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Analytical three-periodic solutions of Korteweg–de Vries-type equations

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Based on the direct method of calculating the periodic wave solution proposed by Nakamura, we give an approximate analytical three-periodic solutions of Korteweg–de Vries (KdV)-type equations by perturbation method for the first time. Limit methods have been used to establish the asymptotic relationships between the three-periodic solution separately and another three solutions, the soliton solution, the one- and the two-periodic solutions. Furthermore, it is found that the asymptotic three-soliton solution presents the same repulsive phenomenon as the asymptotic three-soliton solution during the interaction.

Keywords: Hirota bilinear method, Riemann theta function, three-periodic solution

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1. Introduction

There are many methods to study analytical solutions of nonlinear partial differential equations (NLPDEs), such as inverse scattering transform,[1,2] Painlevé analysis,[3,4] Darboux transformation,[5–9] deep learning method,[10] Hirota bilinear method, and so on,[11–15] among which Hirota bilinear method is an important method to obtain the exact solution of NLPDEs. In particular, Hirota bilinear method is extended to construct periodic solutions of some NLPDEs.

The periodic solution of NLPDEs is one of the hot topics in soliton theory. The quasi-periodic solutions is first studied by Dubrovin, Matveevet, Novikov *et al.*, and they can also be called finite gap solutions.[16–18] Nakamura proposed a method of constructing quasi-periodic solutions based on Hirota bilinear method[19,20] and this method can be sued to obtain the algebraic relations among frequencies, phase shifts, amplitudes, and wave numbers. The existence condition for N-periodic solution of partial differential equations has been given, and when $N \geq 3$, this existence condition is an over-determined system. A few decades later, the two-periodic solutions of the Toda lattice equation, Nizhnik–Novikov–Veselov equation, the Fokas equation, the nonlocal Boussinesq equation, and KdV equation are obtained.[21–25] Recently, some three-periodic solutions of NLPDEs have been proved to exist, such as KdV-type equations and Tzitzeica equation,[26,27] but no one has yet given analytical expressions of their three-periodic solutions. We will construct the three-periodic solutions of two KdV-type equations in this article.

We called KdV-type equations when the equation

$$L(u, u_t, u_{xx}, \ldots) = 0$$

can be transformed into bilinear form

$$F(D_t, D_x, \ldots, C) f \cdot f = 0$$

by dependent variable transformation

$$u = 2(\ln f)_{xx},$$

where F is an even function of D_t, D_x, …, and C is an integration constant. The standard identities for the Hirota D-operators:

$$\begin{aligned} & D_x^m D_t^k f(x,t) \cdot g(x,t) \\ = \; & (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^k f(x,t) g(x',t')|_{x'=x, t'=t}. \end{aligned}$$

The three-periodic solutions of the KdV and the Hietarinta equations, have been shown to exist, and their bilinear equations are as follows:[27]

$$\mathcal{H}_{\mathrm{KdV}} := (D_x D_t + D_x^4 + C) f \cdot f = 0, \tag{1}$$

$$\mathcal{H}_{\mathrm{Hie}} := (D_x^3 D_t + \sqrt{3} D_x^2 D_t^2 + a D_x^2 + b D_x D_t + C) f \cdot f = 0. \tag{2}$$

In this paper, we construct the three-periodic solutions of the KdV and the Hietarinta equations and analyze the asymptotic relations between the periodic solutions and the soliton solutions. In addition, we also analyze the interactions of the three-periodic solutions. In Section 2, 8 algebraic equations satisfying the three-periodic solutions of the Hietarinta and the KdV equations are respectively obtained based on Riemann theta function and Hirota bilinear method. In Section 3, we construct the three-periodic solution of KdV equation, and the asymptotic relationship between the three-periodic solution and the three soliton solution is constructed. In Section 4, the

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three-periodic solution of Hietarinta equation and its asymptotic three soliton are constructed, then we find the periodic solutions and their asymptotic solitons can exhibit the same characteristic during the nonlinear interaction, namely, repulsive phenomenon. In Section 5, a summary is given.

2. The three-periodic solutions of KdV-type equations

2.1. Condition for N-periodic solutions

Based on the following multi-periodic Riemann theta function in genus N, we consider the multi-periodic solutions of the KdV and the Hietarinta equations:

$$u = 2(\ln\vartheta)_{xx},$$

with

$$\vartheta = \vartheta(\boldsymbol{\xi},\boldsymbol{\tau}) = \sum_{\boldsymbol{n}\in Z^N} \mathrm{e}^{-\pi\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle+2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{n}\rangle}, \tag{3}$$

where $\boldsymbol{n} = (n_1,n_2,\ldots n_N)^{\mathrm{T}} \in Z^N$, $\boldsymbol{\xi} = (\xi_1,\xi_2,\ldots\xi_N)^{\mathrm{T}} \in C^2$, $\xi_i = \alpha_i x + \omega_i t + \xi_i^0$, $i = 1,2,\ldots,N$,

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1N} \\ \tau_{21} & \tau_{22} & \cdots & \tau_{23} \\ \vdots & \vdots & \vdots & \vdots \\ \tau_{N1} & \tau_{N2} & \cdots & \tau_{NN} \end{pmatrix}, \quad \tau_{ii} > 0, \quad \tau_{ij} = \tau_{ji}, \quad |\tau| > 0.$$

$\boldsymbol{\tau}$ is a positive definite real valued symmetric $N\times N$ matrix. When we substitute Eq. (3) into Eq. (1) or Eq. (2), we can get

$$\begin{aligned}
\mathcal{H}f\cdot f &= \mathcal{H}(D_x,D_t)f\cdot f \\
&= \mathcal{H}(D_x,D_t)\sum_{\boldsymbol{n}\in Z^N}\mathrm{e}^{-\pi\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle+2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{n}\rangle}\cdot\sum_{\boldsymbol{m}\in Z^N}\mathrm{e}^{-\pi\langle\boldsymbol{\tau}\boldsymbol{m},\boldsymbol{m}\rangle+2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{m}\rangle} \\
&= \sum_{\boldsymbol{n},\boldsymbol{m}\in Z^N}\mathcal{H}[2\pi\mathrm{i}\langle\boldsymbol{n}-\boldsymbol{m},\boldsymbol{\alpha}\rangle,2\pi\mathrm{i}\langle\boldsymbol{n}-\boldsymbol{m},\boldsymbol{\omega}\rangle]\,\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{n}+\boldsymbol{m}\rangle-\pi(\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle+\langle\boldsymbol{\tau}\boldsymbol{m},\boldsymbol{m}\rangle)} \\
&\overset{\boldsymbol{m}=\boldsymbol{l}-\boldsymbol{n}}{=} \sum_{\boldsymbol{n},\boldsymbol{l}\in Z^N}\mathcal{H}[2\pi\mathrm{i}\langle2\boldsymbol{n}-\boldsymbol{l},\boldsymbol{\alpha}\rangle,2\pi\mathrm{i}\langle2\boldsymbol{n}-\boldsymbol{l},\boldsymbol{\omega}\rangle]\,\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{l}\rangle-\pi(\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle+\langle\boldsymbol{\tau}(\boldsymbol{n}-\boldsymbol{l}),\boldsymbol{n}-\boldsymbol{l}\rangle)} \\
&= \sum_{\boldsymbol{l}\in Z^N}\widetilde{\mathcal{H}}(l_1,l_2,\cdots,l_N)\,\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{l}\rangle} = 0,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
\widetilde{\mathcal{H}}(l_1,l_2,\ldots,l_N) &= \sum_{n_j=-\infty}^{\infty}\mathcal{H}[2\pi\mathrm{i}\sum_{j=1}^{N}(2n_j-l_j)\alpha_j,2\pi\mathrm{i}\sum_{j=1}^{N}(2n_j-l_j)\omega_j]\exp\left[-\pi\sum_{j,k=1}^{N}(n_j-l_j)\tau_{jk}(n_k-l_k)+n_j\tau_{jk}n_k\right] \\
&\overset{n_j=n_j+\delta_{jz}}{=} \sum_{n_j=-\infty}^{\infty}\mathcal{H}[2\pi\mathrm{i}\sum_{j=1}^{N}(2n_j-(l_j-2\delta_{jz}))\alpha_j,2\pi\mathrm{i}\sum_{j=1}^{N}(2n_j-(l_j-2\delta_{jz}))\omega_j] \\
&\quad\times\exp\left[-\pi\sum_{j,k=1}^{N}(n_j-(l_j-2\delta_{jz}))\tau_{jk}(n_k-(l_k-2\delta_{kz}))+n_j\tau_{jk}n_k\right],
\end{aligned} \tag{5}$$

