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Citation: Chin. Phys. B, 2024, 33 (**2**): 020307. DOI: 10.1088/1674-1056/ace15f
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Genuine entanglement under squeezed generalized amplitude damping channels with memory

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(Received 3 May 2023; revised manuscript received 14 June 2023; accepted manuscript online 25 June 2023)

We study genuine entanglement among three qubits undergoing a noisy process that includes dissipation, squeezing, and decoherence. We obtain a general solution and analyze the asymptotic quantum states. We find that most of these asymptotic states can be genuinely entangled depending upon the parameters of the channel, memory parameter, and the parameters of the initial states. We study Greenberger–Horne–Zeilinger (GHZ) states and W states, mixed with white noise, and determine the conditions for them to be genuinely entangled at infinity. We find that for these mixtures, it is possible to start with a bi-separable state (with a specific mixture of white noise) and end with genuine entangled states. However, the memory parameter μ must be very high. We find that in contrast to the two-qubit case, none of the three-qubit asymptotic states for $n \to \infty$ are genuinely entangled.

Keywords: genuine entanglement, squeezed thermal baths, multipartite states

PACS: 03.65.Yz, 03.65.Ud, 03.67.Mn **DOI:** 10.1088/1674-1056/ace15f

1. Introduction

Quantum correlations among several particles not only lead to counterintuitive predictions but also have a key role in future technologies.[1] There is growing interest among researchers to study quantum correlations in both theory and experiments.[2–6] Entanglement is a precious resource and is important for several applications. However, one of the big challenges is to preserve quantum correlations in quantum states. Quantum states interact with their environments and it is well-known that such interactions can degrade entanglement.[7–10] Many authors have studied the effect of the environment on entanglement for both bipartite and multipartite systems.[11–30]

Quantum channels describe the physical situations in which a given quantum state is transformed into another quantum state as a dynamical process. These processes are regarded as maps that preserve trace and are also completely positive, which means that a quantum state undergoing a quantum channel is again a valid quantum state. One such channel is the amplitude damping channel (ADC), which models spontaneous emission from excited atoms or energy dissipation at zero temperature.[31] This channel has been generalized to model dissipation at finite temperature[32] and is called the generalized amplitude damping (GAD) channel. Another generalization of this model takes squeezing into account, this model is called squeezed generalized amplitude damping (SGAD) channel.[33] The effect on qubit–qubit entanglement for this channel was studied,[34] where it was found that this channel cannot preserve entanglement. All of these channels are considered as memoryless and the effect of the channel can be extended to N qubits simply as $\Phi = \Phi_1^{\otimes N}$. There are situations where this simplification is not true, and $\Phi \neq \Phi_1^{\otimes N}$[35] and channel is said to have memory. The effect of memory on spin chains,[36] entanglement-enhanced transmission of classical information in Pauli channels,[37] classical and quantum capacities of correlated ADC,[38] and others,[39,40] have been studied. Recently, qubit–qubit entanglement and quantum discord[41] under SGAD channel with memory has been studied.[42]

An array of beam-splitters can be used to model a squeezed reservoir.[43] Laser-cooled trapped ions can also mimic the dynamics of an atom with a squeezed vacuum bath.[44] Another method to mimic coupling is to use a squeezed bath via four-level atoms driven by weak laser fields.[45] Generalized ADCs are studied in experiments for decoherence and decay of atomic states.[46,47]

In this work, we study the effect of memory on genuine entanglement for three-qubit systems, which to our knowledge has not been studied before. We consider three qubits sent by the consecutive use of quantum channel with memory. We determine the asymptotic states for a most general initial state of three qubits. We find that squeezing parameter m is not present in the asymptotic states. This observation is in agreement with the fact that a squeezed thermal bath can suppress quantum decoherence[34] but it is unable to preserve quantum entanglement.[48,49] We then analyze these asymptotic states for various initial states and find that in most cases they can be genuinely entangled depending on memory and the parameters of the initial states. We found that even if we start with bi-separable states, and then depending upon the thermal parameter n (must not be very large) and the memory parameter μ (must be very large), asymptotic states can be genuinely

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entangled. Although the degree of genuine entanglement for these asymptotic states is quite small, it is an interesting feature of this dynamical process.

