

Quasi-synchronization of fractional-order complex networks with random coupling via quantized control

Hongwei Zhang(张红伟), Ran Cheng(程然), and Dawei Ding(丁大为)
Citation: Chin. Phys. B, 2023, 32 (**11**): 110501. DOI: 10.1088/1674-1056/acedf4
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Explosive synchronization of multi-layer complex networks based on star connection between layers with delay

Yan-Liang Jin(金彦亮), Qian-Yuan Han(韩钱源), Run-Zhu Guo(郭润珠), Yuan Gao(高塬), and Li-Quan Shen(沈礼权)
Chin. Phys. B, 2023, 32 (**10**): 100507. DOI: 10.1088/1674-1056/acbaf1

Stability and multistability of synchronization in networks of coupled phase oscillators

Yun Zhai(翟云), Xuan Wang(王璇), Jinghua Xiao(肖井华), and Zhigang Zheng(郑志刚)
Chin. Phys. B, 2023, 32 (**6**): 060503. DOI: 10.1088/1674-1056/acc808

Synchronization of stochastic complex networks with time-delayed coupling

Duolan(朵兰), Linying Xiang(项林英), and Guanrong Chen(陈关荣)
Chin. Phys. B, 2023, 32 (**6**): 060502. DOI: 10.1088/1674-1056/acc44e

Improved functional-weight approach to oscillatory patterns in excitable networks

Tao Li(李涛), Lin Yan(严霖), and Zhigang Zheng(郑志刚)
Chin. Phys. B, 2022, 31 (**9**): 090502. DOI: 10.1088/1674-1056/ac7458

Explosive synchronization: From synthetic to real-world networks

Atiyeh Bayani, Sajad Jafari, and Hamed Azarnoush
Chin. Phys. B, 2022, 31 (**2**): 020504. DOI: 10.1088/1674-1056/ac3cb0

Quasi-synchronization of fractional-order complex networks with random coupling via quantized control

Hongwei Zhang(张红伟), Ran Cheng(程然), and Dawei Ding(丁大为)†

School of Electronics and Information Engineering, Anhui University, Hefei 230601, *China*

(Received 18 April 2023; revised manuscript received 26 July 2023; accepted manuscript online 8 August 2023)

We investigate the quasi-synchronization of fractional-order complex networks (FCNs) with random coupling via quantized control. Firstly, based on the logarithmic quantizer theory and the Lyapunov stability theory, a new quantized feedback controller, which can make all nodes of complex networks quasi-synchronization and eliminate the disturbance of random coupling in the system state, is designed under non-delay conditions. Secondly, we extend the theoretical results under non-delay conditions to time-varying delay conditions and design another form of quantization feedback controller to ensure that the network achieves quasi-synchronization. Furthermore, the error bound of quasi-synchronization is obtained. Finally, we verify the accuracy of our results using two numerical simulation examples.

Keywords: complex network, fractional-order, random coupling, time-varying delay, quasi-synchronization, quantized control

PACS: 05.45.Xt, 02.30.Yy, 45.10.Hj **DOI:** 10.1088/1674-1056/acedf4

1. Introduction

As critical dynamic systems composed of a large collection of interactive dynamical nodes,[1] complex networks (CNs) have received considerable attention in the past decade because of their potential applications in biological networks,[2] social networks,[3] neural networks,[4] and other fields.[5,6] Therefore, certain characteristics of CNs, such as stability,[7] coordination,[8] optimization,[9] and synchronization[10] have achieved good research results in these fields with increasing efforts.

Synchronization is a significant dynamic property of CNs and is the focus of this study. In many studies, researchers have focused on setting the controller to make the error of the complex network system approach zero with the advancement of time.[11–14] However, because of the existence of system disturbance in practical applications and because this disturbance is difficult to eliminate completely, sometimes the synchronization error cannot fully converge to zero.[15] Therefore, based on the above considerations, it is particularly necessary to study how to control the upper limit of the synchronization error by setting a controller to achieve quasi-synchronization. Recent research on quasi-synchronization has also made progress.[16,17] In Ref. [16], Xu *et al.* studied a fractional-order CN model with delay and obtained global quasi-synchronization results by setting an appropriate intermittent controller. In Ref. [17], the network model was extended from a real number domain to a complex number domain, and the quasi-synchronization problem of fractional-order neural networks in complex-valued systems was solved by separating the systems.

The concepts of fractional calculus and integer-order calculus were proposed simultaneously.[18] As a favorable mathematical tool, fractional calculus has the advantage that integer-order calculi such as genetics and memory can describe the system.[19,20] In recent years, they have been widely used by researchers to process physical systems.[21–24] Therefore, it is necessary to use fractional calculus to describe CNs in the study of synchronization phenomena. For example, in Ref. [25], Bao *et al.* studied adaptive synchronization for a class of fractional-order coupled neural networks (FCNNs) with output coupling. In Ref. [26], the authors investigated the stability and synchronization problems of fractional-order delayed multilink complex networks with nonlinear hybrid coupling.

For CNs, the coupling interference and latency between systems are inevitable due to channel crosstalk and transmission rate limitations. Undoubtedly, these two factors will impact the stability of the system.[27–29] Therefore, it is important to consider random coupling disturbances and time delays in the synchronization of complex networks. For example, in Refs. [30,31] and in the above-mentioned literature, only time delay is often considered to be constant, but in an actual system, with the evolution of network dynamics characteristics, the time delay often changes with time rather than being immutable, and may break the stability of the complex networks dynamics system. Therefore, it is more practical to consider time-varying delays in a network-dynamics system.

In recent years, researchers have proposed numerous synchronization control methods, such as adaptive control,[32] periodic intermittent control,[33] and pinning control.[34] In these control methods, researchers typically regarded a control sig-

†Corresponding author. E-mail: dwding@ahu.edu.cn

© 2023 Chinese Physical Society and IOP Publishing Ltd

nal as a continuous distortion-free signal without preprocessing it. However, in real life, when a signal is transmitted in a channel, signal attenuation distortion is often caused by the influence of the channel bandwidth or the channel itself.[35,36] To solve this problem, we quantify the signal before transmission such that the processed signal has good stability and anti-interference during the transmission process. This method, adopted for the control of CNs, is generally referred to as quantized control. In Ref. [37], Shi *et al.*, through the quantized control method, studied the finite- and fixed-time synchronization problems of complex networks using the theory of linear matrix inequality. In Ref. [38], Fan *et al.* studied the synchronization of delayed neural networks using the quantized control theory.

Based on the above considerations, quasi-synchronization of FCNs via quantized control was the main focus of this study. The main contributions of this study are as follows:

1. Our research is centered around addressing the issue of quasi-synchronization in CNs by incorporating fractional calculus techniques.