$\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_{\mathrm{KdV}}$ or $\widetilde{\mathcal{H}}_{\mathrm{Hie}}$, $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_N)^{\mathrm{T}}$, $\boldsymbol{\omega} = (\omega_1,\ldots,\omega_N)^{\mathrm{T}}$, $\boldsymbol{m} = (m_1,\ldots,m_N)^{\mathrm{T}}$, and $\boldsymbol{l} = (l_1,\ldots,l_N)^{\mathrm{T}}$, $z = 1,2,\ldots,N$. δ_{jz} is the Kronecker's symbol, which denotes that

$$\delta_{jz} = \begin{cases} 1, & z = j, \\ 0, & z \neq j. \end{cases}$$

From Eq. (5) we can obtain

$$\widetilde{\mathcal{H}}(l_1,l_2,\ldots,l_j,\ldots,l_N) = \widetilde{\mathcal{H}}(l_1,l_2,\ldots,l_j-2,\ldots,l_N), z = j.$$

By inductive analysis, we can get

$$\begin{aligned}
&\widetilde{\mathcal{H}}\underbrace{(0,0,\ldots,0)}_{N} \\
&= \widetilde{\mathcal{H}}(1,0,\ldots,0) = \widetilde{\mathcal{H}}(0,1,\ldots,0) = \widetilde{\mathcal{H}}(0,0,1,\ldots,0) \\
&= \cdots = \widetilde{\mathcal{H}}(1,1,\ldots,1) = 0.
\end{aligned} \tag{6}$$

From Eq. (6), when $n = N$ we have has 2^N equations. The wave numbers $(\alpha_1,\alpha_2,\ldots,\alpha_N)$ and the amplitude parameters $(\tau_{11},\tau_{22},\ldots,\tau_{NN})$ of the Riemann theta function of genus N are taken as arbitrarily given parameters, then the number of unknown parameters $(\omega_1,\omega_2,\ldots,\omega_N)$, τ_{ij} $(1\le i,j\le N, i\neq j)$, and C is $(N^2+N+2)/2$. So when $N\ge 3$, equation (6) is an over determined system.

2.2. The three-periodic wave solutions of KdV-type equations

Based on Eq. (3), the three-periodic solution of the KdV equation or the Hietarinta equation is

$$u_3 = 2(\ln\vartheta_3)_{xx}, \tag{7}$$

with

$$\vartheta_3 = \vartheta_3(\xi_1,\xi_2,\xi_3,\boldsymbol{\tau}) = \sum_{\boldsymbol{n}\in Z^3} \mathrm{e}^{-\pi\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle+2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{n}\rangle}, \tag{8}$$

where $\boldsymbol{n} = (n_1,n_2,n_3)^{\mathrm{T}} \in Z^3$, $\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3)^{\mathrm{T}} \in C^3$, $\xi_i = \alpha_i x + \omega_i t + \xi_i^0$, $(i = 1,2,3)$,

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}, \quad \tau_{11} > 0, \quad \tau_{22} > 0,$$
$$\tau_{33} > 0, \quad \tau_{21} = \tau_{12}, \quad \tau_{31} = \tau_{13}, \quad \tau_{23} = \tau_{32}, \quad |\tau| > 0.$$

$\boldsymbol{\tau}$ is a positive-definite real valued symmetric 3×3 matrix, which we name the periodic matrix of the Riemann theta function. The three-periodic solution exhibits the interaction of three periodic waves. τ_{ii} represents the amplitude of the i-th wave, and τ_{ij} $(i \neq j)$ represents the phase shift of the interaction between the i-th wave and the j-th wave. Based on Eq. (6), we can obtain 8 equations,

$$\begin{aligned}\widetilde{\mathcal{H}}(0,0,0) &= \widetilde{\mathcal{H}}(1,0,0) = \widetilde{\mathcal{H}}(0,1,0) = \widetilde{\mathcal{H}}(0,0,1) = \widetilde{\mathcal{H}}(0,1,1)\\ &= \widetilde{\mathcal{H}}(1,0,1) = \widetilde{\mathcal{H}}(1,1,0) = \widetilde{\mathcal{H}}(1,1,1) = 0,\end{aligned}$$

where $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_{\mathrm{KdV}}$ or $\widetilde{\mathcal{H}}_{\mathrm{Hie}}$. Next we will use perturbation method to obtain the unknown parameters of the three-periodic solution of the KdV equation or the Hietarinta equation.

3. KdV equation

3.1. The three-periodic solution of KdV equation

In order to make the Riemann ϑ_3 function (8) satisfy the bilinear Eq. (1), that is,

$$\begin{aligned}&\mathcal{H}_{\mathrm{KdV}}(D_x,D_t)\vartheta_3\cdot\vartheta_3\\ &= \sum_{\boldsymbol{n}\in Z^3}\sum_{\boldsymbol{m}\in Z^3} \mathcal{H}_{\mathrm{KdV}}(D_x,D_t)\,\mathrm{e}^{-\pi\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle+2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{n}\rangle}\\ &\quad\times\mathrm{e}^{-\pi\langle\boldsymbol{\tau}\boldsymbol{m},\boldsymbol{m}\rangle+2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{m}\rangle} = 0,\end{aligned}$$

the following 8 equations are obtained

$$\sum_{\boldsymbol{n}\in Z^3} \mathcal{H}_{\mathrm{KdV}}[2\pi\mathrm{i}\langle 2\boldsymbol{n}-\boldsymbol{s}^j,\boldsymbol{\alpha}\rangle, 2\pi\mathrm{i}\langle 2\boldsymbol{n}-\boldsymbol{s}^j,\boldsymbol{\omega}\rangle]\varepsilon_j(\boldsymbol{n}) = 0, \tag{9}$$