This paper is organized as follows. In Section 2, we briefly discuss squeezed generalized ADC for qubits and provide the general solution for an arbitrary state of three qubits. We briefly review the concept of genuine entanglement in Section 3 and present our main findings for two important families of quantum states. Finally, we conclude our work in Section 4.

2. Squeezed generalized amplitude damping channel for qubits

The quantum theory of damping takes a two-level atom (system) as a harmonic oscillator interacting with a reservoir (or bath), which can also be treated set of harmonic oscillators. The Hamiltonian in the interaction picture can be written as[50,51]

$$H=\hbar\sum_k g_k\left[b_k^\dagger\sigma_-\,\mathrm{e}^{-\mathrm{i}(\omega-\nu_k)t}+\sigma_+ b_k\,\mathrm{e}^{\mathrm{i}(\omega-\nu_k)t}\right], \tag{1}$$

where $\sigma_-=|b\rangle\langle a|$ and $\sigma_+=|a\rangle\langle b|$ are atomic lowering and raising operators. b_k ($b_k^\dagger$) are bath annihilation (creation) operators for each mode. $\nu_k=ck$ are density distributed frequencies, ω is atomic transition frequency and g_k are coupling constants. After standard quantum optical approximations and tracing over the reservoir, we get a master equation for the system only. The squeezed generalized ADC is a noisy quantum channel in which a qubit interacts with a bath being initially in a squeezed thermal state with Markov and Born approximations. The master equation in the interaction picture is given as[50,51]

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t}&=-\frac{\Omega(n+1)}{2}\left[\sigma_+\sigma_-\rho+\rho\sigma_+\sigma_--2\sigma_-\rho\sigma_+\right]\\&\quad-\frac{\Omega n}{2}\left[\sigma_-\sigma_+\rho+\rho\sigma_-\sigma_+-2\sigma_+\rho\sigma_-\right]\\&\quad-\Omega m\left[\sigma_+\rho\sigma_++\sigma_-\rho\sigma_-\right],\end{aligned} \tag{2}$$

where n is related to the number of thermal photons and m is the squeezing parameter. The complete positivity demands that for $\Omega\geq 0$, we must have $m^2\leq n(n+1)$. We note that for $m=0$, we have a qubit interacting with thermal reservoirs, and for $n=m=0$, we have (zero-temperature) amplitude damping process (vacuum reservoirs).

We can extend this model for multi-qubits either as uncorrelated noise in which we have to sum these three terms for each qubit separately and then solve the master equation in the interaction picture by taking $\Omega_{\mathrm{A}}=\Omega_{\mathrm{B}}=\cdots=\Omega_N=\Omega$, $n_{\mathrm{A}}=n_{\mathrm{B}}=\cdots=n_N=n$ and $m_{\mathrm{A}}=m_{\mathrm{B}}=\cdots=m_N=m$. This process is the simplest model for the memoryless quantum channel and the stochastic map $\Phi(\rho)$ can be extended to $\Phi^{\otimes N}(\rho)$ for N-systems or uses of quantum channel. The result is that Kraus operators also have structure $K=K_{\mathrm{A}}\otimes K_{\mathrm{B}}\otimes\cdots K_N$. For a single qubit under such uncorrelated noise, the Kraus operators[52] can be written as

$$K_1=\begin{pmatrix}k_1&0\\0&k_2\end{pmatrix}, \tag{3}$$

$$K_2=\begin{pmatrix}0&k_3\\k_4&0\end{pmatrix}, \tag{4}$$

$$K_3=\begin{pmatrix}\sqrt{\mathrm{e}^{-\Omega t(n+1/2)}\cosh(\Omega tm)}&0\\0&\sqrt{\mathrm{e}^{-\Omega t(n+1/2)}\cosh(\Omega tm)}\end{pmatrix}, \tag{5}$$

