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Analytical wave solutions of an electronically and biologically important model via two efficient schemes

Qingbo Huang[1,†], Asim Zafar[2], M. Raheel[2], and Ahmet Bekir[3,‡]

[1] *Chongqing College of Architecture and Technology, Chongqing* 401331, *China*
[2] *Department of Mathematics, COMSATS University Islamabad, Vehari Campus, Pakistan*
[3] *Neighbourhood of Akcaglan, Imarli Street, No.* 28/4, *Eskisehir* 26030, *Turkey*

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We search for analytical wave solutions of an electronically and biologically important model named as the Fitzhugh–Nagumo model with truncated M-fractional derivative, in which the $\exp_a$ function and extended sinh-Gordon equation expansion (EShGEE) schemes are utilized. The solutions obtained include dark, bright, dark-bright, periodic and other kinds of solitons. These analytical wave solutions are gained and verified with the use of Mathematica software. These solutions do not exist in literature. Some of the solutions are demonstrated by 2D, 3D and contour graphs. This model is mostly used in circuit theory, transmission of nerve impulses, and population genetics. Finally, both the schemes are more applicable, reliable and significant to deal with the fractional nonlinear partial differential equations.

Keywords: spacetime fractional Fitzhugh–Nagumo model, truncated M-fractional derivative, $\exp_a$ function scheme, EShGEE scheme, analytical wave solutions

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1. Introduction

Different types of phenomena occurring chemically, biologically, economically and others come from nonlinear partial differential equations (NLPDEs). Many different methods have been developed to gain analytical wave solutions of these NLPDEs, i.e., optical soliton solutions of coupled nonlinear Schrödinger equations have been gained with use of Kudryashov R-function technique,[1] some new kinds of optical soliton solutions of time-fractional perturbed nonlinear Schrödinger equations have been achieved by using the generalized Kudryashov scheme,[2] by applying the modified auxiliary equation technique, optical wave solutions of time-fractional resonant nonlinear Schrödinger equations have been obtained,[3] new optical wave solutions for the time-fractional perturbed nonlinear Schrödinger equations have been achieved by utilizing the improved $\tan[\phi(\zeta/2)]$-expansion scheme,[4] different kinds of optical wave solitons of time-fractional paraxial wave equations have been gained by using the Sardar sub-equation method,[5] various optical wave solutions of three-component coupled nonlinear Schrödinger equations have been attained with the help of generalized exponential rational function scheme,[6] dark, bright, singular and periodic solitary wave solutions of generalized fractional Davey–Stewartson equations have been obtained by applying the generalized projective Riccati equation technique,[7] some exact wave solutions of the Lax equation have been achieved by applying the extended sinh-Gordon expansion technique,[8] kink solitons of the Sharma–Tasso–Olver–Burgers equation have been attained by using Kudryashov and exponential techniques,[9] traveling wave solutions of perturbed Biswas–Milovic equations have been gained with the use of improved F-expansion technique,[10] some new optical wave solutions of complex Korteweg–de Vries equations have been obtained by applying the unified scheme.[11] Similarly, Hirota bilinear method,[12] modified extended tanh expansion method,[13] modified simplest equation technique,[14] extended Jacobi elliptic function scheme,[15] sech and tanh function solutions are obtained by using the sine-Gordon expansion scheme,[16] sinh, cosh, sin and cos involving solutions are gained by utilizing the rational sine-Gordon expansion technique[17] and many other techniques.[18–24]

In this study, we use two schemes, i.e., the $\exp_a$ function and extended sinh-Gordon equation expansion (EShGEE) schemes. The general $\exp_a$ function scheme was first proposed by Ali and Hassan in Ref. [25]. Yan firstly proposed the extended sinh-Gordon equation expansion method in Ref. [26]. They are two different techniques used for solving nonlinear partial differential equations. Both the methods are proposed to find the vast categories of exact wave solutions. Many applications are closely related to these two schemes. For example, different types of solitons of two nonlinear Schrödinger equations have been attained with the application of $\exp_a$ function scheme,[27] analytical wave solitons of the modified Camassa–Holm model are gained with the use of this scheme.[28] Some optical solitons of a new Hamiltonian amplitude model have been achieved by applying the extended sinh-Gordon equa-

†Corresponding author. E-mail: hw913987@126.com
‡Corresponding author. E-mail: bekirahmet@gmail.com

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tion expansion scheme in Ref. [29]. A new type of optical wave solutions of the Biswas–Arshed equation have been gained by using the EShGEE scheme.[30] Different kinds of soliton solutions including dark, bright, dark-bright, singular and other wave solutions of the Kundu–Eckhaus equation have been gained.[31]

We employ the Fitzhugh–Nagumo model that is much more important in biology and electronics. Different techniques have been applied to search the exact wave solutions of this model, e.g., some different types of exact wave solutions of this model have been gained by extended tanh-coth technique in Ref. [32], solitary and periodic wave solutions have been obtained by the exp-function scheme in Ref. [33], a new kind of analytical wave solitons has been achieved by this model with the help of sine-Gordon expansion technique in Ref. [34].

The fundamental purpose of the work is to search the new analytical wave solutions of the spacetime fractional Fitzhugh–Nagumo model via the $\exp_a$ function and extended sinh-Gordon equation expansion schemes.

The motivation of this paper is to explain the effect of M-fractional derivative on the solutions of the spacetime fractional Fitzhugh–Nagumo model, which are gained with the use of the $\exp_a$ function and extended sinh-Gordon equation expansion schemes. Significance of M-fractional derivative is that it fulfills both the properties of integer and fractional order derivatives. By using the extended sinh-Gordon equation expansion scheme, we can obtain the sech, csch, tanh and coth functions involving solutions. Using our schemes, we can observe some elementary relationship between fractional NLPDEs and other simple nonlinear ordinary differential equations. It has been found that with the use of simple schemes and solvable ordinary differential equations, different types of exact wave solutions of some complicated fractional NLPDEs can be easily obtained. The solutions attained are newer than the existing solutions in the literature. The achieved results may behave similarly to biological neurons which can be implementable on digital platforms.

The rest of this paper is organized as follows: In Section 2, we describe the model and present its mathematical analysis. In Section 3, we present the methods. In Section 4, we provide exact wave solitons of the model. In Section 5, we illustrate the model with graphics. Section 6, we provide conclusions.