where $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\alpha_3)^{\mathrm{T}}$, $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)^{\mathrm{T}}$, $\boldsymbol{s}^1 = (0,0,0)^{\mathrm{T}}$, $\boldsymbol{s}^2 = (1,0,0)^{\mathrm{T}}$, $\boldsymbol{s}^3 = (0,1,0)^{\mathrm{T}}$, $\boldsymbol{s}^4 = (0,0,1)^{\mathrm{T}}$, $\boldsymbol{s}^5 = (1,1,0)^{\mathrm{T}}$, $\boldsymbol{s}^6 = (0,1,1)^{\mathrm{T}}$, $\boldsymbol{s}^7 = (1,0,1)^{\mathrm{T}}$, $\boldsymbol{s}^8 = (1,1,1)^{\mathrm{T}}$, $\varepsilon_j(\boldsymbol{n}) = \mathrm{e}^{-\pi[\langle(\boldsymbol{s}^j-\boldsymbol{n})\boldsymbol{\tau},\boldsymbol{s}^j-\boldsymbol{n}\rangle+\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle]}$, $\boldsymbol{s}^j = (s_1^j,s_2^j,s_3^j)^{\mathrm{T}}$, $j = 1,2,3,\ldots,8$. Suppose $\lambda_1 = \mathrm{e}^{-\pi\tau_{11}}$, $\lambda_2 = \mathrm{e}^{-\pi\tau_{22}}$, $\lambda_3 = \mathrm{e}^{-\pi\tau_{33}}$, $\lambda_4 = \mathrm{e}^{-2\pi\tau_{12}}$, $\lambda_5 = \mathrm{e}^{-2\pi\tau_{13}}$, and $\lambda_6 = \mathrm{e}^{-2\pi\tau_{23}}$, then

$$\begin{aligned}\varepsilon_j(\boldsymbol{n}) &= \lambda_1^{n_1^2+(n_1-s_1^j)^2}\lambda_2^{n_2^2+(n_2-s_2^j)^2}\lambda_3^{n_3^2+(n_3-s_3^j)^2}\\ &\quad\times\lambda_4^{n_1n_2+(n_1-s_1^j)(n_2-s_2^j)}\lambda_5^{n_1n_3+(n_1-s_1^j)(n_3-s_3^j)}\\ &\quad\times\lambda_6^{n_2n_3+(n_2-s_2^j)(n_3-s_3^j)}.\end{aligned} \tag{10}$$

λ_1, λ_2, and λ_3, can be regarded as small parameters, and then we denote ω_i, C as the series of λ_1, λ_2, and λ_3,

$$\begin{aligned}\omega_1 &= \sum_{i,j,k\geq 0}\omega_{1,i,j,k}\lambda_1^i\lambda_2^j\lambda_3^k,\\ \omega_2 &= \sum_{i,j,k\geq 0}\omega_{2,i,j,k}\lambda_1^i\lambda_2^j\lambda_3^k,\\ \omega_3 &= \sum_{i,j,k\geq 0}\omega_{3,i,j,k}\lambda_1^i\lambda_2^j\lambda_3^k,\\ C &= \sum_{i,j,k\geq 0}C_{i,j,k}\lambda_1^i\lambda_2^j\lambda_3^k.\end{aligned} \tag{11}$$

By substitute Eqs. (10) and (11) into Eq. (9), and the nonlinear equations can be transformed into linear equations. Then we collect all the coefficients of $\lambda_1^i\lambda_2^j\lambda_3^k$ in the linear equations and set the coefficients to zero one by one, then we can obtain some coefficient equations about $\omega_{a,i,j,k}$, $C_{i,j,k}$, λ_4, λ_5, and λ_6, $(a = 1,2,3)$, $(i,j,k = 0,1,2,\ldots)$. By solving these coefficient equations, we can obtain the following results:

$$\begin{aligned}&\omega_i = 4\pi^2\alpha_i{}^3 - 96\pi^2\alpha_i{}^3\lambda_i{}^2 + \cdots, \quad (i = 1,2,3),\\ &C = 384\pi^4\left(\lambda_1{}^2\alpha_1{}^4 + \lambda_2{}^2\alpha_2{}^4 + \lambda_3{}^2\alpha_3{}^4\right) + \cdots,\\ &\lambda_{i+k+1} = \frac{\alpha_i{}^2 - 2\,\alpha_i\alpha_k + \alpha_k{}^2}{\alpha_i{}^2 + 2\,\alpha_i\alpha_k + \alpha_k{}^2}, \quad (1\leq i,k\leq 3, i\neq j).\end{aligned} \tag{12}$$

Substituting Eq. (12) into Eq. (7), the three-periodic solution of the KdV equation is obtained.

3.2. The asymptotic soliton of the periodic solution

When $C = 0$ in Eq. (1), the three-soliton solution of KdV equation can be obtained[28]

$$U = 2(\ln f)_{xx} \tag{13}$$

with

$$\begin{aligned}f &= 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_3} + A_{12}\,\mathrm{e}^{\eta_1+\eta_2} + A_{23}\,\mathrm{e}^{\eta_2+\eta_3}\\ &\quad + A_{13}\,\mathrm{e}^{\eta_1+\eta_3} + A_{123}\,\mathrm{e}^{\eta_1+\eta_2+\eta_3},\end{aligned} \tag{14}$$

where $\eta_i = \kappa_i x + \mu_i t + \eta_i^0$, $\mu_i = -\kappa_i^3$ $(i = 1,2,3)$, $A_{jk} = \frac{\kappa_j{}^2 - 2\,\kappa_j\kappa_k + \kappa_k{}^2}{\kappa_j{}^2 + 2\,\kappa_j\kappa_k + \kappa_k{}^2}$ $(1\leq i,j\leq 3, i\neq j)$, $A_{123} = A_{12}A_{13}A_{23}$. Now we will prove that the three-periodic solution of the KdV equation can be reduced to its three-soliton solution when $\lambda_1 = \mathrm{e}^{-\pi\tau_{11}}$, $\lambda_2 = \mathrm{e}^{-\pi\tau_{22}}$, $\lambda_3 = \mathrm{e}^{-\pi\tau_{33}} \to 0$. We rewrite Riemann theta function ϑ_3 (8),

$$\begin{aligned}\vartheta_3 &= 1 + (\mathrm{e}^{2\pi\mathrm{i}\xi_1} + \mathrm{e}^{-2\pi\mathrm{i}\xi_1})\mathrm{e}^{-\pi\tau_{11}} + (\mathrm{e}^{2\pi\mathrm{i}\xi_2} + \mathrm{e}^{-2\pi\mathrm{i}\xi_2})\\ &\quad\times\mathrm{e}^{-\pi\tau_{22}} + (\mathrm{e}^{-2\pi\mathrm{i}\xi_3} + \mathrm{e}^{2\pi\mathrm{i}\xi_3})\mathrm{e}^{-\pi\tau_{33}}\\ &\quad + (\mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_2)} + \mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_2)})\mathrm{e}^{-\pi(\tau_{11}+\tau_{22}+2\tau_{12})}\\ &\quad + (\mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_3)} + \mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_3)})\mathrm{e}^{-\pi(\tau_{11}+\tau_{33}+2\tau_{13})}\\ &\quad + (\mathrm{e}^{-2\pi\mathrm{i}(\xi_2+\xi_3)} + \mathrm{e}^{2\pi\mathrm{i}(\xi_2+\xi_3)})\mathrm{e}^{-\pi(\tau_{22}+\tau_{33}+2\tau_{23})}\\ &\quad + (\mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_2+\xi_3)} + \mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_2+\xi_3)})\\ &\quad\times\mathrm{e}^{-\pi(\tau_{11}+\tau_{22}+\tau_{33}+2\tau_{12}+2\tau_{13}+2\tau_{23})} + \cdots.\end{aligned} \tag{15}$$