$$K_4=\begin{pmatrix}0&\sqrt{\mathrm{e}^{-\Omega t(n+1/2)}\sinh(\Omega tm)}\\\sqrt{\mathrm{e}^{-\Omega t(n+1/2)}\sinh(\Omega tm)}&0\end{pmatrix}, \tag{6}$$

where

$$k_1=\sqrt{\frac{n}{2n+1}+\frac{n+1}{2n+1}\mathrm{e}^{-2\Omega t(n+1/2)}-\mathrm{e}^{-\Omega t(n+1/2)}\cosh(\Omega tm)}, \tag{7}$$

$$k_2=\sqrt{\frac{n}{2n+1}\mathrm{e}^{-2\Omega t(n+1/2)}+\frac{n+1}{2n+1}-\mathrm{e}^{-\Omega t(n+1/2)}\cosh(\Omega tm)}, \tag{8}$$

$$k_3=\sqrt{\frac{n}{2n+1}(1-\mathrm{e}^{-2\Omega t(n+1/2)})-\mathrm{e}^{-\Omega t(n+1/2)}\sinh(\Omega tm)}, \tag{9}$$

$$k_4=\sqrt{\frac{n+1}{2n+1}(1-\mathrm{e}^{-2\Omega t(n+1/2)})-\mathrm{e}^{-\Omega t(n+1/2)}\sinh(\Omega tm)}. \tag{10}$$

The Kraus operators satisfy the normalization condition $\sum_i K_i^\dagger(t)K_i(t)=I$. For three qubits, there are 64 such operators, i.e., $M_1=K_1^{\mathrm{A}}K_1^{\mathrm{B}}K_1^{\mathrm{C}}$, $M_2=K_1^{\mathrm{A}}K_1^{\mathrm{B}}K_2^{\mathrm{C}},\ldots,M_{64}=K_4^{\mathrm{A}}K_4^{\mathrm{B}}K_4^{\mathrm{C}}$ with $\sum_{i=1}^{64}M_i^\dagger(t)M_i(t)=I_8$. We have omitted the tensor product symbol between these operators. The stochastic map for SGAD with uncorrelated noise is

$$\Phi_{\mathrm{u}}(\rho)=\rho_{\mathrm{u}}(t)=\sum_{j=1}^{64}M_j\rho M_j^\dagger. \tag{11}$$

For correlated noise $\Phi(\rho)\neq\Phi^{\otimes N}(\rho)$, and for three qubits, we have the correlated version of the master equation

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t}&=-\frac{\Omega(n+1)}{2}\left[\sigma_+^{\otimes 3}\sigma_-^{\otimes 3}\rho+\rho\sigma_+^{\otimes 3}\sigma_-^{\otimes 3}-2\sigma_-^{\otimes 3}\rho\sigma_+^{\otimes 3}\right]\\&\quad-\frac{\Omega n}{2}\left[\sigma_-^{\otimes 3}\sigma_+^{\otimes 3}\rho+\rho\sigma_-^{\otimes 3}\sigma_+^{\otimes 3}-2\sigma_+^{\otimes 3}\rho\sigma_-^{\otimes 3}\right]\\&\quad-\Omega m\left[\sigma_+^{\otimes 3}\rho\sigma_+^{\otimes 3}+\sigma_-^{\otimes 3}\rho\sigma_-^{\otimes 3}\right],\end{aligned} \tag{12}$$

where $\sigma_+^{\otimes 3}=\sigma_+\otimes\sigma_+\otimes\sigma_+$. The stochastic map for the process can be written either in terms of Kraus operators (which are not in tensor product format like uncorrelated case) or simply as a solution of master Eq. (12). In any case, it can be written as

$$\Phi_{\mathrm{c}}(\rho)=\rho_{\mathrm{c}}(t)=\sum_j X_j\rho X_j^\dagger, \tag{13}$$

where $X_j\neq X_{\mathrm{A}}\otimes X_{\mathrm{B}}\otimes\cdots X_N$. The density matrix for an arbitrary state of three qubits has a simple solution in this case

