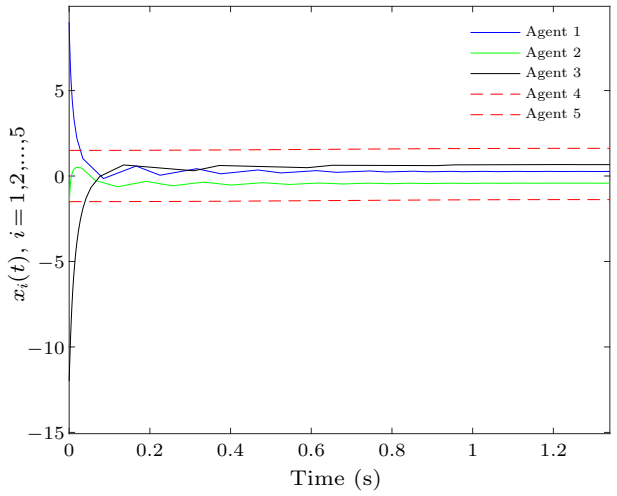



**Fig. 2.** The process of state change under the DETC mechanism.

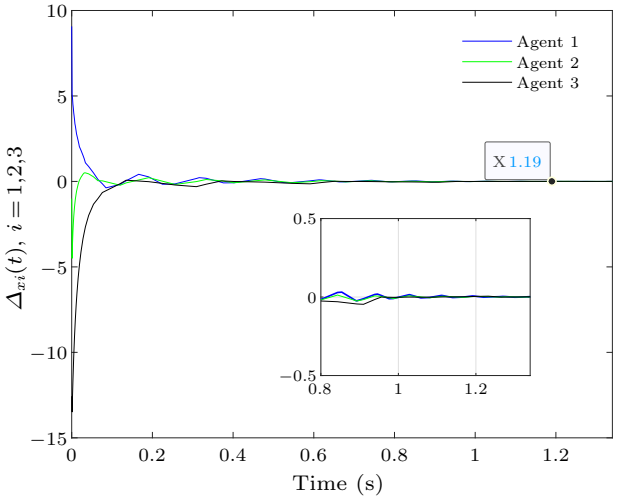



**Fig. 3.** DETC-based containment error.

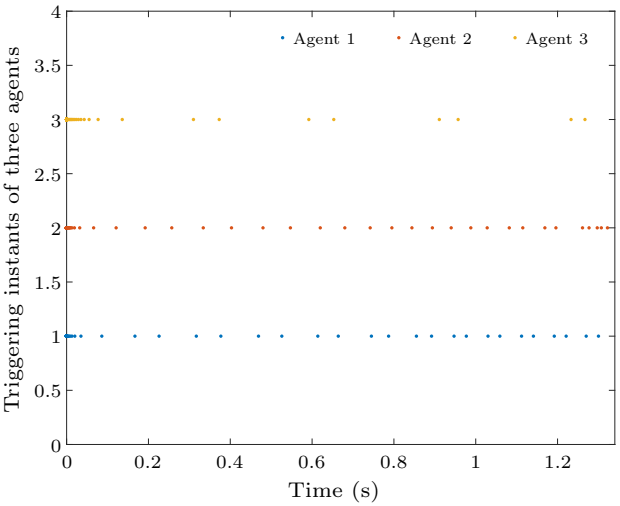



**Fig. 4.** Trigger moments for DETC-based agents.

All trigger moments of the follower agent from 0 to 1.34 s are shown in Fig. 4. It is easy to see that the triggering of the agents is not synchronized and the Zeno phenomenon does not occur. If only symbolic functions are used in Eq. (3), then the trigger moment for each agent is shown in Fig. 5, which may result in triggers being triggered constantly.