If we denote

$$\begin{aligned}
\beta_i &= 2\pi \mathrm{i}\alpha_i, \quad \gamma_i = 2\pi \mathrm{i}\omega_i = -\beta_i^3, \quad \theta_i^0 = 2\pi \mathrm{i}\xi_i^0 - \pi\tau_{ii}, \\
a_{ij} &= \lambda_{i+j+1} = \frac{\alpha_i{}^2 - 2\,\alpha_i\alpha_k + \alpha_k{}^2}{\alpha_i{}^2 + 2\,\alpha_i\alpha_k + \alpha_k{}^2} \\
&= \frac{\beta_i{}^2 - 2\,\beta_i\beta_k + \beta_k{}^2}{\beta_i{}^2 + 2\,\beta_i\beta_k + \beta_k{}^2}, \\
\tau_{12} &= -\frac{\ln a_{12}}{2\pi}, \quad \tau_{13} = -\frac{\ln a_{13}}{2\pi}, \\
\tau_{23} &= -\frac{\ln a_{23}}{2\pi}, \quad a_{123} = a_{12}a_{13}a_{23}, \quad (1 \le i < j \le 3).
\end{aligned}$$

Suppose $\theta_i = \beta_i x + \gamma_i t + \theta_i^0$ and $C = 0$ then equation (15) changes into

$$\begin{aligned}
\vartheta(\xi,\tau) &= 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + a_{12}(\mathrm{e}^{\theta_1+\theta_2}) \\
&\quad + a_{13}(\mathrm{e}^{\theta_1+\theta_3}) + a_{23}(\mathrm{e}^{\theta_2+\theta_3}) \\
&\quad + a_{123}(\mathrm{e}^{\theta_1+\theta_2+\theta_3}) \qquad (16)
\end{aligned}$$

at $\lambda_1 = \mathrm{e}^{-\pi\tau_{11}}$, $\lambda_2 = \mathrm{e}^{-\pi\tau_{22}}$, $\lambda_3 = \mathrm{e}^{-\pi\tau_{33}} \to 0$. Equations (16) and (14) are identical, so we can conclude that the three-periodic solution (7) of KdV equation can be reduced to its three-soliton solution.

We choose appropriate parameters to ensure that the three waves of the three-periodic solution have the same velocity direction. We find that during the interaction, the asymptotic three-soliton solution presents the same repulsive phenomenon as the three-periodic solution. The interactions of the three-periodic solution and its asymptotic three-soliton soulition are shown in Figs. 1(a) and 1(b) respectively. As can be seen from Fig. 1, When the three branches of the three-periodic solution approach to each other, the tall branch jumps forward and the other low branches jump back, forming a repulsion phenomenon. The three branches of the asymptotic three-soliton solution also present repulsion phenomenon during the interaction.

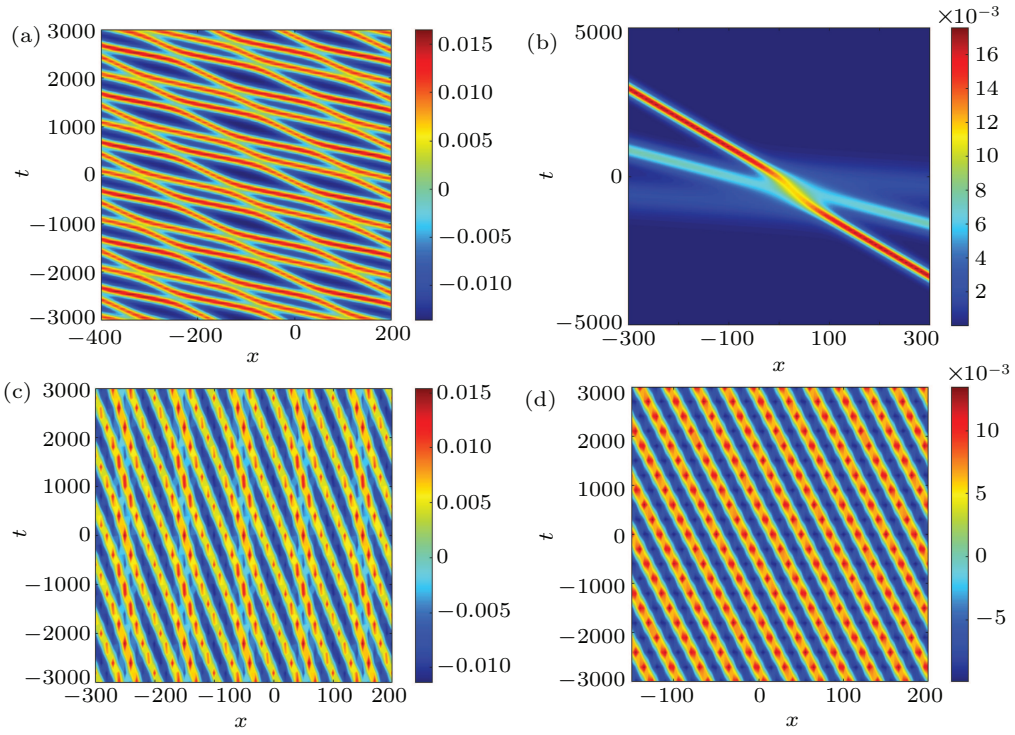


Fig. 1. Choosing $\alpha_1 = 0.01$, $\alpha_2 = 0.02$, $\alpha_3 = 0.03$, $C = 0.1$, and $\xi_1^0 = \xi_2^0 = \xi_3^0 = 0$. (a) The three-periodic soulition with parameters $\tau_{1,1} = 0.6$, $\tau_{2,2} = 0.7$, $\tau_{3,3} = 0.8$. (b) The three-periodic solution in panel (a) degenerates into the three-soliton solution with $\tau_{ii} \to +\infty\,(i = 1,2,3)$. (c) The three-periodic soulition of the KdV equation with parameters $\tau_{1,1} = 0.6$, $\tau_{2,2} = 17$, $\tau_{3,3} = 0.8$. (d) The three-periodic soulition of the KdV equation with parameters $\tau_{1,1} = 16$, $\tau_{2,2} = 17$, $\tau_{3,3} = 0.8$.

The three-periodic solution degenerates to the two-periodic solution when any one of τ_{11}, τ_{22}, and τ_{33} is large enough, and the one-periodic solution when any two of them are large enough. Next, we give two examples. If τ_{22} is large enough, that is $\lambda_2 \ll \lambda_1, \lambda_3$, then equation (15) has the following form:

$$\begin{aligned}
\vartheta_3 &\approx 1 + (\mathrm{e}^{2\pi\mathrm{i}\xi_1} + \mathrm{e}^{-2\pi\mathrm{i}\xi_1})\lambda_1 + (\mathrm{e}^{-2\pi\mathrm{i}\xi_3} + \mathrm{e}^{2\pi\mathrm{i}\xi_3})\lambda_3 \\
&\quad + (\mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_3)} + \mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_3)})\lambda_1\lambda_3\lambda_5 + \cdots . \qquad (17)
\end{aligned}$$

So equation (17) is the Riemann theta function of genus $N = 2$, and the approximate two-periodic solution of KdV equation can be obtained by Eqs. (7) and (17) (see Fig. 1(c)). When τ_{11} and τ_{33} are large enough, that is $\lambda_1, \lambda_3 \ll \lambda_2$,

$$\vartheta_3 \approx 1 + (\mathrm{e}^{-2\pi\mathrm{i}\xi_2} + \mathrm{e}^{2\pi\mathrm{i}\xi_2})\lambda_2 + \cdots . \qquad (18)$$

Similarly, we can obtain an approximate one-periodic solution of the KdV equation from Eqs. (7) and (18) (see Fig.1(d)).

