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Efficient method to calculate the eigenvalues of the Zakharov–Shabat system

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A numerical method is proposed to calculate the eigenvalues of the Zakharov–Shabat system based on Chebyshev polynomials. A mapping in the form of $\tanh(ax)$ is constructed according to the asymptotic of the potential function for the Zakharov–Shabat eigenvalue problem. The mapping can distribute Chebyshev nodes very well considering the gradient for the potential function. Using Chebyshev polynomials, $\tanh(ax)$ mapping, and Chebyshev nodes, the Zakharov–Shabat eigenvalue problem is transformed into a matrix eigenvalue problem. This method has good convergence for the Satsuma–Yajima potential and the convergence rate is faster than the Fourier collocation method. This method is not only suitable for simple potential functions but also converges quickly for a complex Y-shape potential. It can also be further extended to other linear eigenvalue problems.

Keywords: Zakharov–Shabat system, eigenvalue, numerical method, Chebyshev polynomials

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1. Introduction

The nonlinear Schrödinger (NLS) equation is an important integrable equation that is derived from hydrodynamics and has been used to describe the propagation of optical solitons, Langmuir waves in plasma physics, Bose–Einstein condensation, and other physical phenomena.[1–4] Considering the initial value problem for the nonlinear Schrödinger (NLS) equation

$$\begin{cases} \mathrm{i}q_t + q_{xx} + 2\lambda|q|^2 q = 0, \\ q(x,0) = q_0(x), \end{cases} \tag{1}$$

where $q_0(x)$ is a given function defined in Schwartz space $\mathcal{S}(\mathbb{R})$, subscripts x and t represent the partial derivatives with respect to space and time, respectively. When $\lambda = 1$, equation (1) is called a focusing NLS equation, and when $\lambda = -1$, equation (1) is called a defocusing NLS equation. The NLS equation (1) is integrable and admits the following Lax pair:

$$\begin{cases} \psi_x = \begin{pmatrix} -\mathrm{i}k & q \\ -\lambda\bar{q} & \mathrm{i}k \end{pmatrix}\psi, & (2) \\ \psi_t = \begin{pmatrix} -2\mathrm{i}k^2 + \lambda\mathrm{i}|q|^2 & 2kq + \lambda\mathrm{i}q_x \\ -2k\lambda\bar{q} + \lambda\mathrm{i}\bar{q}_x & 2\mathrm{i}k^2 - \lambda\mathrm{i}|q|^2 \end{pmatrix}\psi, & (3) \end{cases}$$

where ψ is a column vector with a shape of 2×1, k is a spectral parameter defined in complex field $\mathbb{C}$, q is the potential function, and $\bar{q}$ represents the complex conjugation of q. A long but standard computation shows that the compatibility condition $\psi_{xt} = \psi_{tx}$ for the eigenfunction ψ is equivalent to the NLS equation $\mathrm{i}q_t + q_{xx} + 2\lambda|q|^2 q = 0$ for classical solutions q.[6] Equation (2) is called the Zakharov–Shabat (ZS) system, which is the x-part of the Lax pair. The properties of the solution for the NLS equation are determined by the eigenvalues for the ZS system.[6]

The inverse scattering transform is an important method for solving integrable equations[5] and was a milestone of mathematical physics in the twentieth century. The inverse scattering transform of the NLS equation was proposed by Zakharov and Shabat.[6] The inverse scattering transform consists of direct scattering and inverse scattering. In direct scattering, we need to solve the ZS system and calculate the scattering data. The ZS system (2) is an eigenvalue problem and we need to find the k so that the ZS system exists as a nontrivial solution, where k represents the eigenvalue and ψ is the corresponding eigenfunction. The eigenvalues for the ZS system consist of the discrete eigenvalues κ_j, $\bar{\kappa}_j$ $(j = 1,2,\ldots,n_1)$ and the continuous eigenvalues, where $\kappa_j \in \mathbb{C}\setminus\mathbb{R}$. The continuous spectrum of the ZS system is the real axis $\mathbb{R}$, so we only need to calculate the discrete eigenvalues of the ZS system. The number of solitons that emerged in $q(x,0)$ for the NLS equation is equal to n_1. It is worth noting that the discrete eigenvalue of the ZS system can be an empty set and the solution will not evolve into solitons at this time. The amplitude of the soliton is determined by the imaginary part of κ_j. The velocity of the soliton is determined by the real part of κ_j.