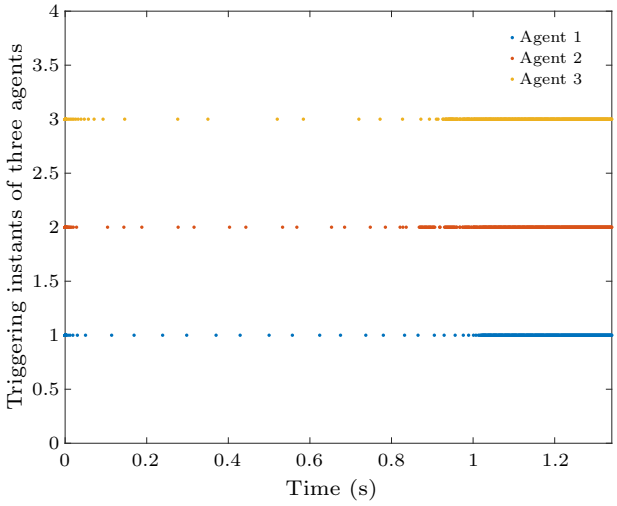



**Fig. 5.** The trigger moment of the DETC-based agent using only the symbolic function sign$(\cdot)$.

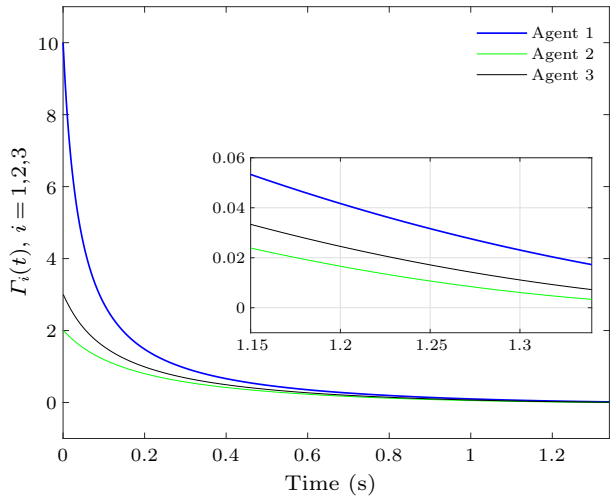



**Fig. 6.** Evolutionary process of dynamic variables.

Figure 6 gives a graph of the evolution of $\Gamma_i(t)$ satisfying Eq. (9), showing that all $\Gamma_i(t)$ are not less than 0, which further confirms the validity of Lemma 6. Take agent 3 as an example and observe the change in control input throughout the process, as shown in Fig. 7, it is clear that this is a continuous function. Moreover, taking agent 2 as an example, the evolution of $\Gamma_i(t)$ and $\vartheta(|\varepsilon_2(t)|-\sigma\phi_2(t))$ is shown in Fig. 8, and it can be inferred that Zeno behavior never occurs when the containment error tends to zero.

**Example 2** Containment control under team-triggered mechanism.

Choosing the same initial conditions as in Example 1, and similar to Fig. 2, Fig. 9 gives the state evolution process for all

agents based on the team trigger conditions (44). The evolution of the containment error generated by the leader and the follower is given in Fig. 10, and it is easy to see that achieving the containment control goal in about 0.844 s has a faster convergence rate compared to using the DETC mechanism.