2. Through logarithmic quantizer, the appropriate quantization controller is designed to make the complex network achieve quasi-synchronization. Simultaneously, this study considers the quasi-synchronization state of the complex network without delay and time-varying delay and designs different quantization controllers. Compared with Ref. [16], it undoubtedly expands the practicability of the results of the study.

3. In FCNs, both random coupling and external disturbance are simultaneously considered. In the absence of delay and time-varying delay, we have successfully achieved quasi-synchronization, ensuring system stability under the influence of these two types of disturbances. Furthermore, we are able to accurately calculate the maximum upper bound of quasi-synchronization error, and by adjusting the parameters of the controller, we can control the upper limit of synchronization error.

The rest of this paper is organized as follows: In Section 2, we present the concepts necessary for the discussion, lemmas, and the FCNs model. Based on the fractional Lyapunov stability theory, we present the results of quasi-synchronization in both non-delay and time-delay scenarios in Section 3. Section 4 presents the correctness of the theoretical results obtained from numerical simulations. Finally, Section 5 summarizes this study.

Notions Let $\mathbb{R}$ and $\mathbb{R}^n$ be real number set and n-dimensional Euclidean space, respectively. I_n and $||\cdot||$ denote the unit matrix and Euclidean norm, respectively; and $||\cdot||_\infty$ represents the infinity norm, $||\cdot||_1$ represents the 1-norm; $\otimes$ denotes the Kronecker product, and $n\times n$ set of matrices is represented by $\mathbb{R}_{n\times n}$. Let H be a square matrix of order n, if for any nonzero column vector x, there is $x^{\mathrm{T}}Hx>0$, where x^{T} denotes the transpose of x, then H is called a positive definite matrix, otherwise, H is called a negative definite matrix, and $\lambda_{\max}(H)$ is the maximum eigenvalue of matrix H.

2. Preliminaries

In this section, for preferably setting the problem, some essential definitions of the logarithmic quantizer and Caputo fractional calculus are reviewed, helpful lemmas are presented, and a mathematical model of FCNs is developed.

Definition 1[38] In the fractional derivative of order α for a given function $f(t)$, its fractional derivative of order α is defined as

$$ {}_{t_0}^{c}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f^{(m)}(s)}{(t-s)^{\alpha-m+1}}\,\mathrm{d}s, $$

where $\alpha\in\mathbb{R}_+, \Gamma(\cdot)$ is a gamma function, m denotes a positive integer, and $0<m-\alpha<1$. Generally, $\alpha\in(0,1)$ and $m=1$. We have

$$ {}_{t_0}^{c}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(s)}{(t-s)^{\alpha}}\,\mathrm{d}s. $$

The properties of the two common fractional calculus operations are as follows:

- If c is a constant, ${}_{t_0}^{c}D_t^{\alpha} c = 0$.
- For any functions $f_1(t), f_2(t)$ and constants a, b, we have ${}_{t_0}^{c}D_t^{\alpha} a f_1(t) + {}_{t_0}^{c}D_t^{\alpha} b f_2(t) = a\,{}_{t_0}^{c}D_t^{\alpha} f_1(t) + b\,{}_{t_0}^{c}D_t^{\alpha} f_2(t)$.

Definition 2[20] Given a continuous function τ, its related logarithmic quantizer $q(\tau):\mathbb{R}\to\Omega$ is defined as

$$ q(\tau) = \begin{cases} \omega_p, & \text{if} \quad \frac{1}{1+\zeta}\omega_p \le \tau \le \frac{1}{1-\zeta}\omega_p, \\ 0, & \text{if} \quad \tau = 0, \\ -q(-\tau), & \text{if} \quad \tau \le 0, \end{cases} $$

where $\Omega = \{\pm\rho_p : \omega_p = \rho^p\omega_0,\ p = 0,\pm1,\pm2,\cdots\}\cup\{0\}$, $\rho\in[0,1]$ denotes the quantized density, and $\zeta = \frac{1-\rho}{1+\rho}$. For any $\tau(\cdot)$, we can build the equation

$$ q(\tau) = (1+\Delta)\tau, $$

where $\Delta\in[-\zeta,\zeta]$ denotes the Filippov solution.

Definition 3[26] The general form of the Mittag–Leffler function is as follows:

$$ E_{\alpha,\beta}(r) = \sum_{k=0}^{\infty} \frac{r^k}{\Gamma(\alpha k+\beta)}, $$

where $r,\beta\in\mathbb{R}$ and $\alpha>0$. For $\beta=1$, the one-parameter form is as follows:

$$ E_{\alpha,1}(r) = \sum_{k=0}^{\infty} \frac{r^k}{\Gamma(\alpha k+1)}, $$

Specifically, $E_{1,1}(r) = \mathrm{e}^r$.

2.1. Model description

Considering FCNs with N nodes, the dynamic formulas for each node can be expressed as follows:

$$\begin{aligned} {}_{t_0}^{c}D_t^{\alpha}x_i(t) = f(x_i(t)) + c(t)\sum_{j=1}^{N} l_{ij}Gx_j(t) \\ + \omega_i(t) + u_i(t), \end{aligned} \tag{1}$$

where $\alpha \in (0,1)$, $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^{\mathrm{T}} \in \mathbb{R}^n$, $i = 1,2\ldots,N$, represents the state vector of i-th node, $f(x_i(t)) : \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions. If a connection between j-th node to i-th node $(i \neq j)$ exists, then $l_{ij} \geq 0$; otherwise, $l_{ij} = 0$ and the diagonal elements $l_{ii} = -\sum_{j=1, j\neq i}^{N} l_{ij}$, $(i = 1,2,\ldots,N)$. $G = \mathrm{diag}(g_1, g_2, \ldots, g_m) \in \mathbb{R}^n$ denotes the internal coupling strength matrix, which refers to the coupling relationship between different dimensional parameters in the network nodes. Here, $c(t)$ denotes the random coupling strength representing the variation of the external coupling strength between nodes in a complex network over time; $\omega_i(t)$ denotes an external disturbance, and $u_i(t)$ represents the quantized controller.

The dynamic formula for an isolated node is as follows:

$${}_{t_0}^{c}D_t^{\alpha}s(t) = f(s(t)), \tag{2}$$

where $s(t) = (s_1(t), s_2(t), \ldots, s_n(t))^{\mathrm{T}} \in \mathbb{R}^n$.

We define the synchronization errors between systems (1) and (2) as $e(t) = (e_1(t), e_2(t), \ldots, e_N(t)), e_i(t) = x_i(t) - s(t)$, then we can obtain the error formula

$$\begin{aligned} {}_{t_0}^{c}D_t^{\alpha}e_i(t) = f(x_i(t)) - f(s(t)) + c(t)\sum_{j=1}^{N} l_{ij}Ge_j(t) \\ + \omega_i(t) + u_i(t). \end{aligned} \tag{3}$$

Definition 4[16] System (1) is said to be quasi-synchronized with state (2) for any initial value if there exists a small error bound $\varepsilon > 0$ such that

$$\lim_{t\to+\infty} ||x_i(t) - s(t)|| = \lim_{t\to+\infty} ||e_i(t)|| \leq \varepsilon,$$

where $t \geq t_0$.