It is important to develop simple and effective methods to calculate the eigenvalues of the ZS system. In most cases, the eigenvalues of the ZS system (2) cannot be obtained analytically. Therefore, it is difficult to calculate the eigenvalues of the ZS system analytically. Calculating the eigenvalues of the ZS system is an important step in the inverse scatter-

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ing transform. If we cannot efficiently calculate the discrete eigenvalues of the ZS system, then we will not be able to solve the equation by the inverse scattering transform. Meanwhile, calculating the eigenvalues of the ZS equation is of great significance for the study of the evolution of solutions. By analyzing the number and magnitude of discrete eigenvalues, we can obtain the number and properties of solitons, which are closely related to the "soliton resolution conjecture".[7] The numerical implementation of the inverse scattering transform attracted special attention when the NLS equation soliton solutions were proposed as potential candidates for fiber optical transmission. At present, increasing the accuracy and efficiency of computational methods for solving the direct ZS system remains an urgent problem in nonlinear optics.

Several numerical methods have been proposed to calculate the eigenvalues of the ZS system. Boffetta and Osborne developed a numerical algorithm for computing the direct scattering transform for the NLS equation.[8] Bronski considered the semi-classical limit of the ZS eigenvalue problem.[9] The finite difference method was used to compute the ZS eigenvalue problem numerically.[10,12] Hill's method can be used to calculate the eigenvalues of the ZS system.[11] The Fourier collocation method (FCM) is an effective method to calculate the eigenvalues of the ZS system.[19] Vasylchenkova *et al.* summarized several nonlinear Fourier transform (NFT) methods, and compared their quality and performance.[13] These methods can be divided into two types: the first is the iterative method for the zero point of Jost function, and the second is to solve the matrix eigenvalue problem.[14]

In this paper, we proposed an efficient numerical method for calculating the eigenvalues for the ZS system. We use Chebyshev polynomials and $\tanh(ax)$ mapping to extract the key information of the potential function, and then transform the ZS eigenvalue problem into a matrix eigenvalue problem. By solving the matrix eigenvalue problem, we can get the eigenvalues for the ZS system. This method not only has good convergence for the Satsuma–Yajima potential but also converges quickly for complex Y-shape potential. In addition, this method can be further extended to other linear systems.

This paper is structured as follows. In Section 2, the theoretical knowledge of Chebyshev polynomials is presented and our numerical method is presented in detail. In Section 3, our method is used to calculate the eigenvalues of the ZS system with the Satsuma–Yajima potential, the $\operatorname{sech}(2\epsilon x)\,\mathrm{e}^{\mathrm{i}\operatorname{sech}(2\epsilon x)/\epsilon}$ potential, and the $\exp(-\mathrm{i}x)\operatorname{sech}(x)$ potential. The convergence of our method is analyzed. Our method has spectral accuracy and its convergence rate is fast. Finally, some discussions are given in Section 4.

2. Methodology

The details of our method will be introduced in this section. Our method is summarized in the following steps. For the ZS system (2), Chebyshev polynomials are used to approximate the eigenfunction ψ and the potential function q with the help of mapping $H(x)=\tanh(ax)$ $(a>0)$. Using Chebyshev nodes, we turn the ZS eigenvalue problem into a matrix eigenvalue problem. We can then obtain the eigenvalues of the ZS system by calculating the matrix eigenvalue problem.

We define the n Chebyshev nodes by

$$\boldsymbol{\chi}=\left(-1,\ \cos\left(\frac{n-2}{n-1}\pi\right),\ \ldots\ \cos\left(\frac{1}{n-1}\pi\right),1\right)^{\mathrm{T}}.$$

For the given function $f(x)$ defined in unit interval $\mathbb{I}$, we can approximate $f(x)$ by

$$f(x)=T(x)\mathcal{F}f(\boldsymbol{\chi}), \tag{4}$$

where

$$T(x)=[T_0(x),\ldots,T_{n-1}(x)],$$
$$f(\boldsymbol{\chi})=\left(f\left(-1,\cos\left(\frac{n-2}{n-1}\pi\right)\right),\ldots,f(1)\right)^{\mathrm{T}},$$

$\mathcal{F}$ is obtained by an appropriately scaled discrete cosine transform of $f(\boldsymbol{\chi})$, $\mathcal{F}=[T(\boldsymbol{\chi})]^{-1}$.