$$\rho_{\rm c}(t)=\begin{pmatrix}\rho_{11}(t)&\rho_{12}(t)&\rho_{13}(t)&\rho_{14}(t)&\rho_{15}(t)&\rho_{16}(t)&\rho_{17}(t)&\rho_{18}(t)\\ \rho_{21}(t)&\rho_{22}&\rho_{23}&\rho_{24}&\rho_{25}&\rho_{26}&\rho_{27}&\rho_{28}(t)\\ \rho_{31}(t)&\rho_{32}&\rho_{33}&\rho_{34}&\rho_{35}&\rho_{36}&\rho_{37}&\rho_{38}(t)\\ \rho_{41}(t)&\rho_{42}&\rho_{43}&\rho_{44}&\rho_{45}&\rho_{46}&\rho_{47}&\rho_{48}(t)\\ \rho_{51}(t)&\rho_{52}&\rho_{53}&\rho_{54}&\rho_{55}&\rho_{56}&\rho_{57}&\rho_{58}(t)\\ \rho_{61}(t)&\rho_{62}&\rho_{63}&\rho_{64}&\rho_{65}&\rho_{66}&\rho_{67}&\rho_{68}(t)\\ \rho_{71}(t)&\rho_{72}&\rho_{73}&\rho_{74}&\rho_{75}&\rho_{76}&\rho_{77}&\rho_{78}(t)\\ \rho_{81}(t)&\rho_{82}(t)&\rho_{83}(t)&\rho_{84}(t)&\rho_{85}(t)&\rho_{86}(t)&\rho_{87}(t)&\rho_{88}(t)\end{pmatrix}, \tag{14}$$

where

$$\begin{aligned}\rho_{1s}(t)&=\rho_{1s}(0)\,{\rm e}^{-\frac{\Omega t(n+1)}{2}},\\ \rho_{s8}(t)&=\rho_{s8}(0)\,{\rm e}^{-\frac{\Omega tn}{2}},\\ \rho_{11}(t)&=\frac{n(\rho_{11}+\rho_{88})}{2n+1}+\frac{\rho_{11}+n(\rho_{11}-\rho_{88})}{2n+1}\,{\rm e}^{-(2n+1)\Omega t},\\ \rho_{18}(t)&=\frac{1}{2}\left[\rho_{18}+\rho_{81}+{\rm e}^{2m\Omega t}(\rho_{18}-\rho_{81})\right]{\rm e}^{-(n+m+1/2)\Omega t},\\ \rho_{88}(t)&=\frac{(1-{\rm e}^{-(2n+1)\Omega t})(1+n)\rho_{11}}{2n+1}\\ &\quad+\frac{1+n(1+{\rm e}^{-(2n+1)\Omega t})\rho_{88}}{2n+1},\end{aligned} \tag{15}$$

with $s=2,3,\ldots,7$. Finally, the stochastic map for three qubits sent by three consecutive uses of channel with memory can be written as

$$\rho(t)=\mu\Phi_{\rm c}(\rho)(t)+(1-\mu)\Phi_{\rm u}(\rho)(t), \tag{16}$$

where $0\leq\mu\leq1$ is degree of channel memory, which means that the noise is correlated with probability μ. We have the most general solution for the system and we can study the asymptotic states by taking $t\to\infty$. The examples that follow refer to such states as $\rho(\infty)$. Although individual matrix elements have quite lengthy expressions, it is possible to study the evolution of entanglement numerically using MATLAB (as described in the following section).