4. Hietarinta equation

4.1. The three-periodic solution of Hietarinta equation

In order to make the Riemann ϑ_3 function (8) satisfy the bilinear equation (2), that is,

$$\begin{aligned}&\mathcal{H}_{\rm Hie}(D_x,D_t)\vartheta_3\cdot\vartheta_3\\ &=\sum_{\boldsymbol{n}\in Z^3}\sum_{\boldsymbol{m}\in Z^3}\mathcal{H}_{\rm Hie}(D_x,D_t)\,{\rm e}^{-\pi\langle\boldsymbol{\tau n},\boldsymbol{n}\rangle+2\pi{\rm i}\langle\boldsymbol{\xi},\boldsymbol{n}\rangle}\\ &\times{\rm e}^{-\pi\langle\boldsymbol{\tau m},\boldsymbol{m}\rangle+2\pi{\rm i}\langle\boldsymbol{\xi},\boldsymbol{m}\rangle}=0,\end{aligned}$$

the following 8 equations are obtained

$$\sum_{\boldsymbol{n}\in Z^3}\mathcal{H}_{\rm Hie}[2\pi{\rm i}\langle 2\boldsymbol{n}-\boldsymbol{P}^j,\boldsymbol{\alpha}\rangle,2\pi{\rm i}\langle 2\boldsymbol{n}-\boldsymbol{P}^j,\boldsymbol{\omega}\rangle]\varepsilon_j(\boldsymbol{n})=0,\tag{19}$$

where $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)^{\rm T}$, $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)^{\rm T}$ $\boldsymbol{P}^1=(0,0,0)^{\rm T}$, $\boldsymbol{P}^2=(1,0,0)^{\rm T}$, $\boldsymbol{P}^3=(0,1,0)^{\rm T}$, $\boldsymbol{P}^4=(0,0,1)^{\rm T}$, $\boldsymbol{P}^5=(1,1,0)^{\rm T}$, $\boldsymbol{P}^6=(0,1,1)^{\rm T}$, $\boldsymbol{P}^7=(1,0,1)^{\rm T}$, $\boldsymbol{P}^8=(1,1,1)^{\rm T}$, $\varepsilon_j(\boldsymbol{n})={\rm e}^{-\pi[\langle(\boldsymbol{P}^j-\boldsymbol{n})\boldsymbol{\tau},\boldsymbol{P}^j-\boldsymbol{n}\rangle+\langle\boldsymbol{\tau n},\boldsymbol{n}\rangle]}$, $\boldsymbol{P}^j=(P_1^j,P_2^j,P_3^j)^{\rm T}$, $(j=1,2,3,\ldots,8)$. We use the same way as the method to get the solution of KdV equation to obtain the unknown parameters ω_1, ω_2, ω_3, C, τ_{12}, τ_{23}, τ_{13}, of the three-periodic solution of Hietarinta equation by using the perturbation method. We assume

$$\begin{aligned}A_i&=\sqrt{-\pi^2{\alpha_i}^2+b},\\ T_{ij}&=\pi^2{\alpha_i}^2+2\pi^2{\alpha_j}^2-\frac{9}{4}C_2,\\ \alpha_{ij}&=\left(\pi^2{\alpha_i}^2-\frac{3}{4}b\right)\left(\pi^2{\alpha_j}^2-\frac{b}{4}\right)\left(\pi^2{\alpha_j}^2-b\right),\\ K_{ij}&=\pi^4{\alpha_i}^4{\alpha_j}^2-\frac{5}{4}\left(\pi^2{\alpha_j}^2-\frac{b}{20}\right)b{\alpha_i}^2+\frac{3}{16}b^2{\alpha_j}^2,\\ B_{ij}&=\pi^2\left(\pi^2{\alpha_j}^2-\frac{b}{2}\right){\alpha_i}^4\\ &\quad+\frac{1}{4}b\left(\pi^2{\alpha_j}^2+\frac{3}{4}b\right){\alpha_i}^2-\frac{3}{16}b^2{\alpha_j}^2,\\ R_{ij}&=\left(\pi^4{\alpha_j}^4-\frac{1}{2}\pi^2b{\alpha_j}^2-\frac{1}{8}b^2\right){\alpha_i}^4\\ &\quad-\frac{3}{4}b\left(\pi^2{\alpha_j}^2-\frac{7b}{12}\right){\alpha_j}^2{\alpha_i}^2-\frac{1}{16}b^2{\alpha_j}^4,\\ Q_{ij}&=\frac{1}{2}\left(\pi^2{\alpha_j}^2+\frac{b}{4}\right)\pi^2{\alpha_i}^4\\ &\quad+\left(\pi^4{\alpha_j}^4-\frac{9\pi^2b{\alpha_j}^2}{8}-\frac{3b^2}{32}\right){\alpha_i}^2\\ &\quad-\frac{1}{2}\left(\pi^2{\alpha_j}^2-\frac{3}{4}b\right)b{\alpha_j}^2,\\ L_{ij}&=\pi^2\left(\pi^4{\alpha_i}^4-\pi^2b{\alpha_i}^2+\frac{1}{8}b^2\right){\alpha_j}^4\\ &\quad-\frac{1}{4}\left(\pi^4{\alpha_i}^4-\frac{9}{4}\pi^2b{\alpha_i}^2+\frac{3}{4}b^2\right)b{\alpha_j}^2\\ &\quad-\frac{7b^2{\alpha_i}^2\left(\pi^2{\alpha_i}^2-\frac{3}{7}b\right)}{16},\end{aligned}\tag{20}$$

and then the unknown parameters are expressed as follows:

$$\lambda_{i+j+1}=-\frac{(\alpha_i-\alpha_j)\left(\pi\left(\alpha_i-\alpha_j\right)\left(-2\pi^2\alpha_i\alpha_j-2A_iA_j+b\right)\sqrt{3}-2\left(\alpha_i+\alpha_j\right)\left(-A_2\alpha_i+A_i\alpha_j\right)\pi^2+3b\left(A_i-A_j\right)\right)}{\left(\pi\left(\alpha_i+\alpha_j\right)\left(2\pi^2\alpha_i\alpha_j+2A_iA_j+b\right)\sqrt{3}+2\left(\alpha_i-\alpha_j\right)\left(A_i\alpha_j-A_j\alpha_i\right)\pi^2+3b\left(A_i+A_j\right)\right)\left(\alpha_i+\alpha_j\right)},\tag{21}$$