$T_k(x)$ $(k=0,1,\ldots,n-1)$ is the Chebyshev polynomial of the first kind,

$$\begin{aligned}T_0(x)&=1,\\ T_1(x)&=x,\\ &\vdots\\ T_k(x)&=2xT_{k-1}(x)-T_{k-2}(x).\end{aligned}$$

Chebyshev polynomials and their derivatives satisfy the relationship[15]

$$\frac{\partial}{\partial x}[T_0(x),T_1(x),\ldots,T_{n-1}(x)]=[T_0(x),T_1(x),\ldots,T_{n-1}(x)]\cdot\mathcal{D}, \tag{5}$$

where

$$\mathcal{D}=\begin{pmatrix}0&1&0&3&\cdots&n-1\\0&0&4&0&\cdots&0\\0&0&0&6&\cdots&2(n-1)\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\cdots&2(n-1)\\0&0&0&0&\cdots&0\end{pmatrix}_{n\times n}\quad\text{for odd } n,$$

$$\mathcal{D}=\begin{pmatrix}0&1&0&3&\cdots&0\\0&0&4&0&\cdots&2(n-1)\\0&0&0&6&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\cdots&2(n-1)\\0&0&0&0&\cdots&0\end{pmatrix}_{n\times n}\quad\text{for even } n.$$

Using Eqs. (4) and (5), the function $\partial f/\partial x$ can be approximated by Chebyshev polynomials

$$\frac{\partial f}{\partial x} = T(x)\mathcal{D}\mathcal{F}f(\boldsymbol{\chi}), \text{ for } x \in \mathbb{I}. \tag{6}$$

For a given function $g(x)$ defined in real field $\mathbb{R}$, we can approximate $g(x)$ by Chebyshev polynomials and mapping $H(x) = \tanh(ax)$

$$g(x) = T^{\mathbb{R}}(x)\mathcal{F}g(H^{-1}(\boldsymbol{\chi})), \quad x \in \mathbb{R}, \tag{7}$$

where $T^{\mathbb{R}}(x) = T(H(x)) = [T_0(H(x)), \ldots, T_{n-1}(H(x))]$, H^{-1} represent the inverse mapping of $H(x)$. $H(x)$ is a one-to-one mapping, which maps the real field $\mathbb{R}$ to the unit interval $\mathbb{I}$. The results of mapping $H(x)$ are shown in Fig. 1.

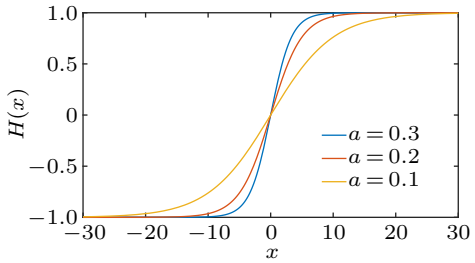


Fig. 1. Results of mapping $H(x)$ about different a.

By using Eqs. (5) and (7), and the chain rule, $\partial g/\partial x$ can be approximated by Chebyshev polynomials

$$\frac{\partial g(x)}{\partial x} = \frac{\partial H(x)}{\partial x}T^{\mathbb{R}}(x)\mathcal{D}\mathcal{F}g(H^{-1}(\boldsymbol{\chi})), \quad \text{for } x \in \mathbb{R}. \tag{8}$$

In this way, the function $g(x)$ and its derivatives $\partial g/\partial x$ are approximated by Chebyshev polynomials.

If a given function changes rapidly in a certain region, then we call this interval its 'rapid-changed interval.' $H(x) = \tanh(ax)$ changes near 0 rapidly, and its rapid-changed interval is expressed as $[-L_1, L_1]$. L_1 is obtained by solving

$$\tanh(aL_1) = a_1,$$

where a_1 is a real number close to 1. Taking $a_1 = 0.9951$ as an example, the rapid-changed interval of $H(x) = \tanh(0.3x)$ is $[-10, 10]$, the rapid-changed interval of $H(x) = \tanh(0.2x)$ is $[-15, 15]$, and the rapid-changed interval of $H(x) = \tanh(0.1x)$ is $[-30, 30]$.

The mapping $H(x)$ distributes more Chebyshev nodes in the rapid-changed interval and distributes fewer Chebyshev nodes outside the rapid-changed interval. So in the rapid-changed interval, we can effectively identify the key information of the given function with the help of $\tanh(ax)$ mapping.