Assumption 1 Suppose that the random coupling $c(t)$ and external disturbance $\omega_i(t)$ in dynamic system (1) satisfy the relations

$$\begin{aligned} 0 < c(t) \leq C_{\max}, \\ 0 < ||\omega_i(t)|| \leq \omega_{\max}, \end{aligned}$$

where $C_{\max} > 0$, $\omega_{\max} > 0$.

Assumption 2 Suppose that the two given functions $x(t), y(t)$ satisfy the following Lipschitz condition:

$$\begin{aligned} (y(t) - x(t))^{\mathrm{T}}(f(y(t)) - f(x(t))) \\ \leq L(y(t) - x(t))^{\mathrm{T}}(y(t) - x(t)), \end{aligned}$$

where $L > 0$, $x(t), y(t) \in \mathbb{R}^n$.

Lemma 1[14] Given $x, y \in \mathbb{R}^n$, there exists $\varepsilon \in (0, +\infty)$ and any $n \times n$ real matrix $Q \in (0, +\infty)$ such that the following formula holds:

$$2x^{\mathrm{T}}y \leq \varepsilon^{-1}x^{\mathrm{T}}Q^{-1}x + \varepsilon y^{\mathrm{T}}Qy.$$

Lemma 2[25] Given a derivable function $x(t)$, there exists $t \geq t_0$ such that the following formula holds:

$$\frac{1}{2}({}_{t_0}^{c}D_t^{\alpha}x^{\mathrm{T}}(t)x(t)) \leq x^{\mathrm{T}}(t){}_{t_0}^{c}D_t^{\alpha}x(t).$$

Lemma 3[17] Let $t > t_0$, $E_\alpha(\lambda(t - t_0)^\alpha)$ be monotonically non-increasing, and $0 \leq E_\alpha(\lambda(t - t_0)^\alpha) \leq 1$ for $\lambda \leq 0$.

Lemma 4[17] Let $\omega : [t_0, \infty) \to \mathbb{R}$ be a differential function such that

$${}_{t_0}^{c}D_t^{\alpha}\omega(t) \leq -a\omega(t) + b, \quad t \geq t_0 \geq 0.$$

If $a > 0$, $b \geq 0$, then

$$\omega(t) \leq \left(\omega(t_0) - \frac{b}{a}\right)E_\alpha(-a(t - t_0)^\alpha) + \frac{b}{a}, \ \ t \geq t_0 \geq 0.$$

3. Main result

In this section, we design the corresponding quantized controller such that the network system in different states can achieve quasi-synchronization, and the quasi-synchronization criteria of FCNs with random coupling are obtained under non-delay and time-varying delays.

3.1. Quasi-synchronization of FCNs

Theorem 1 Assume that Assumptions 1 and 2 hold, if there exists a constant $\varepsilon_1 > 0$, we can gain $k^* >$ such that

$$\lambda_{\max}\{\Pi - 2k^*(1 - \delta) \otimes I_{Nn}\} < 0,$$

where $\Pi = 2L \otimes I_{Nn} + C_{\max}[(G \otimes \Gamma) + (G \otimes \Gamma)^{\mathrm{T}}] + 2\varepsilon_1 \otimes I_{Nn}$, and if the controller $u_i(t)$ is designed as

$$u_i(t) = -k^* q(e_i(t)),$$

then the system is quasi-synchronized. We have

$$\lim_{t\to+\infty} ||e(t)|| \leq \sqrt{\frac{B}{|A|}},$$

where

$$\begin{aligned} A = \lambda_{\max}\{\Pi - 2k^*(1 - \delta) \otimes I_{Nn}\}, \\ B = 2\left[\frac{N}{\varepsilon_1}||\omega_{\max}||^2\right]. \end{aligned}$$

Proof Consider the following Laypunov function:

$$V(t) = \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t). \tag{4}$$

According to Assumptions 1 and 2 and Lemma 1, we can obtain the fractional derivative of $V(t)$:

$$
\begin{aligned}
{}_{t_0}^{c}D_t^{\alpha}V(t) &\le \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\,{}_{t_0}^{c}D_t^{\alpha}e_i(t) + \sum_{i=1}^{N}({}_{t_0}^{c}D_t^{\alpha}e_i(t))^{\mathrm{T}}e_i(t)\\
&= \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)[F(x_i(t)) - F(s(t)) + c(t)\sum_{j=1}^{N} g_{ij}\Gamma e_j(t)\\
&\quad + \omega_i(t) + u_i(t)] + \sum_{i=1}^{N}[F(x_i(t)) - F(s(t))\\
&\quad + c(t)\sum_{j=1}^{N} g_{ij}\Gamma e_j(t) + \omega_i(t) + u_i(t)]^{\mathrm{T}}e_i(t)\\
&\le \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)[Le_i(t) + c(t)\sum_{j=1}^{N} g_{ij}\Gamma e_j(t) + \omega_i(t) + u_i(t)]\\
&\quad + \sum_{i=1}^{N}[Le_i(t) + c(t)\sum_{j=1}^{N} g_{ij}\Gamma e_j(t) + \omega_i(t)\\
&\quad - (1-\delta)k^* e_i(t)]^{\mathrm{T}}e_i(t)\\
&\le 2L\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) + c(t)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\sum_{j=1}^{N} g_{ij}\Gamma e_j(t)\\
&\quad + c(t)\sum_{i=1}^{N}(\sum_{j=1}^{N} g_{ij}\Gamma e_j(t))^{\mathrm{T}}e_i(t)\\
&\quad + \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\omega_i(t) + \sum_{i=1}^{N}\omega_i^{\mathrm{T}}(t)e_i(t)\\
&\quad + \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)u_i(t) + \sum_{i=1}^{N} u_i^{\mathrm{T}}(t)e_i(t)\\
&\le \mathrm{e}^{\mathrm{T}}\{2L\otimes I_{Nn} + C_{\max}[(G\otimes\Gamma) + (G\otimes\Gamma)^{\mathrm{T}}]\}e(t)\\
&\quad + 2[\varepsilon_1\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) + \frac{1}{\varepsilon_1}\sum_{i=1}^{N}\omega_i^{\mathrm{T}}(t)\omega_i(t)]\\
&\quad + 2\varepsilon_2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) - 2k^*(1-\delta)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t)\\
&\le \mathrm{e}^{\mathrm{T}}\{2L\otimes I_{Nn} + C_{\max}[(G\otimes\Gamma) + (G\otimes\Gamma)^{\mathrm{T}}]\}e(t)\\
&\quad + 2[\varepsilon_1\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) + \frac{N}{\varepsilon_1}||\omega_{\max}||^2]\\
&\quad + 2\varepsilon_2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) - 2k^*(1-\delta)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t)\\
&\le -AV(t) + B.
\end{aligned}
$$

As $A > 0$ and $B > 0$,from Lemma 8, we have

$$V(t) \le \left[V(t_0) - \frac{B}{A}\right]E_{\alpha}(-A(t-t_0)^{\alpha}) + \frac{B}{A}. \tag{5}$$

Then, based on the constructed Lypunov function, we obtain

$$||e(t)|| \le \sqrt{\left[V(t_0) - \frac{B}{A}\right]E_{\alpha}(-A(t-t_0)^{\alpha}) + \frac{B}{A}}. \tag{6}$$

Finally, we obtain

$$\lim_{t\to+\infty}\|e(t)\| \le \sqrt{\frac{B}{A}}. \tag{7}$$

The proof is completed.