$$\begin{aligned}\omega_1=&\frac{-\pi\sqrt{3}\alpha_1-A_1}{2\pi}+\frac{6b\left(\pi\sqrt{3}\alpha_1+A_1\right){\lambda_1}^2}{\pi\left(\pi^2{\alpha_1}^2-\pi\sqrt{3}A_1\alpha_1-b\right)}\\ &+\frac{24\left(\pi^2{\alpha_2}^2-b/4\right)\alpha_2\left(\left(B_{21}\sqrt{3}\alpha_1A_1+3R_{21}\pi\right)A_2+\alpha_2\left(-3K_{21}\alpha_1\pi A_1+L_{21}\sqrt{3}\right)\right){\lambda_2}^2}{\left(\alpha_1-\alpha_2\right)\alpha_1\left(-\alpha_2\left(-T_{21}\alpha_1\pi A_1\left(\pi^2{\alpha_1}^2-b/4\right)+\sqrt{3}\alpha_{21}\right)A_2+\alpha_{12}\sqrt{3}\alpha_1A_1+Q_{21}\pi\left(\pi^2{\alpha_1}^2-b\right)\right)\pi\left(\alpha_1+\alpha_2\right)}\\ &+\frac{24\left(\left(B_{31}\sqrt{3}\alpha_1A_1+3R_{31}\pi\right)A_3+\alpha_3\left(-3K_{31}\alpha_1\pi A_1+L_{31}\sqrt{3}\right)\right)\left(\pi^2{\alpha_3}^2-b/4\right)\alpha_3{\lambda_3}^2}{\left(-\left(-T_{31}\alpha_1\pi A_1\left(\pi^2{\alpha_1}^2-b/4\right)+\sqrt{3}\alpha_{31}\right)\alpha_3A_3+\alpha_{13}\sqrt{3}\alpha_1A_1+Q_{31}\pi\left(\pi^2{\alpha_1}^2-b\right)\right)\left(\alpha_1-\alpha_3\right)\alpha_1\pi\left(\alpha_1+\alpha_3\right)}+\cdots,\end{aligned}\tag{22}$$

$$\begin{aligned}\omega_2=&\frac{-\pi\sqrt{3}\alpha_2-A_2}{2\pi}+\frac{6b\left(\pi\sqrt{3}\alpha_2+A_2\right){\lambda_2}^2}{\pi\left(\pi^2{\alpha_2}^2-\pi\sqrt{3}A_2\alpha_2-b\right)}\\ &-\frac{24\left(\left(\alpha_2B_{12}\sqrt{3}A_2+3R_{12}\pi\right)A_1+\left(-3\alpha_2K_{12}\pi A_2+\sqrt{3}L_{12}\right)\alpha_1\right)\alpha_1\left(\pi^2{\alpha_1}^2-b/4\right){\lambda_1}^2}{\alpha_2\left(\alpha_1-\alpha_2\right)\pi\left(\alpha_1+\alpha_2\right)\left(\left(\left(\pi^2{\alpha_2}^2-b/4\right)\alpha_2\pi T_{12}A_2-\alpha_{12}\sqrt{3}\right)\alpha_1A_1+\alpha_2\sqrt{3}\alpha_{21}A_2+2Q_{12}\left(\pi^2{\alpha_2}^2-b\right)\pi\right)}\\ &+\frac{24\left(\pi^2{\alpha_3}^2-b/4\right)\alpha_3\left(\left(B_{32}\sqrt{3}\alpha_2A_2+3R_{32}\pi\right)A_3+\left(-3\alpha_2\pi K_{32}A_2+L_{32}\sqrt{3}\right)\alpha_3\right){\lambda_3}^2}{\left(\alpha_3\left(\alpha_2\pi T_{32}A_2\left(\pi^2{\alpha_2}^2-b/4\right)-\alpha_{32}\sqrt{3}\right)A_3+\alpha_{23}\sqrt{3}\alpha_2A_2+Q_{32}\pi\left(\pi^2{\alpha_2}^2-b\right)\right)\left(\alpha_2-\alpha_3\right)\alpha_2\pi\left(\alpha_2+\alpha_3\right)}+\cdots,\end{aligned}\tag{23}$$

$$\omega_3=\frac{-\pi\sqrt{3}\alpha_3-A_3}{2\pi}+\frac{6b\left(\pi\sqrt{3}\alpha_3+A_3\right){\lambda_3}^2}{\pi\left(\pi^2{\alpha_3}^2-\pi\sqrt{3}A_3\alpha_3-b\right)}$$

$$-\frac{24\left(\left(\alpha_3 B_{13}\sqrt{3}A_3+3R_{13}\pi\right)A_1+\alpha_1\left(-3K_{13}\alpha_3\pi A_3+\sqrt{3}L_{13}\right)\right)\alpha_1\left(\pi^2{\alpha_1}^2-b/4\right){\lambda_1}^2}{\alpha_3\left(\left(\left(\pi^2{\alpha_3}^2-b/4\right)T_{13}\alpha_3\pi A_3-\alpha_{13}\sqrt{3}\right)\alpha_1 A_1+\sqrt{3}\alpha_{31}\alpha_3 A_3+2Q_{13}\left(\pi^2{\alpha_3}^2-b\right)\pi\right)\left(\alpha_1-\alpha_3\right)\pi\left(\alpha_1+\alpha_3\right)}$$
$$-\frac{24\left(\left(\alpha_3 B_{23}\sqrt{3}A_3+3R_{23}\pi\right)A_2+\alpha_2\left(-3\alpha_3 K_{23}\pi A_3+\sqrt{3}L_{23}\right)\right)\alpha_2\left(\pi^2{\alpha_2}^2-b/4\right){\lambda_2}^2}{\alpha_3\left(\alpha_2-\alpha_3\right)\left(\alpha_2\left(\left(\pi^2{\alpha_3}^2-b/4\right)\alpha_3 T_{23}\pi A_3-\alpha_{23}\sqrt{3}\right)A_2+\alpha_3\sqrt{3}\alpha_{32}A_3+2\left(\pi^2{\alpha_3}^2-b\right)Q_{23}\pi\right)\pi\left(\alpha_2+\alpha_3\right)}+\cdots, \tag{24}$$

$$\begin{aligned}C=&\,48\left(\pi\sqrt{3}{\alpha_1}^2+A_1\alpha_1\right)\pi b{\lambda_1}^2\\&+48\left(\pi\sqrt{3}{\alpha_2}^2+A_2\alpha_2\right)\pi b{\lambda_2}^2\\&+48\left(\pi\sqrt{3}{\alpha_3}^2+A_3\alpha_3\right)\pi b{\lambda_3}^2+\cdots.\end{aligned}$$

Equations (22)–(24) show the algebraic relationships among the free constant b, frequencies and wave numbers. It can be seen from the expression that the frequency ω_j not only is related to the j-th wave number but also to the other two wave numbers, in other words, the dispersion relationship of the waves has changed. By substituting Eq. (24) into Eq. (8), the three-periodic solution of Hietarinta equation is obtained.

The three-periodic solution is shown in Fig. 2, and it has the following two characteristics: (i) The three-periodic solution is symmetric about the x and t axes (see Figs. 2(c) and 2(d)). (ii) It has $2N$ fundamental periods $\{\zeta_i, i=1,2,\ldots,N\}$ and $\{\tau_i, i=1,2,\ldots,N\}$ in (ξ_1,ξ_2,ξ_3) with $\zeta_1=(1,0,\ldots,0)^{\mathrm{T}}$, $\ldots$, $\zeta_N=(0,0,\ldots,1)^{\mathrm{T}}$.