2. Governing model and its mathematical treatment

Assume the (1+1)-dimensional nonlinear truncated M-fractional Fitzhugh–Nagumo model given as[32]

$$D_{M,t}^{\alpha,\Upsilon} v = D_{M,2x}^{2\alpha,\Upsilon} v + v(1-v)(v-\delta), \quad t>0,\ 0<\alpha\le 1,\ x\in\Re, \tag{1}$$

where

$$D_{M,t}^{\alpha,\Upsilon} v(t) = \lim_{\tau\to 0} \frac{v(t,\, E_\Upsilon(\tau t^{1-\alpha})) - v(t)}{\tau},\ 0<\alpha\le 1,\ \Upsilon>0, \tag{2}$$

with $E_\Upsilon(\cdot)$ representing the truncated Mittag–Leffler function of one parameter given in Refs. [35,36]; $v = v(x,t)$ shows the wave profile, x and t represent the space and time variables, respectively; δ denotes the parameter. Equation (1) expresses a nonlinear reaction-diffusion model used in the transmission of nerve impulses as well as in biology as population genetics. It is also used in circuit theory.

Let us assume the following wave transformation:

$$v(x,t) = V(\zeta),\ \ \zeta = \frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha - \lambda t^\alpha), \tag{3}$$

where τ and λ are the parameters. Substituting Eq. (3) into Eq. (1), we can gain the following nonlinear ordinary differential equation:

$$\lambda V' + \tau^2 V'' + V(1-V)(V-\delta) = 0. \tag{4}$$

By applying the homogenous balance technique and balancing V'' and V^3, we gain

$$3m = m+2, \tag{5}$$

$$m = 1. \tag{6}$$

We will find the new exact wave solitons using the following schemes.

3. Methods

3.1. The $\exp_a$ function scheme

This scheme is briefly explained in the following. Consider the nonlinear partial differential equation

$$Y(v, v^2 v_t, v_x, v_{tt}, v_{xx}, v_{xt}, \ldots) = 0. \tag{7}$$

It can be transformed to the nonlinear partial differential equation

$$Z(V, V', V'', \ldots,) = 0. \tag{8}$$

Inserting in the wave transformation yields

$$v(x,\, y,\, t) = V(\eta),\quad \eta = ax + \omega y + \delta t. \tag{9}$$

Utilizing the solution of Eq. (8) given in Refs. [37–39], we have

$$V(\eta) = \frac{\alpha_0 + \alpha_1 d^\eta + \cdots + \alpha_m d^{m\eta}}{\beta_0 + \beta_1 d^\eta + \cdots + \beta_m d^{m\eta}},\ d\neq 0,1, \tag{10}$$

where α_i and $\beta_i (0\le i\le m)$ are unknown. Natural number m is obtain by applying the homogeneous balance scheme to Eq. (8). Substituting Eq. (10) into Eq. (8) yields

$$\wp(d^\eta) = \ell_0 + \ell_1 d^\eta + \cdots + \ell_n d^{t\eta} = 0. \tag{11}$$

Substituting ℓ_i $(0 \le i \le n)$ into Eq. (11) and making it equal to zero, a system of equations is gained as follows:

$$\ell_n = 0, \qquad i = 0, \ldots, n, \tag{12}$$

with assistance of the achieved equations, we can gain analytical wave solutions of Eq. (7).

3.2. The EShGEE scheme

There are following main steps in this scheme.

Step 1 Assume a nonlinear fractional partial differential equation

$$Y(v, D_{M,t}^{\alpha,\Upsilon} v^2, v^2 v_x, v_y, v_{yy}, v_{xx}, v_{xy}, v_{xt}, \ldots) = 0, \tag{13}$$

where $v = v(x,y,t)$ represents the wave profile. Consider the following wave transformations

$$v(x,y,t) = V(\eta), \quad \eta = x - \nu y + \frac{\Gamma(1+\Upsilon)}{\alpha}(\kappa t^{\alpha}). \tag{14}$$

Substituting Eq. (14) into Eq. (13) yields

$$T(V(\eta), V^2(\eta)V'(\eta), V''(\eta), \ldots) = 0. \tag{15}$$

Step 2 Consider the solution of Eq. (15) as

$$V(w) = \alpha_0 + \sum_{j=1}^{m} [\beta_j \sinh(w) + \alpha_j \cosh(w)]^j, \tag{16}$$

where α_0, α_j, β_j $(j = 1,2,3,\ldots,m)$ are undetermined. Suppose a function w of η that fulfills the given equation

$$\frac{\mathrm{d}w}{\mathrm{d}\eta} = \sinh(w). \tag{17}$$

The positive number m is calculated by utilizing the homogenously balance method. Equation (17) is obtained from the sinh-Gordon equation as

$$q_{xt} = \kappa \sinh(v). \tag{18}$$

From Ref. [40], we obtain the solutions to Eq. (18) as

$$\sinh w(\eta) = \pm \mathrm{csch}(\eta) \quad \text{or} \quad \cosh w(\eta) = \pm \coth(\eta), \tag{19}$$

$$\sinh w(\eta) = \pm \mathrm{i}\,\mathrm{sech}(\eta) \quad \text{or} \quad \cosh w(\eta) = \pm \tanh(\eta), \tag{20}$$

here $\mathrm{i}^2 = -1$.

Step 3 Substituting Eqs. (16) and (20) into Eq. (15), we achieve the equations consisting of $w'^k(\eta)\sinh^l w(\eta)\cosh^m w(\eta)$ $(k = 0,1;\ l = 0,1;\ m = 0,1,2,\ldots)$. Substituting each coefficient of $w'^k(\eta)\sinh^l w(\eta)\cosh^m w(\eta)$ and make it equal to zero, we gain a set of ν, κ, α_0, α_j, and β_j $(j = 1,2,3,\ldots,m)$.

Step 4 After solving the achieved set of equations, one can gain the results for ν, κ, α_0, α_j, and β_j.

Step 5 From the gained results, i.e., Eqs. (19) and (20), we obtain the analytical wave solutions to Eq. (15) as follows:

$$V(\eta) = \alpha_0 + \sum_{j=1}^{m} [\pm\beta_j \,\mathrm{csch}(\eta) \pm \alpha_j \coth(\eta)]^j, \tag{21}$$

$$V(\eta) = \alpha_0 + \sum_{j=1}^{m} [\pm\mathrm{i}\beta_j \,\mathrm{sech}(\eta) \pm \alpha_j \tanh(\eta)]^j. \tag{22}$$

Using this technique, we can obtain the sech, csch, tanh and coth functions involving solutions.