It is worth noting that the value of a will influence the calculated result. Choosing an appropriate a is important in our numerical method. For the selection of parameter a $(0 < a < 1)$, we give the following recommendation. The value of a affects the range of rapid-changed interval and the range of rapid-changed interval will increase as a decreases. For the potential function defined in Schwartz space, it also has the rapid-changed interval. The rapid-changed interval of the potential function must be included in the rapid-changed interval of $\tanh(ax)$ mapping. If not, then we will be unable to completely extract the information of the potential function.

Rewriting the ZS system $(\lambda = 1)$ into a linear eigenvalue problem gives

$$\begin{pmatrix} -\frac{\partial}{\partial x} & q \\ \bar{q} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \mathrm{i}k \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{9}$$

Using Eqs. (7) and (8), we appropriate the eigenfunction ψ, ψ_x and the potential function q by Chebyshev polynomials with n nodes,

$$\begin{aligned} \psi_j &= T^{\mathbb{R}}(x)\mathcal{F}\psi_j(H^{-1}(\boldsymbol{\chi})), \\ \frac{\partial \psi_j}{\partial x} &= \frac{\partial H(x)}{\partial x}T^{\mathbb{R}}(x)\mathcal{D}\mathcal{F}\psi_j(H^{-1}(\boldsymbol{\chi})), \\ q(x) &= T^{\mathbb{R}}(x)\mathcal{F}q(H^{-1}(\boldsymbol{\chi})), \end{aligned} \tag{10}$$

where $j = 1, 2$.

By substituting Eq. (10) into Eq. (9), we get

$$\begin{aligned} &\begin{pmatrix} -\frac{\partial H(x)}{\partial x}T^{\mathbb{R}}(x)\mathcal{D}\mathcal{F} & T^{\mathbb{R}}(x)\mathcal{F}q^{\mathbb{R}}(\boldsymbol{\chi})T^{\mathbb{R}}(x)\mathcal{F} \\ T^{\mathbb{R}}(x)\mathcal{F}\bar{q}^{\mathbb{R}}(\boldsymbol{\chi})T^{\mathbb{R}}(x)\mathcal{F} & \frac{\partial H(x)}{\partial x}T^{\mathbb{R}}(x)\mathcal{D}\mathcal{F} \end{pmatrix} \\ &\times \begin{pmatrix} \psi_1^{\mathbb{R}}(\boldsymbol{\chi}) \\ \psi_2^{\mathbb{R}}(\boldsymbol{\chi}) \end{pmatrix} = \mathrm{i}k \begin{pmatrix} T^{\mathbb{R}}(x)\mathcal{F}\psi_1^{\mathbb{R}}(\boldsymbol{\chi}) \\ T^{\mathbb{R}}(x)\mathcal{F}\psi_2^{\mathbb{R}}(\boldsymbol{\chi}) \end{pmatrix}, \end{aligned} \tag{11}$$

where $q^{\mathbb{R}}(\boldsymbol{\chi}) = q(H^{-1}(\boldsymbol{\chi}))$, $\bar{q}^{\mathbb{R}}(\boldsymbol{\chi}) = \bar{q}(H^{-1}(\boldsymbol{\chi}))$, $\psi_j^{\mathbb{R}}(\boldsymbol{\chi}) = \psi_j(H^{-1}(\boldsymbol{\chi}))$, $(j = 1, 2)$.

Setting $x = H^{-1}(\boldsymbol{\chi})$, equation (11) is rewritten into

$$\begin{aligned} &\begin{pmatrix} -\mathrm{diag}\left[\frac{\partial H(\boldsymbol{\chi})}{\partial x}\right]\mathcal{F}^{-1}\mathcal{D}\mathcal{F} & \mathrm{diag}\left[q(H^{-1}(\boldsymbol{\chi}))\right] \\ \mathrm{diag}[\bar{q}(H^{-1}(\boldsymbol{\chi}))] & \mathrm{diag}\left[\frac{\partial H(\boldsymbol{\chi})}{\partial x}\right]\mathcal{F}^{-1}\mathcal{D}\mathcal{F} \end{pmatrix}_{2n\times 2n} \\ &\times \begin{pmatrix} \psi_1^{\mathbb{R}}(\boldsymbol{\chi}) \\ \psi_2^{\mathbb{R}}(\boldsymbol{\chi}) \end{pmatrix}_{2n\times 1} = \mathrm{i}k \begin{pmatrix} \psi_1^{\mathbb{R}}(\boldsymbol{\chi}) \\ \psi_2^{\mathbb{R}}(\boldsymbol{\chi}) \end{pmatrix}_{2n\times 1}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} &\mathrm{diag}\left[\frac{\partial H(\boldsymbol{\chi})}{\partial x}\right] \\ &= \begin{pmatrix} \frac{\partial H(x)}{\partial x}|_{x=-1} & 0 & \cdots & 0 \\ 0 & \frac{\partial H(x)}{\partial x}|_{x=\cos(\frac{n-2}{n-1}\pi)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial H(x)}{\partial x}|_{x=1} \end{pmatrix}. \end{aligned}$$