3. Genuine entanglement under SGAD channels

It is appropriate that we first briefly review the ideas about genuine entanglement. Let us consider three qubits to discuss the ideas with this understanding that the same arguments are equally valid for multipartite systems where each subsystem is a qudit. For pure states, we say that a state is fully separable if $|\psi^{\rm ABC}\rangle=|\psi^{\rm A}\rangle\otimes|\psi^{\rm B}\rangle\otimes|\psi^{\rm C}\rangle$. For mixed states, a state is fully separable if $\rho=\sum_j p_j\rho_j^{\rm A}\otimes\rho_j^{\rm B}\otimes\rho_j^{\rm C}$. Bi-separable states can be written as $\rho^{\rm sep}_{\rm A|BC}=\sum_j q_j|\phi_{\rm A}^j\rangle\langle\phi_{\rm A}^j|\otimes|\phi_{\rm BC}^j\rangle\langle\phi_{\rm BC}^j|$, with similar construction for $\rho^{\rm sep}_{\rm B|AC}$ and $\rho^{\rm sep}_{\rm C|AB}$. Their mixtures are also bi-separable $\rho^{\rm bs}=p_1\rho^{\rm sep}_{\rm A|BC}+p_2\rho^{\rm sep}_{\rm B|AC}+p_3\rho^{\rm sep}_{\rm C|AB}$. A multipartite state is said to be genuinely entangled if it is not fully separable and not bi-separable. We should keep in mind that there are quantum states that have negative partial transpose (NPT) under each partition but they are bi-separable.[3]

Genuine entanglement can be detected and quantified via positive partial transpose (PPT) mixtures.[53–55] PPT mixtures are characterized by semidefinite programming (SDP) and this approach can be used to quantify genuine entanglement.[53] For bipartite systems, this procedure is equivalent to negativity,[56] so we can call it genuine negativity. For multi-qubits, it is bounded by $E(\rho)\leq1$ with upper bound for certain pure states.[57] For mixed states, it is always less than 1. If this measure is positive, then the state is guaranteed to be genuinely entangled, otherwise we are not sure unless some other procedure indicates its entanglement properties.

It is well-known that for three qubits there are two inequivalent families of quantum states, namely, GHZ states and W states, given as

$$\begin{aligned}|{\rm GHZ}_1\rangle&=\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle),\\ |W\rangle&=\frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle).\end{aligned} \tag{17}$$

Other equivalent GHZ states are $|{\rm GHZ}_2\rangle=1/\sqrt{2}(|001\rangle+|110\rangle)$, $|{\rm GHZ}_3\rangle=1/\sqrt{2}(|010\rangle+|101\rangle)$, and $|{\rm GHZ}_4\rangle=1/\sqrt{2}(|011\rangle+|100\rangle)$. The W state equivalent is $|\tilde{W}\rangle=1/\sqrt{3}(|011\rangle+|101\rangle+|110\rangle)$. The GHZ states have maximum value of genuine negativity, i.e., $E(|{\rm GHZ}_j\rangle\langle{\rm GHZ}_j|)=1$, whereas for the W state, $E(|W\rangle\langle W|)=E(|\tilde{W}\rangle\langle\tilde{W}|)\approx0.886$.[53]

As a first example, we consider ${\rm GHZ}_1$ state mixed with white noise

$$\rho_1=\alpha|{\rm GHZ}_1\rangle\langle{\rm GHZ}_1|+\frac{1-\alpha}{8}I_8, \tag{18}$$

where $0\leq\alpha\leq1$. It is known that these states are genuinely entangled for $0.429\leq\alpha\leq1$.[53] An interesting property of SGAD channel with memory is that for these states, the only non-zero matrix elements are only on the main diagonal and main off-diagonal of the density matrix, whereas all other elements remain zero (X structure).[58] A result of the detection of genuine entanglement states that the inequality

$$|\rho_{18}|\leq\sqrt{\rho_{22}\rho_{77}}+\sqrt{\rho_{33}\rho_{66}}+\sqrt{\rho_{44}\rho_{55}} \tag{19}$$

is satisfied by bi-separable states and the violation implies genuine entanglement.[59] This criterion is a necessary and sufficient condition for GHZ-diagonal states.[59] Application of this result to $\rho_1(\infty)$ gives us the condition that

$$3\sqrt{\left(\frac{n^2(1+n)(1-\mu)}{(1+2n)^3}+\mu(1-\alpha)/8\right)\left(\frac{n(1+n)^2(1-\mu)}{(1+2n)^3}+\mu(1-\alpha)/8\right)} \geq \frac{(n(1+n))^{3/2}\alpha(1-\mu)}{(1+2n)^3}. \tag{20}$$

This condition is always satisfied for $n \to \infty$ because it simplifies to $\mu \leq (3-\alpha)/2\alpha$. Hence, none of the asymptotic states with $n \to \infty$ are genuinely entangled. However, it is possible that the condition is violated for certain values of α, n and μ. Therefore, it is possible to have asymptotic states that are genuinely entangled.