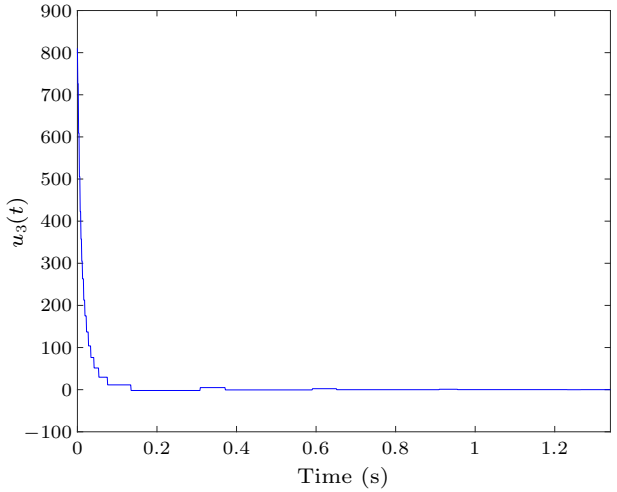



**Fig. 7.** Control input evolution process for agent 3.

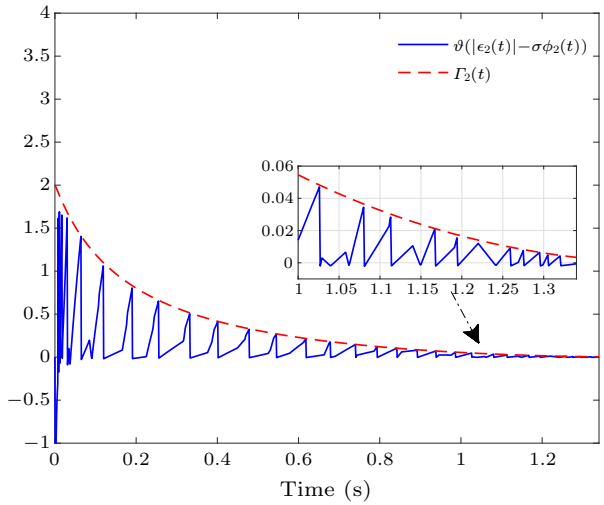



**Fig. 8.** Evolution of $\vartheta(|\varepsilon_2(t)|-\sigma\phi_2(t))$ and $\Gamma_2(t)$.

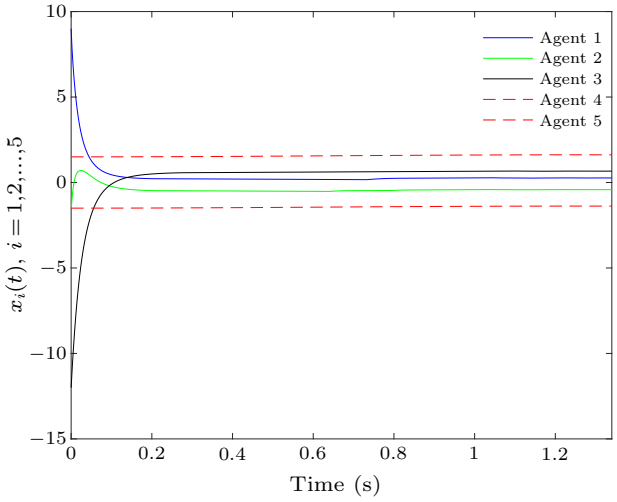



**Fig. 9.** Process of state change under the team-triggered mechanism.

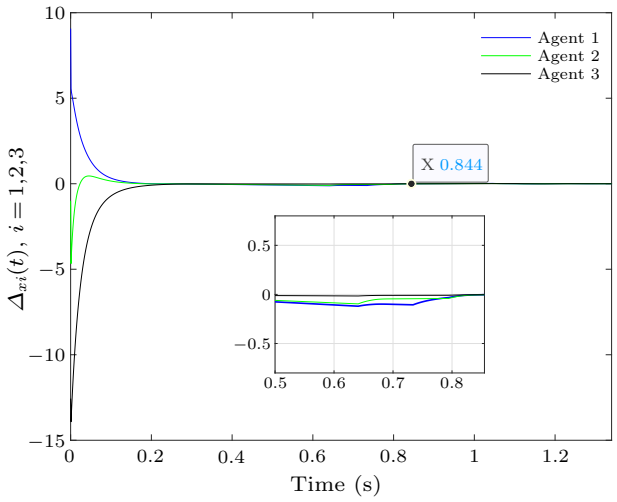



**Fig. 10.** Team-based containment error.

Figure 11 shows the trigger moments for each agent and counts them in Table 1, implying that Zeno behavior is excluded, in addition we find that more triggers are needed to compare to Example 1, which confirms Remark 10 such that in a team-based triggering mechanism, more triggers are needed to compensate for the effects of discontinuous communication, but faster convergence can be obtained, therefore team-based triggering strategy makes a trade-off among convergence speed, communication cost and trigger time.

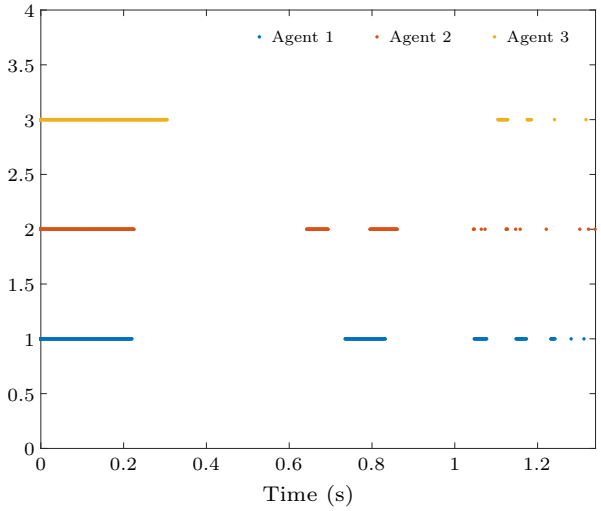



**Fig. 11.** Trigger moments for team-based agents.

**Table 1.** Comparison of data under different trigger conditions.

| Agent | | 1 | 2 | 3 |
|---|---|---|---|---|
| DETC-based | trigger count | 30 | 37 | 24 |
| | convergence time | | 1.19 | |
| Team-based | trigger count | 386 | 359 | 341 |
| | convergence time | | 0.844 | |