4. Analytical wave solutions

4.1. Using the $\exp_a$ function scheme

For $m = 1$, Eq. (10) reduces to

$$V(\zeta) = \frac{\alpha_0 + \alpha_1 d^{\zeta}}{\beta_0 + \beta_1 d^{\zeta}}. \tag{23}$$

Substituting Eq. (23) into Eq. (4), we can obtain a set of equations, then we gain some sets.

Set 1

$$\Big\{\alpha_0 = \beta_0,\ \alpha_1 = 0,\ \beta_0 = \beta_0,\ \beta_1 = \beta_1,$$
$$\lambda = \frac{1-2\delta}{2\log(d)},\ \tau = \mp\frac{1}{\sqrt{2}\log(d)}\Big\}, \tag{24}$$

$$v_1(x,t) = \frac{\beta_0}{\beta_0 + \beta_1 d^{[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha} - \lambda t^{\alpha})]}}. \tag{25}$$

Note that here λ and τ are given in Eq. (24).

Set 2

$$\Big\{\alpha_0 = \delta\beta_0,\ \alpha_1 = 0,\ \beta_0 = \beta_0,\ \beta_1 = \beta_1,$$
$$\lambda = \frac{(\delta-2)\delta}{2\log(d)},\ \tau = \mp\frac{1}{\sqrt{2}\log(d)}\Big\}, \tag{26}$$

$$v_2(x,t) = \frac{\delta\beta_0}{\beta_0 + \beta_1 d^{[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha} - \lambda t^{\alpha})]}}, \tag{27}$$

where λ and τ are given in Eq. (26).

Set 3

$$\Big\{\alpha_0 = 0,\ \alpha_1 = \beta_1,\ \beta_0 = \beta_0,\ \beta_1 = \beta_1,$$
$$\lambda = \frac{2\delta-1}{2\log(d)},\ \tau = \mp\frac{1}{\sqrt{2}\log(d)}\Big\}, \tag{28}$$

$$v_3(x,t) = \frac{\beta_1 d^{[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha} - \lambda t^{\alpha})]}}{\beta_0 + \beta_1 d^{[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha} - \lambda t^{\alpha})]}}, \tag{29}$$

where λ and τ are given in Eq. (28).

Set 4

$$\Big\{\alpha_0 = \beta_0\delta,\ \alpha_1 = \beta_1,\ \lambda = \frac{\delta^2-1}{2\log(d)},\ \tau = \mp\frac{\delta-1}{\sqrt{2}\log(d)}\Big\}, \tag{30}$$

$$v_4(x,t)=\frac{\beta_0\delta+\beta_1 d^{[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)]}}{\beta_0+\beta_1 d^{[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)]}}, \tag{31}$$

where λ and τ are given in Eq. (30).

Set 5

$$\left\{\alpha_0=0,\ \alpha_1=\beta_1\delta,\ \lambda=\frac{2\delta-\delta^2}{2\log(d)},\ \tau=\mp\frac{\delta}{\sqrt{2}\log(d)}\right\}, \tag{32}$$

$$v_5(x,t)=\frac{\beta_1\delta d^{[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)]}}{\beta_0+\beta_1 d^{[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)]}}, \tag{33}$$

where λ and τ are given in Eq. (32).

Set 6

$$\left\{\alpha_0=\beta_0,\ \alpha_1=\beta_1\delta,\ \lambda=\frac{1-\delta^2}{2\log(d)},\ \tau=\mp\frac{\delta-1}{\sqrt{2}\log(d)}\right\}, \tag{34}$$

$$v_6(x,t)=\frac{\beta_0+\beta_1\delta d^{[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)]}}{\beta_0+\beta_1 d^{[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)]}}, \tag{35}$$

where λ and τ are given in Eq. (34).

4.2. Using the EShGEE scheme

For $m=1$, Eqs. (21), (22), and (16) reduce to

$$V(\zeta)=\alpha_0\pm\beta_1\,\mathrm{csch}(\zeta)\pm\alpha_1\coth(\zeta), \tag{36}$$

$$V(\zeta)=\alpha_0\pm\mathrm{i}\beta_1\,\mathrm{sech}(\zeta)\pm\alpha_1\tanh(\zeta), \tag{37}$$

$$V(\zeta)=\alpha_0+\beta_1\sinh(w)+\alpha_1\cosh(w), \tag{38}$$

where α_0, α_1, and β_1 are undetermined. Substituting Eq. (38) into Eq. (4) we gain algebraic equations containing α_0, α_1, β_1, and other parameters. By using the Mathematica tool, we obtain the following sets.

Set 1

$$\left\{\alpha_0=\frac{1}{2},\ \alpha_1=-\frac{1}{2},\ \beta_1=-\frac{1}{2},\ \lambda=\delta-\frac{1}{2},\ \tau=-\frac{1}{\sqrt{2}}\right\}, \tag{39}$$

$$v_1(x,t)=\frac{1}{2}\left\{1\mp\left[\coth\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)+\mathrm{csch}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\right]\right\}, \tag{40}$$

$$v_2(x,t)=\frac{1}{2}\left\{1\mp\left[\mathrm{i}\,\mathrm{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)+\tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\right]\right\}, \tag{41}$$

where λ and τ are given in Eq. (39).

Set 2

$$\left\{\alpha_0=\frac{1}{2},\ \alpha_1=-\frac{1}{2},\ \beta_1=-\frac{1}{2},\ \lambda=\delta-\frac{1}{2},\ \tau=\frac{1}{\sqrt{2}}\right\}, \tag{42}$$

$$v_1(x,t)=\frac{1}{2}\left\{1\mp\left[\coth\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)+\mathrm{csch}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\right]\right\}, \tag{43}$$

$$v_2(x,t)=\frac{1}{2}\left\{1\mp\left[\mathrm{i}\,\mathrm{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)+\tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\right]\right\}, \tag{44}$$

where λ and τ are given in Eq. (42).

Set 3

$$\left\{\alpha_0=\frac{1}{2},\ \alpha_1=\frac{1}{2},\ \beta_1=-\frac{1}{2},\ \lambda=\frac{1}{2}-\delta,\ \tau=-\frac{1}{\sqrt{2}}\right\}, \tag{45}$$

$$v_1(x,t)=\frac{1}{2}\left\{1\mp\mathrm{csch}\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\pm\coth\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\right\}, \tag{46}$$

$$v_2(x,t)=\frac{1}{2}\left\{1\mp\mathrm{i}\,\mathrm{sech}\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\pm\tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{47}$$

where λ and τ are given in Eq. (45).