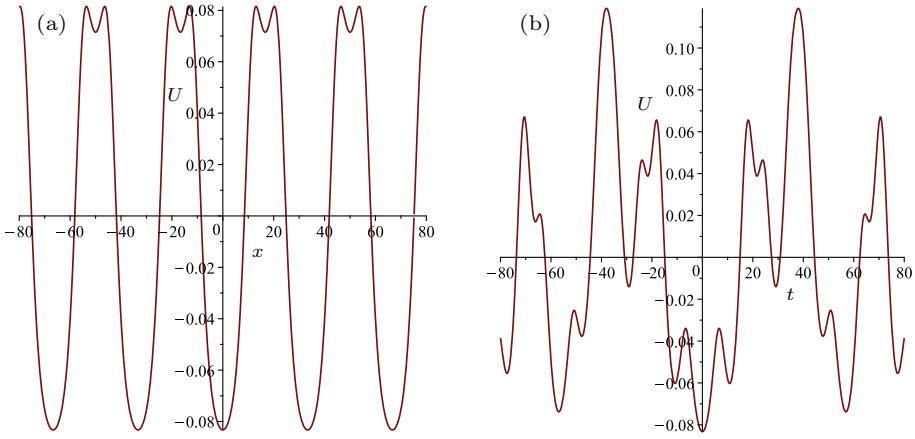


Fig. 2. The three-periodic solution of the Hietarinta equation with parameters $\alpha_1=0.03$, $\alpha_2=0.06$, $\alpha_3=0.09$, $\tau_{1,1}=0.85$, $\tau_{2,2}=1.05$, $\tau_{3,3}=1.25$, $\lambda_1=0.06922773125$, $\lambda_2=0.03693217019$, $\lambda_3=0.01970287297$, $C=0.1$, and $\xi_1^0=\xi_2^0=\xi_3^0=0$. (a) The three-periodic solution of the Hietarinta equation at $t=0$. (b) The three-periodic solution of the Hietarinta equation at $x=0$.

4.2. The asymptotic soliton of the periodic solution

When $C=0$ in the bilinear equation (2), the three soliton solution of Hietarinta equation can be obtained by using Hirota bilinear method

$$U_{\mathrm{H}}=2(\ln f)_{xx}, \tag{25}$$

with

$$\begin{aligned}f=&\,1+\mathrm{e}^{\psi_1}+\mathrm{e}^{\psi_2}+\mathrm{e}^{\psi_3}+m_{12}\mathrm{e}^{\psi_1+\psi_2}\\&+m_{23}\mathrm{e}^{\psi_2+\psi_3}+m_{13}\mathrm{e}^{\psi_1+\psi_3}+m_{123}\mathrm{e}^{\psi_1+\psi_2+\psi_3},\end{aligned} \tag{26}$$

where $\psi_i=\iota_i x+\nu_i t+\psi_i^0$, $\nu_i=\sqrt{3}\iota_i/2+\sqrt{-\iota_i^2-4b}/2$ and

$$m_{ij}=-\frac{(\iota_i-\iota_j)}{(\iota_i+\iota_j)}\times\frac{\left(\left(\sqrt{3}(\iota_i-\iota_j)\sqrt{-\iota_j^2-4b}+\iota_i\iota_j+{\iota_j}^2+6b\right)\sqrt{-\iota_i^2-4b}-\left({\iota_i}^2+\iota_i\iota_j+6b\right)\sqrt{-\iota_j^2-4b}+\sqrt{3}(\iota_i-\iota_j)(\iota_i\iota_j+2b)\right)}{\left(\left(-\sqrt{3}\left(\iota_i+\iota_j\right)\sqrt{-\iota_j^2-4b}-\iota_i\iota_j+{\iota_j}^2+6b\right)\sqrt{-\iota_i^2-4b}+\left({\iota_i}^2-\iota_i\iota_j+6b\right)\sqrt{-\iota_j^2-4b}+\sqrt{3}(-\iota_i\iota_j+2b)(\iota_i+\iota_j)\right)}$$

$(1\le i,j\le 3)$ and $m_{123}=m_{12}m_{13}m_{23}$. Now we will prove that the three-periodic wave solution of the Hietarinta equation can be reduced to its three-soliton solution at $\lambda_1=\mathrm{e}^{-\pi\tau_{11}}$, $\lambda_2=\mathrm{e}^{-\pi\tau_{22}}$, $\lambda_3=\mathrm{e}^{-\pi\tau_{33}}\to 0$. We rewrite ϑ_3 as

$$\begin{aligned}\vartheta_3=&\,1+(\mathrm{e}^{2\pi\mathrm{i}\xi_1}+\mathrm{e}^{-2\pi\mathrm{i}\xi_1})\lambda_1+(\mathrm{e}^{2\pi\mathrm{i}\xi_2}+\mathrm{e}^{-2\pi\mathrm{i}\xi_2})\lambda_2+(\mathrm{e}^{-2\pi\mathrm{i}\xi_3}+\mathrm{e}^{2\pi\mathrm{i}\xi_3})\lambda_3+(\mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_2)}\\&+\mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_2)})\lambda_1\lambda_2\lambda_4+(\mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_3)}+\mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_3)})\lambda_1\lambda_3\lambda_5+(\mathrm{e}^{-2\pi\mathrm{i}(\xi_2+\xi_3)}+\mathrm{e}^{2\pi\mathrm{i}(\xi_2+\xi_3)})\lambda_2\lambda_3\lambda_6\\&+(\mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_2+\xi_3)}+\mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_2+\xi_3)})\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6+\cdots.\end{aligned} \tag{27}$$

If we denote

$$\iota_i = 2\pi \mathrm{i}\alpha_i, \quad \nu_i = 2\pi \mathrm{i}\omega_i = -\frac{1}{2}\sqrt{-\alpha_1{}^2 - 4b} - \frac{1}{2}\sqrt{3}\alpha_1,$$
$$\theta_i^0 = 2\pi \mathrm{i}\xi_i^0 - \pi\tau_{ii},$$

then

$$\lambda_{i+j+1} = m_{ij}, \quad \tau_{12} = -\frac{\ln m_{12}}{2\pi}, \quad \tau_{13} = -\frac{\ln m_{13}}{2\pi},$$
$$\tau_{23} = -\frac{\ln m_{23}}{2\pi}, \quad m_{123} = m_{12}m_{13}m_{23}, \quad (1 \le i < j \le 3).$$

Suppose $\theta_i = \iota_i x + \nu_i t + \theta_i^0$ and $C = 0$ then equation (27) changes into

$$\vartheta(\xi,\tau) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + m_{12}(\mathrm{e}^{\theta_1+\theta_2}) + m_{13}(\mathrm{e}^{\theta_1+\theta_3}) + m_{23}(\mathrm{e}^{\theta_2+\theta_3}) + m_{123}(\mathrm{e}^{\theta_1+\theta_2+\theta_3}) \tag{28}$$

at $\lambda_1 = \mathrm{e}^{-\pi\tau_{11}}$, $\lambda_2 = \mathrm{e}^{-\pi\tau_{22}}$, $\lambda_3 = \mathrm{e}^{-\pi\tau_{33}} \to 0$. Therefore, the equation (28) shows that the three-periodic solution (27) of Hietarinta equation can be reduced to its three-soliton solution (26).