3.2. Quasi-synchronization of FCNs with time-varying delays

Consider the following FCNs composed of N identical nodes with time-varying delays, described by

$$
\begin{aligned}
{}_{t_0}^{c}D_t^{\alpha}x_i(t) &= f(x_i(t)) + c(t)\sum_{j=1}^{N} l_{ij}Gx_j(t-\tau(t))\\
&\quad + \omega_i(t) + u_i(t),
\end{aligned} \tag{8}
$$

the interpretation of some mathematical symbols in the above equation is the same as those in Eq. (1). Additionally, $\tau(t)$ denotes the time-varying delay.

We assume that the dynamic behavior of the isolated node is the same as that in Eq. (2), and the dynamic error system for network (8) can be obtained as follows:

$$
\begin{aligned}
{}_{t_0}^{c}D_t^{\alpha}e_i(t) &= f(x_i(t)) - f(s(t)) + c(t)\sum_{j=1}^{N} l_{ij}Ge_j(t-\tau(t))\\
&\quad + \omega_i(t) + u_i(t).
\end{aligned} \tag{9}
$$

Theorem 2 Assume that Assumptions 1 and 2 hold, if there are constants $\varepsilon_2 > 0$, we can gain $k_0 > 0$, and matrix Λ, the following inequalities and linear matrix inequalities hold:

$$L + \varepsilon_2 - (1-\delta)k_0 < 0,$$

$$\Lambda = \begin{bmatrix} C_{\max}d_G - (1-\delta)k_1 & 0\\ 0 & C_{\max}b_G - (1-\delta)k_2 \end{bmatrix} \le 0,$$

where $d_G = \frac{1}{2}||G||_{\infty}$, $b_G = \frac{1}{2}||G||_1$, and k_1 and k_2 are positive constants. If the following controller $u_i(t)$ is designed as

$$
\begin{aligned}
u_i(t) = &-(k_0+k_1)q(e_i(t))\\
&- k_2\mathrm{sign}(q(e_i(t)))\frac{||q(e_i(t-\tau(t)))||^2}{|q(e_i(t))|},
\end{aligned}
$$

then the system is quasi-synchronized. We have

$$\lim_{t\to+\infty}\|e(t)\| \le \sqrt{\frac{D}{C}},$$

where

$$
\begin{aligned}
C &= |L + \varepsilon_2 - (1-\delta)k_0|,\\
D &= \frac{N}{\varepsilon_2}||\omega_{\max}||^2.
\end{aligned}
$$

Proof The following Laypunov function is constructed:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t). \tag{10}$$

From Assumptions 1 and 2 and Lemma 2, we can calculate the fractional derivative of $V(t)$ as

$${}_{t_0}^{c}D_t^{\alpha}V(t) \le \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\,{}_{t_0}^{c}D_t^{\alpha}e_i(t)$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)[f(x_i t) - f(s(t)) \\
&\quad + c(t)\sum_{j=1}^{N} g_{ij}\Gamma e_j(t-\tau(t)) + \omega_i(t) + u_i(t)] \\
&\le L\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) \\
&\quad + c(t)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\sum_{j=1}^{N} g_{ij}\Gamma e_j(t-\tau(t)) \\
&\quad + \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\omega_i(t) \\
&\quad - (1-\delta)(k_0+k_1)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) \\
&\quad - (1-\delta)k_2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t) \\
&\quad \times \left(\mathrm{sign}(e_i(t))\frac{||e_i(t-\tau(t))||^2}{|e_i(t)|}\right) \\
&= V_1(t) + V_2(t).
\end{aligned} \tag{11}
$$

From Eq. (11), we have

$$
\begin{aligned}
V_1(t) &= L\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) + \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\omega_i(t) \\
&\quad - (1-\delta)k_0\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t), \\
V_2(t) &= c(t)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\sum_{j=1}^{N} g_{ij}\Gamma e_j(t-\tau(t)) \\
&\quad - (1-\delta)k_2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t) \\
&\quad \times \left(\mathrm{sign}(e_i(t))\frac{||e_i(t-\tau(t))||^2}{|e_i(t)|}\right) \\
&\quad - (1-\delta)k_1\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t).
\end{aligned}
$$

For $V_1(t)$, we can rewrite it as

$$
\begin{aligned}
V_1(t) &\le L\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) + \varepsilon_2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) \\
&\quad + \frac{1}{\varepsilon_2}\sum_{i=1}^{N}\omega_i^{\mathrm{T}}(t)\omega_i(t) - (1-\delta)k_0\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) \\
&= (L+\varepsilon_2-(1-\delta)k_0)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) \\
&\quad + \frac{1}{\varepsilon_2}\sum_{i=1}^{N}\omega_i^{\mathrm{T}}(t)\omega_i(t) \\
&\le (L+\varepsilon_2-(1-\delta)k_0)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) \\
&\quad + \frac{N}{\varepsilon_2}||\omega_{\max}||^2.
\end{aligned}
$$

Then, according to condition (19), we obtain $V_1 \le 0$.

For $V_2(t)$, we can obtain:
when $i \ne j$

$$
\begin{aligned}
c(t)g_{ij}e_i^{\mathrm{T}}(t)\Gamma e_j(t) &\le C_{\max}g_{ij}e_i^{\mathrm{T}}(t)e_i(t) \\
&\quad + C_{\max}g_{ij}e_j^{\mathrm{T}}(t-\tau(t))e_j(t-\tau(t)),
\end{aligned}
$$

when $i = j$

$$
\begin{aligned}
c(t)g_{ii}e_i^{\mathrm{T}}(t)\Gamma e_i(t) &\le C_{\max}|g_{ii}|e_i^{\mathrm{T}}(t)e_i(t) \\
&\quad + C_{\max}|g_{ii}|e_i^{\mathrm{T}}(t-\tau(t))e_i(t-\tau(t)).
\end{aligned}
$$