Set 4

$$\left\{\alpha_0=\frac{1}{2},\ \alpha_1=\frac{1}{2},\ \beta_1=-\frac{1}{2},\ \lambda=\frac{1}{2}-\delta,\ \tau=\frac{1}{\sqrt{2}}\right\}, \tag{48}$$

$$v_1(x,t)=\frac{1}{2}\left\{1\mp\mathrm{csch}\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\pm\coth\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\right\}, \tag{49}$$

$$v_2(x,t)=\frac{1}{2}\left\{1\mp\mathrm{i}\,\mathrm{sech}\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\pm\tanh\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{50}$$

where λ and τ are given in Eq. (48).

Set 5

$$\left\{\alpha_0=\frac{1}{2},\ \alpha_1=-\frac{1}{2},\ \beta_1=0,\ \lambda=\frac{1}{4}(2\delta-1),\ \tau=-\frac{1}{2\sqrt{2}}\right\}, \tag{51}$$

$$v_1(x,t)=\frac{1}{2}\left\{1\mp\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{52}$$

$$v_2(x,t)=\frac{1}{2}\left\{1\mp\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{53}$$

where λ and τ are given in Eq. (51).

Set 6

$$\left\{\alpha_0=\frac{1}{2},\ \alpha_1=-\frac{1}{2},\ \beta_1=0,\ \lambda=\frac{1}{4}(2\delta-1),\ \tau=\frac{1}{2\sqrt{2}}\right\}, \tag{54}$$

$$v_1(x,t)=\frac{1}{2}\left\{1\mp\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{55}$$

$$v_2(x,t)=\frac{1}{2}\left\{1\mp\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{56}$$

where λ and τ are given in Eq. (54).

Set 7

$$\left\{\alpha_0=\frac{1}{2},\ \alpha_1=\frac{1}{2},\ \beta_1=0,\ \lambda=\frac{1}{4}(1-2\delta),\ \tau=-\frac{1}{2\sqrt{2}}\right\}, \tag{57}$$

$$v_1(x,t)=\frac{1}{2}\left\{1\pm\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{58}$$

$$v_2(x,t)=\frac{1}{2}\left\{1\pm\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{59}$$

where λ and τ are given in Eq. (57).

Set 8

$$\left\{\alpha_0=\frac{1}{2},\ \alpha_1=\frac{1}{2},\ \beta_1=0,\ \lambda=\frac{1}{4}(1-2\delta),\ \tau=\frac{1}{2\sqrt{2}}\right\}, \tag{60}$$

$$v_1(\zeta)=\frac{1}{2}\left\{1\pm\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{61}$$

$$v_2(x,t)=\frac{1}{2}\left\{1\pm\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{62}$$

where λ and τ are given in Eq. (60).

Set 9

$$\left\{\alpha_0=\frac{\delta}{2},\ \alpha_1=-\frac{\delta}{2},\ \beta_1=0,\ \lambda=-\frac{1}{4}(\delta-2)\delta,\ \tau=-\frac{\delta}{2\sqrt{2}}\right\}, \tag{63}$$

$$v_1(x,t)=\frac{1}{2}\delta\left\{1\mp\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{64}$$

$$v_2(x,t)=\frac{1}{2}\delta\left\{1\mp\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{65}$$

where λ and τ are given in Eq. (63).

Set 10

$$\left\{\alpha_0=\frac{\delta}{2},\ \alpha_1=-\frac{\delta}{2},\ \beta_1=0,\ \lambda=-\frac{1}{4}(\delta-2)\delta,\ \tau=\frac{\delta}{2\sqrt{2}}\right\}, \tag{66}$$

$$v_1(x,t)=\frac{1}{2}\delta\left\{1\mp\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{67}$$

$$v_2(x,t)=\frac{1}{2}\delta\left\{1\mp\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{68}$$

where λ and τ are given in Eq. (66).

Set 11

$$\left\{\alpha_0=\frac{\delta}{2},\ \alpha_1=\frac{\delta}{2},\ \beta_1=0,\ \lambda=\frac{1}{4}(\delta-2)\delta,\ \tau=-\frac{\delta}{2\sqrt{2}}\right\}, \tag{69}$$

$$v_1(x,t)=\frac{1}{2}\delta\left\{1\pm\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{70}$$

$$v_2(x,t)=\frac{1}{2}\delta\left\{1\pm\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{71}$$

where λ and τ are given in Eq. (69).

Set 12

$$\left\{\alpha_0=\frac{\delta}{2},\ \alpha_1=\frac{\delta}{2},\ \beta_1=0,\ \lambda=\frac{1}{4}(\delta-2)\delta,\ \tau=\frac{\delta}{2\sqrt{2}}\right\}, \tag{72}$$

$$v_1(x,t)=\frac{1}{2}\delta\left\{1\pm\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{73}$$

$$v_2(x,t)=\frac{1}{2}\delta\left\{1\pm\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right]\right\}, \tag{74}$$

where λ and τ are given in Eq. (72).

Set 13

$$\left\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{1-\delta}{2},\ \beta_1=0,\ \lambda=\frac{1}{4}(1-\delta^2),\ \tau=-\frac{\delta-1}{2\sqrt{2}}\right\}, \tag{75}$$

$$v_1(x,t)=\frac{\delta+1}{2}\mp\frac{1}{2}(\delta-1)\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right], \tag{76}$$

$$v_2(x,t)=\frac{\delta+1}{2}\mp\frac{1}{2}(\delta-1)\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right], \tag{77}$$

where λ and τ are given in Eq. (75).

Set 14

$$\left\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{1-\delta}{2},\ \beta_1=0,\ \lambda=\frac{1}{4}(1-\delta^2),\ \tau=\frac{\delta-1}{2\sqrt{2}}\right\}, \tag{78}$$

$$v_1(x,t)=\frac{\delta+1}{2}\mp\frac{1}{2}(\delta-1)\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right], \tag{79}$$

$$v_2(x,t)=\frac{\delta+1}{2}\mp\frac{1}{2}(\delta-1)\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right], \tag{80}$$

where λ and τ are given in Eq. (78).