It can be seen from Fig. 3(a) that the three branches of the three-periodic solution of the Hietrinta equation, and they move in the same direction. The three-periodic solution of Hietrinta equation degenerates into its three-soliton solution as $\lambda_1, \lambda_2, \lambda_3, \to 0$ (see Fig. 3(b)). Similar to the periodic and soliton solutions of the KdV equation, it can be seen from Figs. 3(a) and 3(b) that when the three branches of the three-periodic solution or the three-soltion solution of the Hietrinta equation approach to each other, the high beanch jumps forward and the other low branches jump backward, forming a repulsive phenomenon. As can be seen from Figs. 3(a), 3(c), and 3(d), when any two or one of τ_{11}, τ_{22}, and τ_{33} are large enough, the three-periodic soulition can degenerate to two- or one-periodic soulitions.

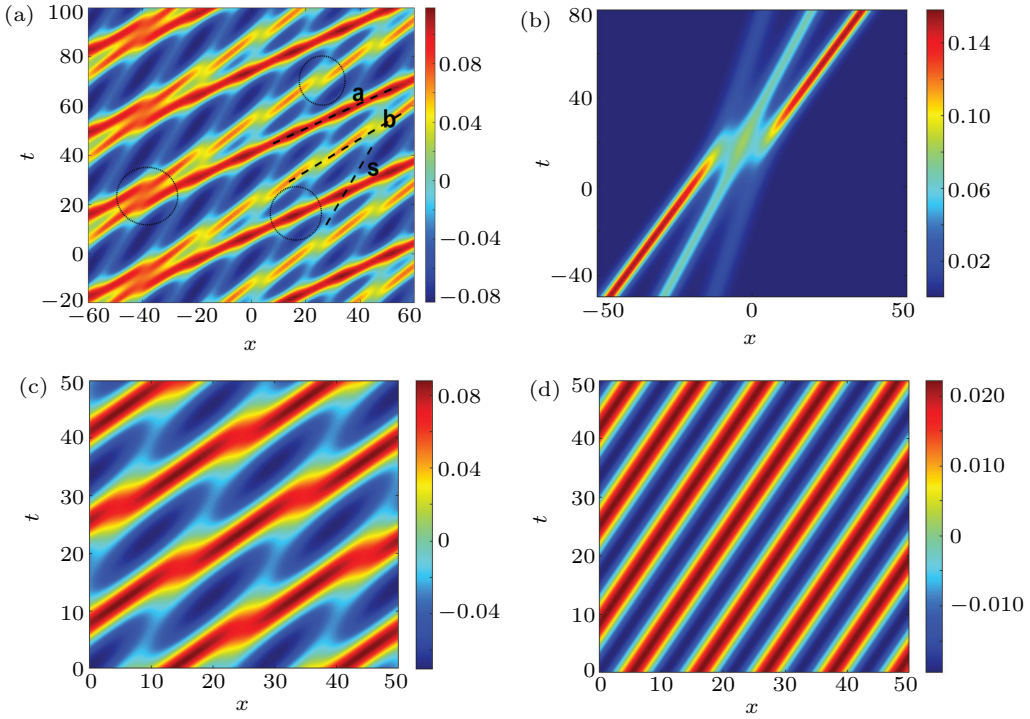


Fig. 3. (a) The three-periodic soulition of the Hietrinta equation with parameters $p_1 = 0.03$, $p_2 = 0.06$, $p_3 = 0.09$, $\tau_{1,1} = 0.85$, $\tau_{2,2} = 1.05$, $\tau_{3,3} = 1.25$, $\lambda_1 = 0.06922773125$, $\lambda_2 = 0.03693217019$, $\lambda_3 = 0.01970287297$, $C = 0.1$, and $\xi_1^0 = \xi_2^0 = \xi_3^0 = 0$. (b) As λ_1, λ_2, $\lambda_3 \to 0$, the three-periodic solution degenerates into the three-soliton. (c) The parameter $\tau_{2,2} = 25$ and other parameters remain the same, the three-periodic solution degenerates to a two-periodic solution. (d) The parameters $\tau_{1,1} = 85$, $\tau_{3,3} = 25$ and other parameters remain the same, the three-periodic solution degenerates to a one-periodic solution.

5. Conclusion

We gave the approximate analytical three-periodic solutions of the KdV equation and the Hietarinta equation for the first time by using the perturbation method. We obtained the algebraic relations among phase shift, wave numbers, frequency, and amplitudes after tedious calculations. By a limit method, that is, $\tau_{1,1}$, $\tau_{2,2}$, $\tau_{3,3} \to +\infty$, the three-periodic solutions of the KdV equation and the Hietarinta equation degenerate into three-soliton solutions. We find that the three-periodic solution presents the same repulsive phenomenon as the asymptotic three-soliton solution during the interaction. Moreover, we find that when one or two of $\tau_{1,1}$, $\tau_{2,2}$, $\tau_{3,3}$ are large enough, the three-periodic solution can degenerate into the one- or two-periodic solution.

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References

[1] Ablowitz M J and Musslimani Z H 2016 *Nonlinearity* **29** 915
[2] Ablowitz M J, Luo X D and Musslimani Z H 2018 *J. Math. Phys.* **59** 011501
[3] Huang L L, Qiao Z J and Chen Y 2018 *Chin. Phys. B* **27** 020201
[4] Park Q H and Shin H J 1999 *Phys. Rev. E* **59** 2373
[5] Wen X Y and Gao Y T 2010 *Commun. Theor. Phys.* **53** 825
[6] Chen M and Wang Z 2020 *Chin. Phys. B* **29** 120201
[7] Zhang Z, Yang X Y and Li B 2020 *Appl. Math. Lett.* **103** 106168
[8] Feng B F and Ling L 2022 *Physica D* **437** 133332
[9] Feng B F, Ling L and Takahashi D A 2020 *Stud. Appl. Math.* **144** 46
[10] Pu J, Li J and Chen Y 2021 *Chin. Phys. B* **30** 060202
[11] Hirota R 1985 *J. Phys. Soc. Jpn.* **54** 2409
[12] Hirota R, Hu X B and Tang X Y 2003 *J. Math. Anal. Appl.* **288** 326
[13] Hirota R 2004 *The Direct Method in Soliton Theory* (Cambridge: Cambridge University Press)
[14] Wazwaz A M 2013 *Ocean Eng.* **60** 95
[15] Wang Y and Chen Y 2012 *J. Math. Phys.* **53** 123504
[16] Novikov S P 1974 *Funct. Anal. Appl.* **8** 236
[17] Dubrovin B A 1975 *Funct. Anal. Appl.* **9** 215
[18] Its A R and Matveev V B 1975 *Funct. Anal. Appl.* **9** 65
[19] Nakamura A 1979 *J. Phys. Soc. Jpn.* **47** 1701
[20] Nakamura A 1980 *J. Phys. Soc. Jpn.* **48** 1365
[21] Fan E G and Hon Y C 2008 *Phys. Rev. E* **78** 036607
[22] Fan E G 2009 *J. Phys. A: Math. Theor.* **42** 095206
[23] Wang X B, Tian S F, Feng L L and Zhang T T 2018 *J. Math. Phys.* **59** 073505
[24] Wang Z, Qin Y P and Zou L 2017 *Chin. Phys. B* **26** 050504
[25] Pu J and Chen Y 2022 *Acta Math. Appl. Sin-E* **38** 861
[26] Wang X X, Sun J Q and Zhang Y N 2021 *Numerical Algorithms* **88** 711
[27] Zhang Y N, Hu X B and Sun J Q 2018 *J. Comput. Phys.* **355** 566
[28] Zhou Y and Ma W X 2017 *J. Math. Phys.* **58** 101511