Thus, we can easily obtain

$$
\begin{aligned}
&c(t)\sum_{j=1}^{N} g_{ij}\Gamma e_j(t-\tau(t)) \\
&\le C_{\max}\sum_{i=1}^{N}\sum_{j=1}^{N}\{|g_{ij}|e_i^{\mathrm{T}}(t)e_i(t) \\
&\quad + |g_{ij}|e_j^{\mathrm{T}}(t-\tau(t))e_j(t-\tau(t))\} \\
&\le C_{\max}\sum_{i=1}^{N}\{d_G e_i^{\mathrm{T}}(t)e_i(t) + b_G e_i^{\mathrm{T}}(t-\tau(t))e_i(t-\tau(t))\}.
\end{aligned}
$$

Subsequently, for $V_2(t)$ we obtain

$$
\begin{aligned}
V_2(t) &\le C_{\max}\sum_{i=1}^{N}\{d_G e_i^{\mathrm{T}}(t)e_i(t) + b_G e_i^{\mathrm{T}}(t-\tau(t))e_i(t-\tau(t))\} \\
&\quad - (1-\delta)k_2\sum_{i=1}^{N}||e_i(t-\tau(t))||^2 \\
&\quad - (1-\delta)k_1\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) \\
&\le C_{\max}\sum_{i=1}^{N} d_G||e_i(t)||^2 - (1-\delta)k_1\sum_{i=1}^{N}||e_i(t)||^2 \\
&\quad + C_{\max}\sum_{i=1}^{N} b_G||e(t-\tau(t))||^2 \\
&\quad - (1-\delta)k_2\sum_{i=1}^{N}||e_i(t-\tau(t))||^2 \\
&= \Theta^{\mathrm{T}}(t)\Lambda\Theta(t),
\end{aligned}
$$

where

$$
\Lambda = \begin{bmatrix} C_{\max}d_G - (1-\delta)k_1 & 0 \\ 0 & C_{\max}b_G - (1-\delta)k_2 \end{bmatrix}.
$$

We can properly choose k_1, k_2 and δ, such that $\Lambda \le 0$.

Then, we can easily obtain

$$
\begin{aligned}
V(t) &= V_1 + V_2 \\
&\le (L+\varepsilon_2-(1-\delta)k_0)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) + \frac{N}{\varepsilon_2}||\omega_{\max}||^2 \\
&= -CV(t) + D.
\end{aligned}
$$

As $C > 0$ and $D > 0$, from Lemma 8, we have

$$
V(t) \le \left[V(t_0) - \frac{D}{C}\right]E_\alpha(-C(t-t_0)^\alpha) + \frac{D}{C}. \tag{12}
$$

Then, based on the constructed Lypunov function, we can reach

$$||e(t)|| \leq \sqrt{\left[V(t_0) - \frac{D}{C}\right] E_\alpha(-A(t-t_0)^\alpha) + \frac{D}{C}}. \tag{13}$$

Finally, we obtain

$$\lim_{t\to+\infty} ||e(t)|| \leq \sqrt{\frac{D}{C}}. \tag{14}$$

This completes the proof.

Remark 1 The result of Theorem 2 is based on the FCNs system with time-varying delay. If we change $\tau(t)$ to τ, that is, the type of time delay in system (8) is a fixed time delay, the results of Theorem 2 will be applicable. Thus, the results of Theorem 2 include the consideration of the FCNs system with fixed time delay.

4. Numerical simulations

In this section, we mainly conduct numerical simulations on the two theorems presented in Section 3. By observing the dynamic behavior of the nodes in the given FCNs, we have verified the correctness of Theorems 1 and 2.

Example 1 The FCNs dynamics formula of three-dimensional Lorenz system is given, and the number of nodes is four. We have

$$\begin{aligned} {}_{t0}^{c}D_t^\alpha x_i(t) = &\begin{pmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix} + c(t)\sum_{j=1}^{N} l_{ij} G x_j(t) + \omega_i(t). \end{aligned} \tag{15}$$

The response system reads

$${}_{t0}^{c}D_t^\alpha s(t) = \begin{pmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -s_1 s_3 \\ s_1 s_2 \end{pmatrix}, \tag{16}$$

where $\alpha = 0.995$, $a = 10$, $b = 8/3$, $c = 28$, $\omega_i(t) = (2\sqrt{2}|\sin(t)|, 2\sqrt{2}|\cos(t)|, 0)$, $c(t) = 2 + \cos(t)$, $G = \mathrm{diag}(1,1,1)$. The initial system value $x_i(0) = (-2+0.1i, 1+0.2i, 2+0.3i)$ $(i = 1,2,3,4)$, $s(0) = (1,1,1)$, and the outer coupling matrix is given as follows:

$$L = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}. \tag{17}$$

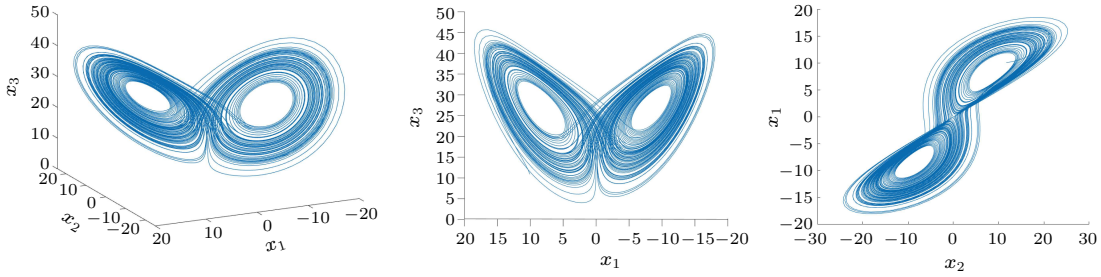


Fig. 1. Phase plot of the Lorenze system ($\alpha = 0.995$).

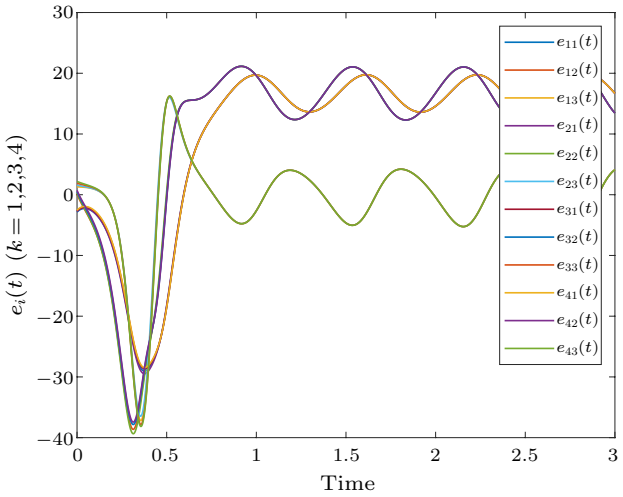


Fig. 2. Profiles of $e_i(t)$ of the error system without control.