Set 15

$$\left\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{\delta-1}{2},\ \beta_1=0,\ \lambda\to\frac{1}{4}(\delta^2-1),\ \tau=-\frac{\delta-1}{2\sqrt{2}}\right\}, \tag{81}$$

$$v_1(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(\delta-1)\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right], \tag{82}$$

$$v_2(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(\delta-1)\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\right], \tag{83}$$

where λ and τ are given in Eq. (81).

Set 16

$$\left\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{\delta-1}{2},\ \beta_1=0,\right.$$

$$\lambda \to \frac{1}{4}(\delta^2-1),\ \tau=\frac{\delta-1}{2\sqrt{2}}\Big\}, \tag{84}$$

$$v_1(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(\delta-1)\coth\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big], \tag{85}$$

$$v_2(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(\delta-1)\tanh\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big], \tag{86}$$

where λ and τ are given in Eq. (84).

Set 17

$$\Big\{\alpha_0=\frac{1}{2},\ \alpha_1=-\frac{1}{2},\ \beta_1=\frac{1}{2},\ \lambda=\delta-\frac{1}{2},\ \tau=-\frac{1}{\sqrt{2}}\Big\} \tag{87}$$

$$v_1(x,t)=\frac{1}{2}\Big\{1\pm\Big[\operatorname{csch}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)-\coth\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)\Big]\Big\}, \tag{88}$$

$$v_2(x,t)=\frac{1}{2}\Big\{1\pm\Big[\operatorname{i}\operatorname{sech}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)-\tanh\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)\Big]\Big\}, \tag{89}$$

where λ and τ are given in Eq. (87).

Set 18

$$\Big\{\alpha_0=\frac{1}{2},\ \alpha_1=-\frac{1}{2},\ \beta_1=\frac{1}{2},\ \lambda=\delta-\frac{1}{2},\ \tau=\frac{1}{\sqrt{2}}\Big\}, \tag{90}$$

$$v_1(x,t)=\frac{1}{2}\Big\{1\pm\Big[\operatorname{csch}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)-\coth\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)\Big]\Big\}, \tag{91}$$

$$v_2(x,t)=\frac{1}{2}\Big\{1\pm\Big[\operatorname{i}\operatorname{sech}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)-\tanh\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)\Big]\Big\}, \tag{92}$$

where λ and τ are given in Eq. (90).

Set 19

$$\Big\{\alpha_0=\frac{1}{2},\ \alpha_1=\frac{1}{2},\ \beta_1=\frac{1}{2},\ \lambda=-\delta+\frac{1}{2},\ \tau=-\frac{1}{\sqrt{2}}\Big\}, \tag{93}$$

$$v_1(x,t)=\frac{1}{2}\Big\{1\pm\Big[\operatorname{csch}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)+\coth\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)\Big]\Big\}, \tag{94}$$

$$v_2(x,t)=\frac{1}{2}\Big\{1\pm\Big[\operatorname{i}\operatorname{sech}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)+\tanh\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)\Big]\Big\}, \tag{95}$$

where λ and τ are given in Eq. (93).

Set 20

$$\Big\{\alpha_0=\frac{1}{2},\ \alpha_1=\frac{1}{2},\ \beta_1=\frac{1}{2},\ \lambda=-\delta+\frac{1}{2},\ \tau=\frac{1}{\sqrt{2}}\Big\}, \tag{96}$$

$$v_1(x,t)=\frac{1}{2}\Big\{1\pm\Big[\operatorname{csch}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)+\coth\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)\Big]\Big\}, \tag{97}$$

$$v_2(x,t)=\frac{1}{2}\Big\{1\pm\Big[\operatorname{i}\operatorname{sech}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)+\tanh\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big)\Big]\Big\}, \tag{98}$$

where λ and τ are given in Eq. (96).

Set 21

$$\Big\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{1-\delta}{2},\ \beta_1=\frac{1-\delta}{2},\ \lambda=-\frac{1}{2}(\delta-1)(\delta+1),\ \tau=-\frac{\delta-1}{\sqrt{2}}\Big\}, \tag{99}$$

$$v_1(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(1-\delta)\Big\{\coth\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big]+\operatorname{csch}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big]\Big\}, \tag{100}$$

$$v_2(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(1-\delta)\Big\{\operatorname{i}\operatorname{sech}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big]+\tanh\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big]\Big\}, \tag{101}$$

where λ and τ are given in Eq. (99).

Set 22

$$\Big\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{1-\delta}{2},\ \beta_1=\frac{1-\delta}{2},\ \lambda=-\frac{1}{2}(\delta-1)(\delta+1),\ \tau=\frac{\delta-1}{\sqrt{2}}\Big\}, \tag{102}$$

$$v_1(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(1-\delta)\Big\{\coth\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big]+\operatorname{csch}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big]\Big\}, \tag{103}$$

$$v_2(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(1-\delta)\Big\{\operatorname{i}\operatorname{sech}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big]+\tanh\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big]\Big\}, \tag{104}$$

where λ and τ are given in Eq. (102).

Set 23

$$\Big\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{\delta-1}{2},\ \beta_1=\frac{1-\delta}{2},\ \lambda=\frac{1}{2}(\delta-1)(\delta+1),\ \tau=-\frac{\delta-1}{\sqrt{2}}\Big\}, \tag{105}$$

$$v_1(x,t)=\frac{\delta+1}{2}+\frac{1}{2}(\delta-1)\Big\{\mp\operatorname{csch}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big]\pm\coth\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\Big]\Big\}, \tag{106}$$

$$v_2(x,t)=\frac{\delta+1}{2}+\frac{1}{2}(\delta-1)\Big\{\mp \mathrm{i}\,\mathrm{sech}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big] \pm\tanh\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big]\Big\}, \tag{107}$$

where λ and τ are given in Eq. (105).

Set 24

$$\Big\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{\delta-1}{2},\ \beta_1=\frac{1-\delta}{2}, \lambda=\frac{1}{2}(\delta-1)(\delta+1),\ \tau=\frac{\delta-1}{\sqrt{2}}\Big\}, \tag{108}$$

$$v_1(x,t)=\frac{\delta+1}{2}+\frac{1}{2}(\delta-1)\Big\{\mp\mathrm{csch}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big] \pm\coth\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big]\Big\}, \tag{109}$$

$$v_2(x,t)=\frac{\delta+1}{2}+\frac{1}{2}(\delta-1)\Big\{\mp\mathrm{i}\,\mathrm{sech}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big] \pm\tanh\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big]\Big\}, \tag{110}$$

where λ and τ are given in Eq. (98).