In Fig. 1, we plot genuine entanglement $E(\rho)$ for asymptotic states with $n = 1$. We observe that the state $\rho_1(\infty)$ with $\alpha = 1$ is always genuinely entangled as long as memory $\mu > 0$. For a mixture with white noise, if we take $\alpha = 0.95$ (dashed line) or $\alpha = 0.9$ (dotted–dashed line), then the asymptotic states are genuinely entangled for larger values of memory μ. We also observe that most of the genuine entanglement is lost because asymptotic states have a much lower degree of genuine entanglement. In this case, our initial states are genuinely entangled and asymptotic states may be genuinely entangled, as seen in Fig. 1.

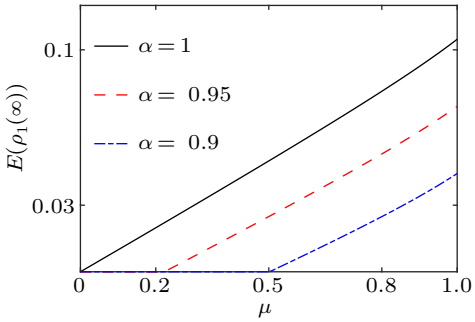


Fig. 1. Genuine entanglement for states at infinity plotted against memory parameter μ and $n = 1$. We see that for $\alpha = 1$, asymptotic states are genuinely entangled for $\mu > 0$. If $\alpha < 1$, then some states are genuinely entangled if we increase parameter μ.

In contrast, if we take initial state $|\mathrm{GHZ}_2\rangle$ mixed with white noise, then we would expect different results. As we saw in Eq. (14), $|\mathrm{GHZ}_2\rangle$ lives in decoherence free space. Therefore, its entanglement is not changing and for $\mu = 1$ and $\alpha = 1$, we must have the same state at infinity. For $\alpha < 1$ and $\mu < 1$, we have to analyze the asymptotic states. In Fig. 2, genuine entanglement is plotted for asymptotic states with $n = 1$ for three choices of parameter α. We can see that for $\mu = 1$ and $\alpha = 1$, the states are stationary and genuine entanglement is fixed at maximum value of 1. However, as memory μ and α are decreased, the effects of uncorrelated noise degrade the genuine entanglement until for a specific value of memory the asymptotic states are no longer genuinely entangled.

The condition for bi-separable states for $n \to \infty$ gives us $\mu \leq (6-\alpha)/(13\alpha)$, which is not true for the whole range of α. It is violated for $\alpha < 0.429$ and the initial states are genuinely entangled for $0.429 \leq \alpha \leq 1$. Since this result is for very large n, we expect that it is possible to start with a bi-separable state and end up with a genuine entangled state, provided that memory μ is quite high and that n is small. In Fig. 3, we observe that all three bi-separable states can end up with genuine entangled states for $\mu \geq 0.97$.

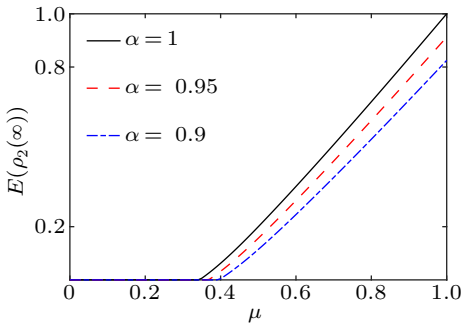


Fig. 2. Multipartite entanglement is plotted for asymptotic states against memory μ. We take $n = 1$ for this plot.