**Example 3** (Fixed time stability analysis) Containment control under DETC mechanism.

Assumed that $x(0) = [6;-2;-8;1;-1]$ and $\Gamma(0) = [10;2;3]$, the state trajectories, containment errors, and agent trigger moments for all agents in the DETC condition (10) are

shown in Figs. 12–14, respectively. It is clear that the convergence time does not change with the initial condition of the agent despite its change, which achieves a fixed time containment control under the DETC condition (10).

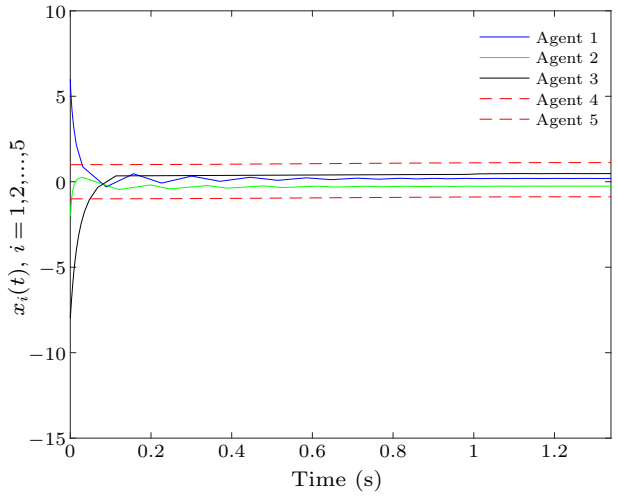



**Fig. 12.** Process of state change under the DETC mechanism.

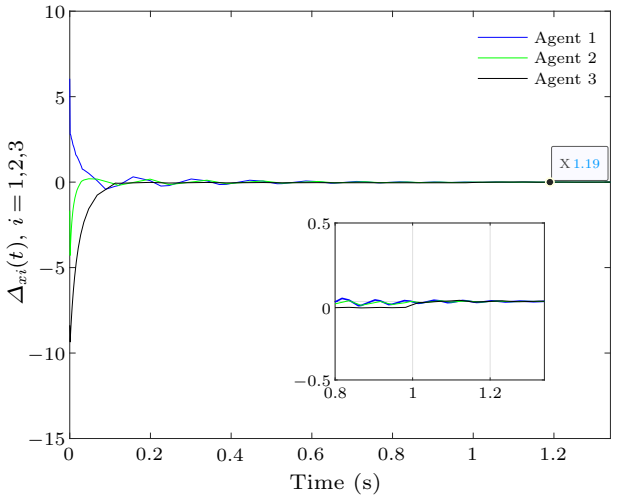



**Fig. 13.** DETC-based containment error.

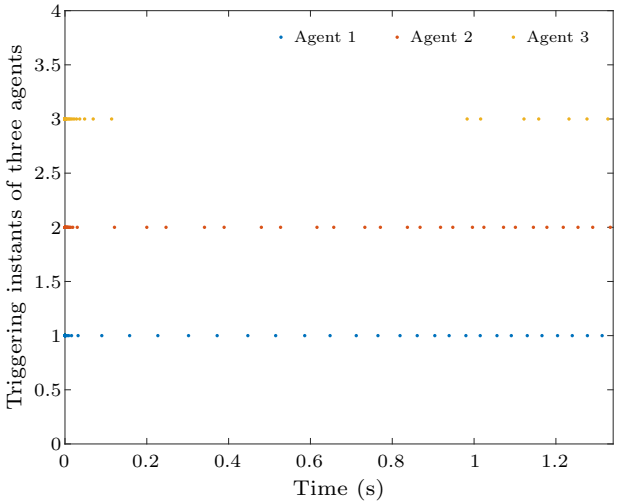



**Fig. 14.** Trigger moments for DETC-based agents.

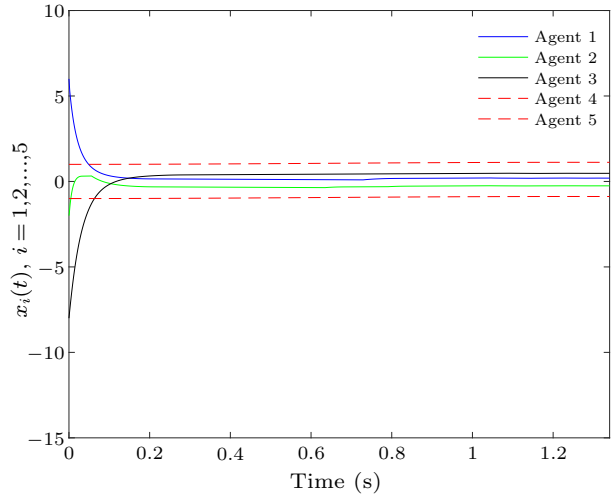



**Fig. 15.** Process of state change under the team-triggered mechanism.

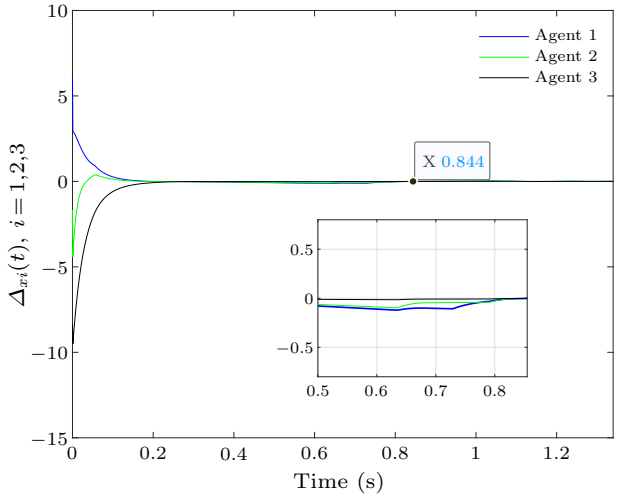



**Fig. 16.** DETC-based containment error.

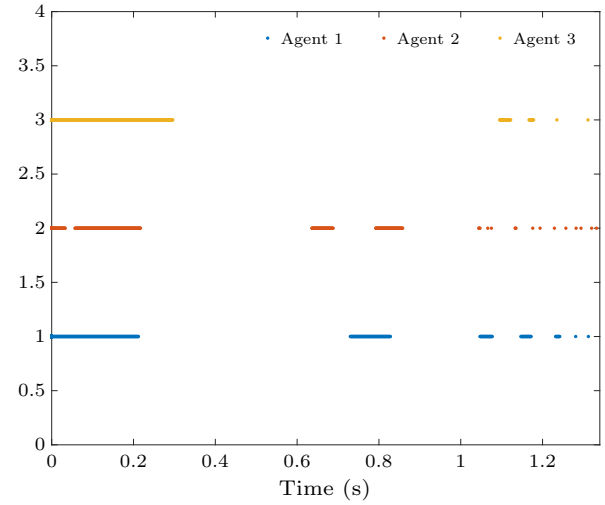



**Fig. 17.** Trigger moments for team-based agents.

**Example 4** (Fixed time stability analysis) Containment control under team-triggered mechanism.

Assumed that $x(0) = [6;-2;-8;1;-1]$ and $\Gamma(0) = [10;2;3]$, similar to Example 3, we also give the state change

process and trigger points under the team trigger conditions (44) as shown in Figs. 15–17. Despite the change in the initial state, the final convergence time is the same as that of Example 2.

## 5. Conclusions

In summary, we have studied the fixed-time containment control problem with disturbances. Two different event triggering algorithms have been proposed to achieve fixed time containment control. The first algorithm successfully implements fixed-time containment control by designing a dynamic variable that reduces the number of controller updates, inspired by the application of dynamic event triggering techniques in consensus control. In order to overcome the problem of continuous communication, a team triggering mechanism has been further proposed, by combining the self-triggering strategy with the dynamic event method proposed in the first algorithm. Team-based triggering mechanisms have been shown to provide a trade-off among convergence time, communication cost, and number of triggers with limited time to achieve containment control, which can also be demonstrated by simulation. In the near future, we will further consider designing a dynamic self-triggering algorithm to achieve fixed-time containment control, where dynamic variables become directly part of the control of the self-triggering algorithm. It is interesting not only to completely eliminate continuous communication but also to reduce the number of triggers. In addition, it is of great potential for solving problems encountered in engineering fields (e.g., heterogeneous network system coordination problems,[34] input-output stability of switching systems).

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