Set 25

$$\Big\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{1-\delta}{2},\ \beta_1=\frac{\delta-1}{2}, \lambda=-\frac{1}{2}(\delta-1)(\delta+1),\ \tau=-\frac{\delta-1}{\sqrt{2}}\Big\}, \tag{111}$$

$$v_1(x,t)=\frac{\delta+1}{2}+\frac{1}{2}(\delta-1)\Big\{\pm\mathrm{csch}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big] \mp\coth\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big]\Big\}, \tag{112}$$

$$v_2(x,t)=\frac{\delta+1}{2}+\frac{1}{2}(\delta-1)\Big\{\pm\mathrm{i}\,\mathrm{sech}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}[\tau x^{\alpha}-\lambda t^{\alpha}]\Big) \mp\tanh[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})]\Big\}, \tag{113}$$

where λ and τ are given in Eq. (111).

Set 26

$$\Big\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{1-\delta}{2},\ \beta_1=\frac{\delta-1}{2}, \lambda=-\frac{1}{2}(\delta-1)(\delta+1),\ \tau=\frac{\delta-1}{\sqrt{2}}\Big\}, \tag{114}$$

$$v_1(x,t)=\frac{\delta+1}{2}+\frac{1}{2}(\delta-1)\Big\{\pm\mathrm{csch}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big] \mp\coth\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big]\Big\}, \tag{115}$$

$$v_2(x,t)=\frac{\delta+1}{2}+\frac{1}{2}(\delta-1)\Big\{\pm\mathrm{i}\,\mathrm{sech}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big] \mp\tanh\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big]\Big\}, \tag{116}$$

where λ and τ are given in Eq. (114).

Set 27

$$\Big\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{\delta-1}{2},\ \beta_1=\frac{\delta-1}{2}, \lambda=\frac{1}{2}(\delta-1)(\delta+1),\ \tau=-\frac{\delta-1}{\sqrt{2}}\Big\}, \tag{117}$$

$$v_1(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(\delta-1)\Big\{\coth\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big] +\mathrm{csch}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big]\Big\}, \tag{118}$$

$$v_2(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(\delta-1)\{\mathrm{i}\,\mathrm{sech}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big] +\tanh\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big]\Big\}, \tag{119}$$

where λ and τ are given in Eq. (117).

Set 28

$$\Big\{\alpha_0=\frac{\delta+1}{2},\ \alpha_1=\frac{\delta-1}{2},\ \beta_1=\frac{\delta-1}{2}, \lambda=\frac{1}{2}(\delta-1)(\delta+1),\ \tau=\frac{\delta-1}{\sqrt{2}}\Big\}, \tag{120}$$

$$v_1(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(\delta-1)\Big\{\coth\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big] +\mathrm{csch}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big]\Big\}, \tag{121}$$

$$v_2(x,t)=\frac{\delta+1}{2}\pm\frac{1}{2}(\delta-1)\Big\{\mathrm{i}\,\mathrm{sech}\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big] +\tanh\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big]\Big\}, \tag{122}$$

where λ and τ are given in Eq. (120).

Set 29

$$\Big\{\alpha_0=\frac{\delta}{2},\ \alpha_1=-\frac{\delta}{2},\ \beta_1=-\frac{\delta}{2},\ \lambda=-\frac{1}{2}(\delta-2)\delta,\ \tau=-\frac{\delta}{\sqrt{2}}\Big\}, \tag{123}$$

$$v_1(x,t)=\frac{1}{2}\delta\Big\{1\mp\Big[\coth\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big) +\mathrm{csch}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big)\Big]\Big\}, \tag{124}$$

$$v_2(x,t)=\frac{1}{2}\delta\Big\{1\mp\Big[\mathrm{i}\,\mathrm{sech}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big) +\tanh\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big)\Big]\Big\}, \tag{125}$$

where λ and τ are given in Eq. (123).

Set 30

$$\Big\{\alpha_0=\frac{\delta}{2},\ \alpha_1=-\frac{\delta}{2},\ \beta_1=-\frac{\delta}{2},\ \lambda=-\frac{1}{2}(\delta-2)\delta,\ \tau=\frac{\delta}{\sqrt{2}}\Big\}, \tag{126}$$

$$v_1(x,t)=\frac{1}{2}\delta\Big\{1\mp\Big(\coth\Big[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big) +\mathrm{csch}\Big(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^{\alpha}-\lambda t^{\alpha})\Big)\Big]\Big\}, \tag{127}$$

$$v_2(x,t)=\frac{1}{2}\delta\Big\{1\mp\Big[\mathrm{i}\,\mathrm{sech}\Big(\tau\frac{\Gamma(1+\Upsilon)}{\alpha}x^{\alpha}-\lambda\frac{\Gamma(1+\Upsilon)}{\alpha}t^{\alpha}\Big) +\tanh\Big(\tau\frac{\Gamma(1+\Upsilon)}{\alpha}x^{\alpha}-\lambda\frac{\Gamma(1+\Upsilon)}{\alpha}t^{\alpha}\Big)\Big]\Big\}, \tag{128}$$

where λ and τ are given in Eq. (126).

Set 31

$$\left\{\alpha_0=\frac{\delta}{2},\ \alpha_1=\frac{\delta}{2},\ \beta_1=-\frac{\delta}{2},\ \lambda=\frac{1}{2}(\delta-2)\delta,\ \tau=-\frac{\delta}{\sqrt{2}}\right\}, \tag{129}$$

$$v_1(x,t)=\frac{1}{2}\delta\left\{1\mp\operatorname{csch}\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right.$$
$$\left.\pm\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{130}$$

$$v_2(x,t)=\frac{1}{2}\delta\left\{1\mp\mathrm{i}\ \operatorname{sech}\left[\tau\frac{\Gamma(1+\Upsilon)}{\alpha}x^\alpha-\lambda\frac{\Gamma(1+\Upsilon)}{\alpha}t^\alpha\right]\right.$$
$$\left.\pm\tanh\left[\tau\frac{\Gamma(1+\Upsilon)}{\alpha}x^\alpha-\lambda\frac{\Gamma(1+\Upsilon)}{\alpha}t^\alpha\right]\right\}, \tag{131}$$

where λ and τ are given in Eq. (129).