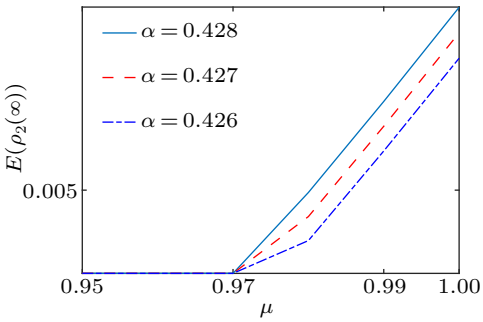


Fig. 3. Genuine entanglement is plotted against memory μ. This plot is for $n = 0.1$. We observe that all three initially bi-separable states may become genuinely entangled as a result of dynamical process.

Let us now take another type of genuine entangled state, namely, W state mixed with white noise, as

$$\rho_W = (1-\beta)|W\rangle\langle W| + \frac{\beta}{8}I_8, \tag{21}$$

where $0 \leq \beta \leq 1$. We want to mention here that $1-\beta = \alpha$ has the same meanings and we could have used α instead of β. These states are genuinely entangled for $0 \leq \beta \leq 0.521$.[53]

Figure 4 depicts genuine entanglement for asymptotic states with initial states as genuinely entangled for $\beta = 0, 0.2$ and bi-separable for $\beta = 0.522$. We observe that even though the initial state is bi-separable (for $\beta = 0.522$), the asymptotic state is genuinely entangled. We observe that all entangled asymptotic states have a very small degree of genuine entanglement. Nevertheless, it is an interesting feature of this channel that it can convert bi-separable states into genuine entangled states.

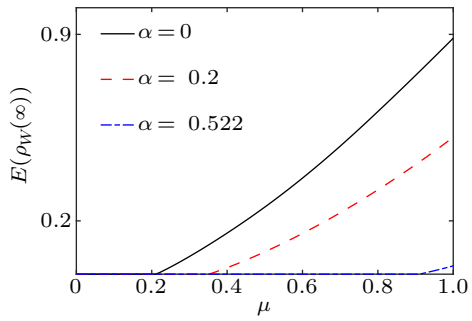


Fig. 4. Genuine entanglement plotted against memory μ with $n = 1$ for initially W states mixed with white noise.

4. Discussion and conclusion

We have studied genuine entanglement under the SGAD channel with memory for three qubits. We have obtained the most general solution for the system. Based on this solution, we are able to analyze the asymptotic states. We have found that in the case of $|\mathrm{GHZ}_1\rangle$ states mixed with white noise and for $n \to \infty$, none of the asymptotic states are genuinely entangled. However, for small values of n and depending on memory μ and initial states, it is possible to have genuinely entangled asymptotic states. We have also taken initial bi-separable states and have found that under certain circumstances the asymptotic states may be genuinely entangled. We found this possibility not only in GHZ states but also for W states mixed with white noise. It is an interesting feature of this dynamical process that it may bring bi-separable states to genuinely entangled states. However for this phenomenon to occur, we must have a very high degree of memory μ in the channel. We also observed that squeezing parameter m is absent only among asymptotic states. It is well-known that expectation values of certain physical quantities at infinity are a function of average thermal photons $\langle n(\omega)\rangle$.[50] This has a simple interpretation. After a long time, the oscillator in contact with a heat bath gets thermalized, with the same average photon number as the thermal average at the oscillator's frequency. The difference between two qubits under SGAD and three qubits under similar noise can be summarized as follows. It was found that for two-qubits singlet state with correlated noise, squeezing parameter m does not affect the dynamics of entanglement and quantum discord.[42] In addition, some of the two-qubit asymptotic states are also not dependent on squeezing parameter m. It is possible that with singlet states as initial states, the asymptotic states with $n \to \infty$ can be entangled even if initial states are separable. For three qubits, we get similar results with some differences. The first main difference is the fact that we can only get genuine entangled states at infinity if n is not too large because we observed that none of the asymptotic states with $n \to \infty$ are genuinely entangled. Similar to two qubits, for correlated noise $\mu = 1$, the GHZ state living in decoherence free space is invariant under dynamics. For three qubits, we have two types of inequivalent genuine entangled states; whereas for two qubits, states are either entangled or separable.

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