To set the quantizer, we select the following parameters: $\rho = 0.6$, $\delta = \frac{1-\rho}{1+\rho} = 0.25$. For the Lorenze system, $L = 25.12$, $C_{\max} = 3$, $\omega_{\max} = 2\sqrt{2}$ such that Assumptions 1 and 2 hold, and we select $\varepsilon_1 = 1$, $k^* = 48$. Then, according to Theorem 1, we can easily obtain $A = \lambda_{\max}\{\Pi - 2k^*(1-\delta)\otimes I_{Nn}\} = -17.1<0$, so $B = 2[\frac{N}{\varepsilon_1}||\omega_{\max}||^2] = 64$ and $\sqrt{\frac{B}{|A|}} = 1.9$. Based on Theorem 1, system (1) is quasi synchronized.

Figure 2 shows the synchronization error $e_i(t)$ fluctuations under uncontrolled conditions. Figure 3 shows the synchronization quantized error $q(e_i(t))$ fluctuations under uncontrolled conditions. Figure 4 shows that the synchronization errors $e_i(t)$ fluctuate within an interval. Figure 5 shows that the synchronization quantized errors $q(e_i(t))$ fluctuate within an interval. Figure 6 displays the quasi-synchronization error $||e||$ under uncontrolled conditions. Figure 7 shows the quasi-synchronization error $||e||$, which indicates that quasi-synchronization is realized. Through a simulation comparison, we observe that the results of Theorem 1 are correct.

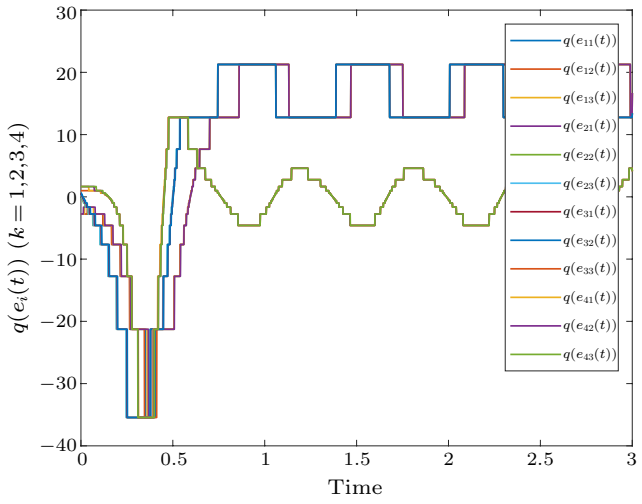


Fig. 3. Profiles of $q(e_i(t))$ of the error system without control.

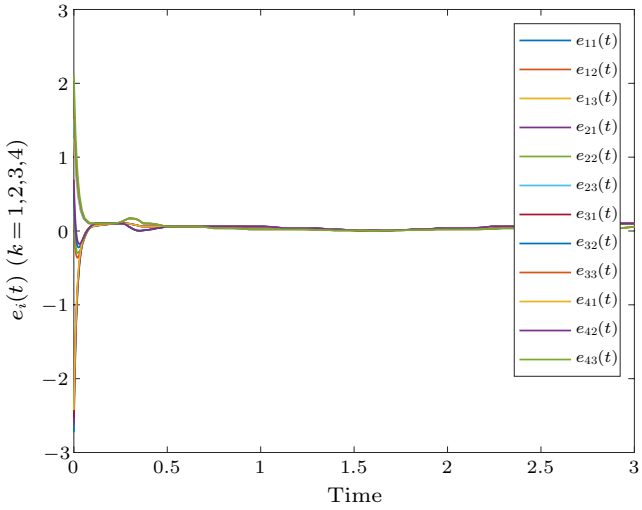


Fig. 4. Profiles of $e_i(t)$ of the error system.

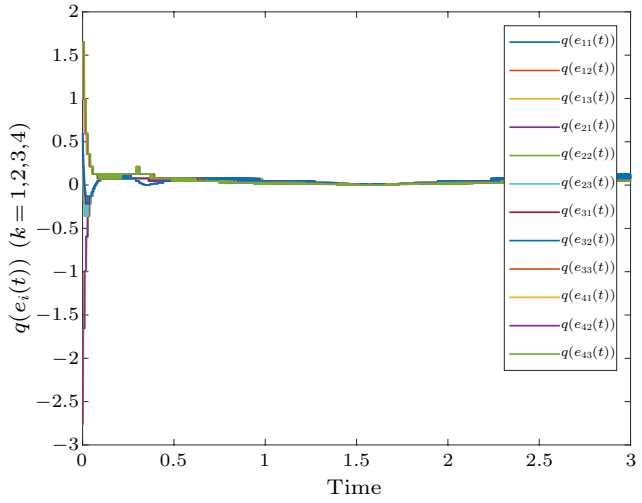


Fig. 5. Profiles of $q(e_i(t))$ of the error system.

Example 2 Considering the three-dimensional FCNs composed of 4 nodes with time-varying delay, the node dynamics system is the same as Example 1:

$$
\begin{aligned}
{}_{t0}^{c}D_t^{\alpha}x_i(t) = &\begin{pmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}\begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} \\
&+\begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix} + c(t)\sum_{j=1}^{N} l_{ij}Gx_j(t-\tau(t)) + \omega_i(t),
\end{aligned} \tag{18}
$$

where $a=10$, $b=8/3$, $c=28$, $\omega_i(t)=(0,2-\mathrm{e}^{-t},0)^{\mathrm{T}}$, $c(t)=|\cos(t)|$, $\tau(t)=|\sin(t)|$. The system initial value is $x_i(0)=(4+0.1i,1+0.1i,4-0.1i)$ $(i=1,2,3,4)$, $s(0)=(1,2,1)$. The settings for the logarithmic quantizers are the same as those in Example 1.

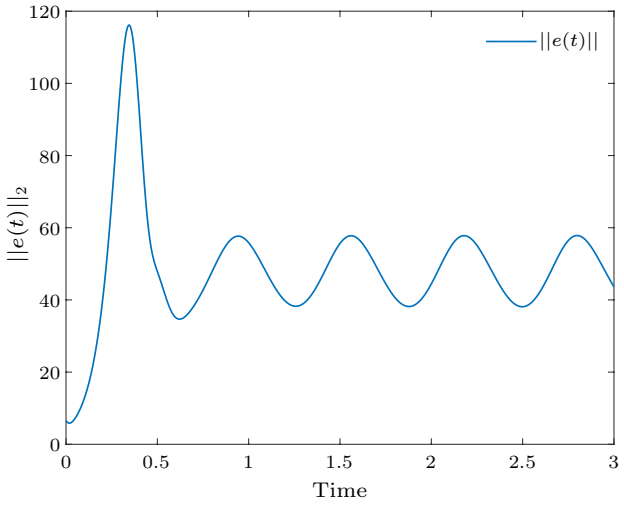


Fig. 6. Profiles of $||e(t)||$ of the error system without control.