Set 32

$$\left\{\alpha_0=\frac{\delta}{2},\ \alpha_1=\frac{\delta}{2},\ \beta_1=-\frac{\delta}{2},\ \lambda=\frac{1}{2}(\delta-2)\delta,\ \tau=-\frac{\delta}{\sqrt{2}}\right\}, \tag{132}$$

$$v_1(x,t)=\frac{1}{2}\delta\left\{1\mp\operatorname{csch}\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right.$$
$$\left.\pm\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{133}$$

$$v_2(x,t)=\frac{1}{2}\delta\left\{1\mp\mathrm{i}\ \operatorname{sech}\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right.$$
$$\left.\pm\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{134}$$

where λ and τ are given in Eq. (132).

Set 33

$$\left\{\alpha_0=\frac{\delta}{2},\ \alpha_1=-\frac{\delta}{2},\ \beta_1=\frac{\delta}{2},\ \lambda=-\frac{1}{2}(\delta-2)\delta,\ \tau=-\frac{\delta}{\sqrt{2}}\right\}, \tag{135}$$

$$v_1(x,t)=\frac{1}{2}\delta\left\{\left[1\pm\operatorname{csch}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\right]\right.$$
$$\left.\mp\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{136}$$

$$v_2(x,t)=\frac{1}{2}\delta\left\{\left[1\pm\mathrm{i}\ \operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\right]\right.$$
$$\left.\mp\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{137}$$

where λ and τ are given in Eq. (135).

Set 34

$$\left\{\alpha_0=\frac{\delta}{2},\ \alpha_1=-\frac{\delta}{2},\ \beta_1=\frac{\delta}{2},\ \lambda=-\frac{1}{2}(\delta-2)\delta,\ \tau=\frac{\delta}{\sqrt{2}}\right\}, \tag{138}$$

$$v_1(x,t)=\frac{1}{2}\delta\left[\left[1\pm\operatorname{csch}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\right]\right.$$
$$\left.\mp\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{139}$$

$$v_2(x,t)=\frac{1}{2}\delta\left\{\left[1\pm\mathrm{i}\ \operatorname{sech}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\right]\right.$$
$$\left.\mp\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{140}$$

where λ and τ are given in Eq. (138).

Set 35

$$\left\{\alpha_0=\frac{\delta}{2},\ \alpha_1=\frac{\delta}{2},\ \beta_1=\frac{\delta}{2},\ \lambda=\frac{1}{2}(\delta-2)\delta,\ \tau=-\frac{\delta}{\sqrt{2}}\right\}, \tag{141}$$

$$v_1(x,t)=\frac{1}{2}\delta\left(1\pm\operatorname{csch}\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right)\right.$$
$$\left.\pm\coth\left(\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{142}$$

$$v_2(x,t)=\frac{1}{2}\delta\left\{1\pm\mathrm{i}\ \operatorname{sech}\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right.$$
$$\left.\pm\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{143}$$

where λ and τ are given in Eq. (141).

Set 36

$$\left\{\alpha_0=\frac{\delta}{2},\ \alpha_1=\frac{\delta}{2},\ \beta_1=\frac{\delta}{2},\ \lambda=\frac{1}{2}(\delta-2)\delta,\ \tau=\frac{\delta}{\sqrt{2}}\right\}, \tag{144}$$

$$v_1(x,t)=\frac{1}{2}\delta\left\{1\pm\operatorname{csch}\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right.$$
$$\left.\pm\coth\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{145}$$

$$v_2(x,t)=\frac{1}{2}\delta\left\{1\pm\mathrm{i}\ \operatorname{sech}\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right.$$
$$\left.\pm\tanh\left[\frac{\Gamma(1+\Upsilon)}{\alpha}(\tau x^\alpha-\lambda t^\alpha)\right]\right\}, \tag{146}$$

where λ and τ are given in Eq. (144).

5. Physical explanation of solutions

In this section, some of our obtained traveling wave solutions are depicted in the following. Figure 1 shows a wave profile of $|v_1(x,t)|$ appearing in Eq. (25) with $d=2$, $\beta_0=0.2$, $\beta_1=0.3$, $\Upsilon=1$, $\delta=0.1$, and $-5<x<5$ in 2D with $\alpha=1$, blue line plotted at $t=0$, orange line plotted at $t=1$, green line plotted at $t=2$, in 3D with $\alpha=0.8$ and $0<t<2$, and in contour with $\alpha=0.8$ and $0<t<2$. Figure 2 shows a wave profile of $|v_4(x,t)|$ appearing in Eq. (31) with $d=5$, $\beta_0=0.3$, $\beta_1=0.5$, $\Upsilon=1$, $\delta=-0.3$, and $-5<x<5$ in 2D with $\alpha=1$, blue line plotted at $t=0$, orange line plotted at $t=1$, green line plotted at $t=2$, in 3D with $\alpha=0.8$ and $0<t<2$, and in contour with $\alpha=0.8$ and $0<t<2$. Figure 3 shows a wave profile of $|v_1(x,t)|$ appearing in Eq. (40) with $\Upsilon=1$, $\delta=-0.3$, and $-5<x<5$ in 2D with $\alpha=1$, blue line plotted at $t=0$, orange line plotted at $t=1$, green line plotted at $t=2$, in 3D with $\alpha=0.8$ and $0<t<2$, and in contour with $\alpha=0.8$ and $0<t<2$. Figure 4 shows a wave profile of $|v_2(x,t)|$ appearing in Eq. (41) with $\Upsilon=1$, $\delta=0.3$, and $-5<x<5$ in 2D with $\alpha=1$, blue line plotted at $t=0$, orange line plotted at $t=1$, green line plotted at $t=2$, in 3D with $\alpha=0.8$ and $0<t<2$, and in contour with $\alpha=0.8$ and $0<t<2$.

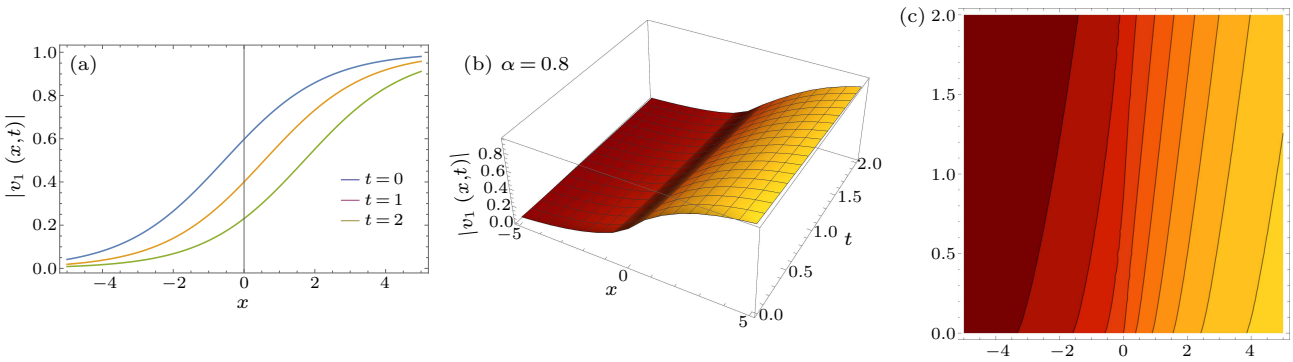


Fig. 1. Wave profile of $|v_1(x,t)|$ appearing in Eq. (25) in 2D, 3D, and contour.

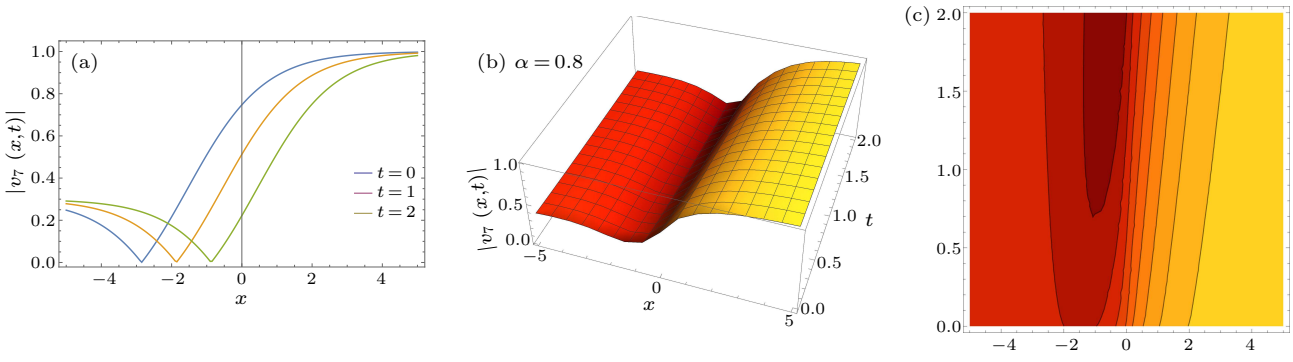


Fig. 2. Wave profile of $|v_7(x,t)|$ appearing in Eq. (31) in 2D, 3D, and contour.

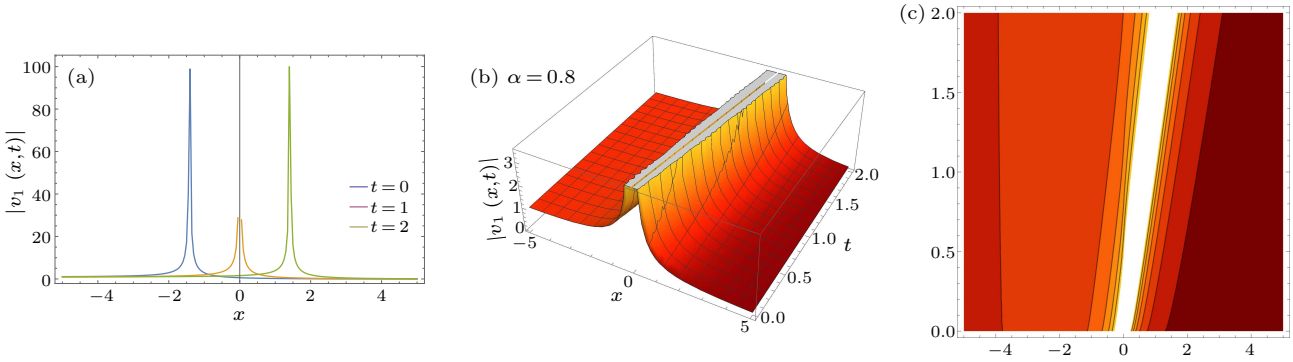


Fig. 3. Wave profile of $|v_1(x,t)|$ appearing in Eq. (40) in 2D, 3D, and contour.

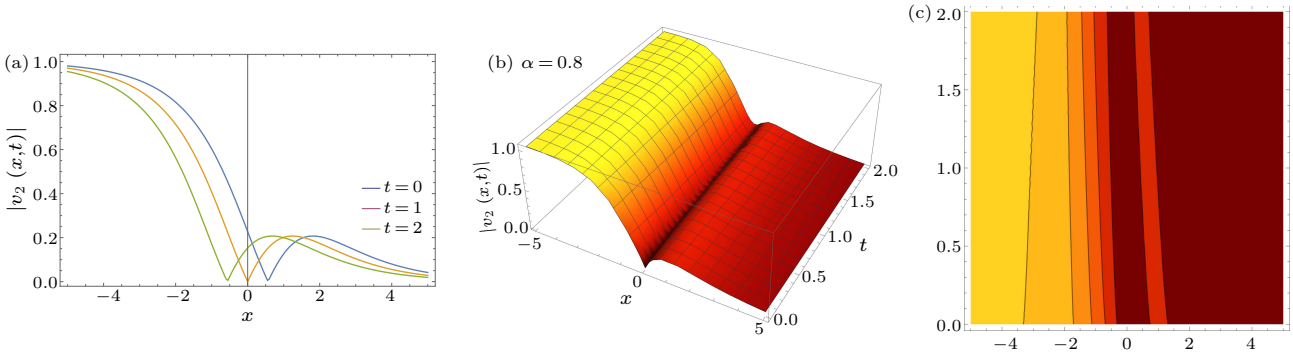


Fig. 4. Wave profile of $|v_2(x,t)|$ appearing in Eq. (41) in 2D, 3D, and contour.

6. Conclusions

In summary, we gain the analytical wave solutions using the truncated M-fractional Fitzhugh–Nagumo model and applying the $\exp_a$ function and EShGEE schemes. The solutions gained involve the sech, csch, tanh and coth functions. These analytical wave solutions are gained and verified with the use of Mathematica software. These solutions do not exist in literature. Some of the solutions are demonstrated by

2D, 3D and contour graphs. This research may be helpful in the telecommunications field and other related waveguides. It is suggested that these two schemes are more applicable, reliable and significant to deal with fractional nonlinear partial differential equations.

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