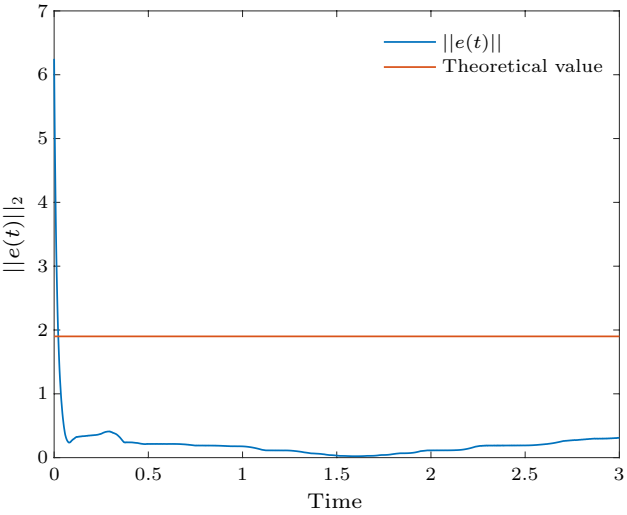


Fig. 7. Profiles of $||e(t)||$ of the error system.

For the Lorenze system, $L=25.12$, $C_{\max}=1$, $\omega_{\max}=2$ such that Assumptions 1 and 2 hold, and we select $\varepsilon_2=2$, $k_0=40$, $k_1=10$, $k_2=10$. Then, according Theorem 2, we can easily obtain $d_g=2.5$, $b_g=3$, $\Lambda=\mathrm{diag}(2.5-0.75\cdot 10, 3-0.75\cdot$

$10) = \mathrm{diag}(-5,-4.5) \le 0$, $C = |25.12+2-0.75\cdot 40| = 2.88$, $D = \frac{N}{\varepsilon_2}||\omega_{\max}||^2 = 8$, then $\sqrt{\frac{D}{C}} = 1.6$. Based on Theorem 2, system (8) is quasi-synchronized.

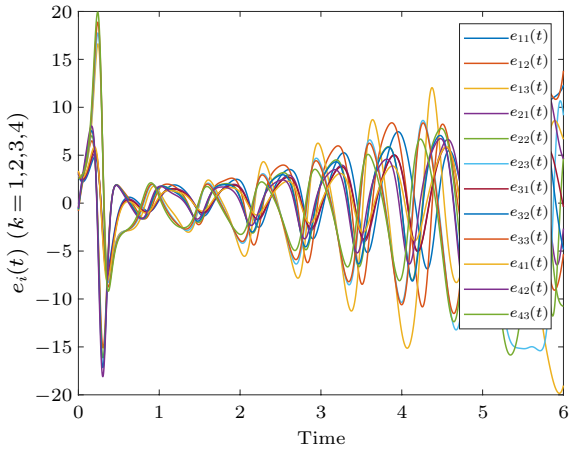


Fig. 8. Profiles of $e_i(t)$ of the time-varying delay error system without control.

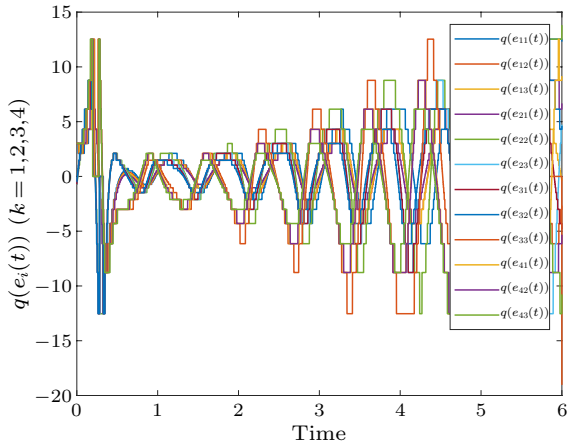


Fig. 9. Profiles of $q(e_i(t))$ of the time-varying delay error system without control.

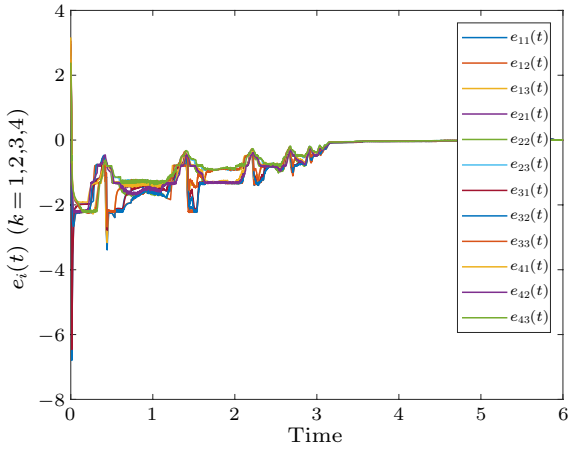


Fig. 10. Profiles of $e_i(t)$ of the error system with time-varying delay.

With a time-varying delay, Fig. 8 shows the synchronization error $e_i(t)$ fluctuations under uncontrolled conditions. Figure 9 shows the synchronization quantized error $q(e_i(t))$ fluctuations under uncontrolled conditions. Figure 10 shows that the synchronization errors $e_i(t)$ fluctuate within an interval. Figure 11 shows that the synchronization quantized errors $q(e_i(t))$ fluctuate within an interval. Figure 12 shows the quasi-synchronization error $||e||$ under uncontrolled conditions. Figure 13 shows the quasi-synchronization error$||e||$, which indicates that quasi-synchronization is realized.

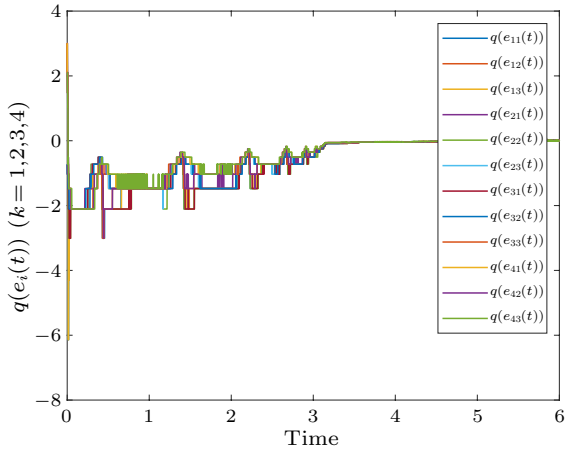


Fig. 11. Profiles of $q(e_i(t))$ of the error system with time-varying delay.

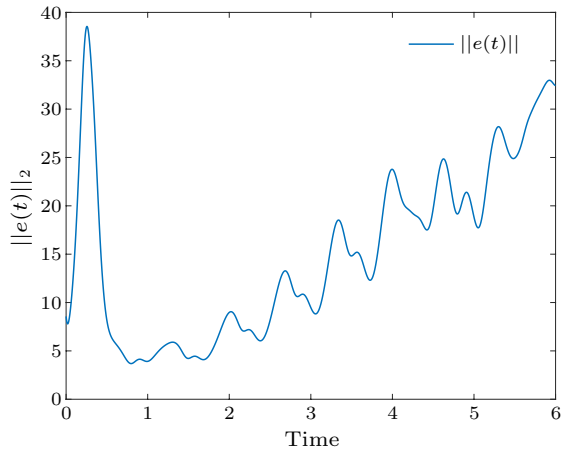


Fig. 12. Profiles of $||e(t)||$ of the error system without control.

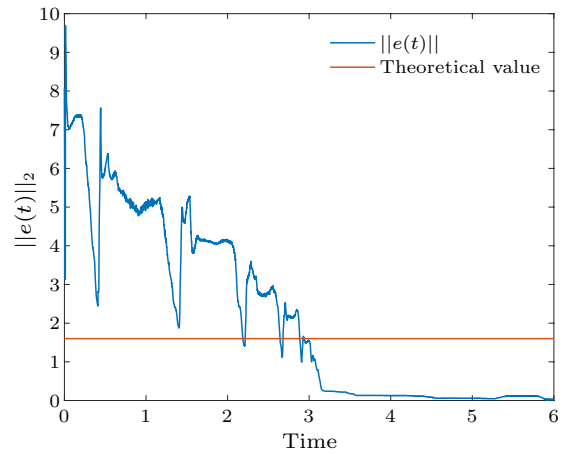


Fig. 13. Profiles of $||e(t)||$ of the error system with time-varying delay.

5. Conclusion

In summary, we have investigated quasi-synchronization of FCNs. Firstly, using the given logarithmic quantizer theory and the Lyapunov stability theory, we design a new quantitative feedback controller and obtain sufficient criteria such that FCNs with random coupling can achieve quasi-synchronization. Secondly, we study an FCN model with random coupling and time-varying delays. Another quantized controller is designed to obtain the quasi-synchronization criterion in this model. Notably, the upper bound of quasi-synchronization in both the cases can be adjusted by setting the relevant controller parameters. Finally, we prove the validity of our theory with two numerical examples. In the future, we will focus on the use of new control methods such that the time upper bound to achieve quasi-synchronization of FCNs can be controlled to achieve finite-time quasi-synchronization of FCNs.

Acknowledgments

The work was supported by the Anhui Provincial Development and Reform Commission New Energy Vehicles and Intelligent Connected Automobile Industry Technology Innovation Project.

References

[1] Albert R and Barabasi A 2002 *Rev. Mod. Phys.* **74** 47
[2] Philipp H, Aline V and Philipp L 2018 *J. Nonlinear Sci.* **30** 2259
[3] Li X R, Liu J Q and Dong J C2021 *IEEE Trans. Circuits Syst. I: Regular Papers* **68** 4268
[4] Huang L L, Zhang Y, Xiang J and Liu J 2022*IEEE Trans. Circuits Syst. II: Express Briefs* **69** 4568
[5] Liu Y Y and Slotine J J 2011 *Nature* **473** 167
[6] Rosvall M and Bergstrom C T 2008 *Proc. Natl. Acad. Sci. USA* **105** 1118
[7] Wang L and Wang P 2015 *Int. J. Mod. Phys. C* **26** 1550052
[8] Russo G 2018 *Int. J. Robust Nonlinear Control* **28** 120
[9] Shakibian H and Charkari N M 2016 *2016 24th Iranian Conference on Electrical Engineering (ICEE)*, May 10–12, 2016, Shiraz, Iran, pp. 682–686
[10] Huang L L, Li W Y, Xiang J H and Zhu G L 2022 *Eur. Phys. J. Spec. Top.* **231** 3109
[11] Yang F, Li H, Guo C, Xia D and Qi H 2018 *Neural Comput. Appl.* **31** 7945
[12] Ding X S, Cao J D and Alsaadi F E 2019 *Int. J. Adaptive Control Signal Process.* **33** 1478
[13] Mwanandiye E S, Wu B and Jia Q 2020 *Neural Comput. Appl.* **32** 11277
[14] Ding D W, Yan J, Wang N and Liang D 2017 *Chaos, Solitons Fractals* **104** 41
[15] He W L, Qian F, Lam J, Chen G and Han Q L 2015 *Automatica* **62** 249
[16] Xu L G, Chu X Y and Hu H X 2021 *Math. Comput. Simul.* **185** 594
[17] Yang S A, Yu J, Cheng H and Jiang H J 2018 *Neural Networks* **104** 104
[18] Machado J T, Kiryakova V and Mainardi F 2011*Commun. Nonlinear Sci. Numerical Simul.* **16** 1140
[19] Aguila C and Duarte M A 2014 *Commun. Nonlinear Sci. Numerical Simul.* **19** 2951
[20] Hartley T T, Lorenzo C F and Qammer H K 1995 *IEEE Trans. Circuits Syst. I: Fundam. Theory Appl.* **42** 485
[21] Saadatmandi A and Dehghan M 2010 *Comput. Math. Appl.* **59** 1326
[22] Sun H G, Zhang Y, Baleanu D, Chen W and Chen Y G 2018 *Commun. Nonlinear Sci. Numerical Simul.* **64** 213
[23] Magin R L 2010 *Comput. Math. Appl.* **59** 1586
[24] Ding D W, Yao X L and Zhang H W 2019 *Mod. Phys. Lett. B* **33** 1950351
[25] Bao H B, Park J H and Cao J D 2021 *IEEE Trans. Neural Networks Learning Syst.* **32** 3230
[26] Xu Y, Wang Q, Li W X and Feng J Q 2021 *Math. Methods Appl. Sci.* **69** 1539
[27] He J J, Chen H, Ge M F and Ding T F 2021 *Neurocomputing* **431** 90
[28] Hu J T, Sui G X and Li X D 2020 *Chaos, Solitons Fractals* **140** 110216
[29] Li C G and Chen G R 2004 *Physica A* **343** 263
[30] Wu Y Q and Liu L 2015 *Entropy* **17** 3097
[31] Hang L H and Wu X R 2019 *Neural Process. Lett.* **50** 2373
[32] Li W Y, Zhou J, Li J Q and Xie T 2021 *IEEE Trans. Circuits Syst. II: Express Briefs* **68** 1338
[33] Zhang L L, Zhong J and Lu J 2021 *Neural Networks* **144** 11
[34] Zhang Q J, Luo J and Li W 2013 *Nonlinear Dyn.* **71** 353
[35] Brockett R and Liberzon D 2000 *IEEE Trans. Automat. Control* **45** 1279
[36] Niu Y G and Daniel W H 2014 *Automatica* **50** 2665
[37] Shi Y J and Ma Y 2021 *Electron. Res. Arch.* **29** 2047
[38] Xu C, Yang X S, Lu J Q and Feng J W 2018 *IEEE Trans. Cybernet.* **48** 3021