Equation (12) is recorded as $A\psi = \mathrm{i}k\psi$, where

$$A = \begin{pmatrix} -A_1 & B \\ B^{\mathfrak{L}} & A_1 \end{pmatrix},$$

where $A_1 = \mathrm{diag}\left[\frac{\partial H(\boldsymbol{\chi})}{\partial x}\right]\mathcal{F}^{-1}\mathcal{D}\mathcal{F}$, $B^{\mathfrak{L}}$ is the Hermitian of B, $B = \mathrm{diag}\left[q(H^{-1}(\boldsymbol{\chi}))\right]$. In fact, equation (12) is a $2n \times 2n$ matrix eigenvalue problem. Note that A_1 is the differentiation

matrix for ∂_x in our method and B is a diagonal matrix that is composed of Chebyshev series for $q(x)$.

The eigenvalue problem (12) can be solved by the quadrature right-triangle (QR) algorithm.[16] In the QR algorithm, the matrix A is decomposed into $A = QR$, where Q is an orthogonal matrix and R is an upper triangular matrix.

The steps of the QR algorithm are as follows:

$$\begin{aligned} &A_1 = A = Q_1 R_1, \\ &A_2 = R_1 Q_1 = Q_1^{\mathrm{T}} A_1 Q_1 = Q_2 R_2, \\ &\quad\vdots \\ &A_n = Q_n R_n = (Q_1 Q_2 \ldots Q_k)^{\mathrm{T}} A (Q_1 Q_2 \ldots Q_k), \end{aligned}$$

diagonal elements of A_n are the eigenvalues of A as $n \to \infty$.

Regarding the accuracy of this method, it does not need to truncate the interval and has spectral accuracy[19] for the smooth potential function. Because we do not truncate the calculated interval, our method will not produce a truncation error for the analytic potential.

3. Numerical results

Our method is used to calculate the eigenvalues of ZS system ($\lambda = 1$) (2) with three potentials and the convergency of the method is analyzed. All of the numerical examples that are reported here are run on an Asustek computer with Intel(R) Core(TM) i7-11800H processor and 16 GB memory.

3.1. The Satsuma–Yajima potential

Our numerical method is used to calculate the eigenvalues of the ZS system ($\lambda = 1$) with Satsuma–Yajima potential. Numerical results are compared with the analytical results and the performance of our numerical method is compared with the performance of the FCM.[19]

When $q = A\mathrm{sech}(x)$, Satsuma and Yajima exactly calculated the discrete eigenvalues of the ZS system.[17] Satsuma and Yajima found that the discrete eigenvalue in the upper-half complex plane $\mathbb{C}_+$ is

$$\kappa_n = \mathrm{i}\left(A + \frac{1}{2} - n\right), \tag{13}$$

where n is a positive number satisfying $n \leq A + \frac{1}{2}$. Due to the symmetry of the discrete eigenvalues,[17] the ZS system with $A\mathrm{sech}(x)$ potential has the discrete eigenvalues $\bar{\kappa}_n$ in $\mathbb{C}_-$.

In the specific calculation, we calculate the eigenvalues of $q(x) = 1.8\mathrm{sech}(x)$. When $q(x) = 1.8\mathrm{sech}(x)$, the ZS system has four discrete eigenvalues $\kappa_1 = 1.3\mathrm{i}$, $\kappa_2 = 0.3\mathrm{i}$, $\bar{\kappa}_1 = -1.3\mathrm{i}$ and $\bar{\kappa}_2 = -0.3\mathrm{i}$. The number of Chebyshev nodes n is set to 200 and the value of a is set to 0.15. The calculation results are shown in Fig. 2. Figure 2(a) shows the calculated eigenvalues of the ZS system with the Satsuma–Yajima potential. There are four discrete eigenvalues in Fig. 2(a), which is consistent with the theoretical result. Figure 2(b) shows the calculated eigenfunction in point κ_1 and figure 2(c) gives the calculated eigenfunction in point κ_2. The absolute error between the calculated κ_1 and the exact κ_1 is 1.85×10^{-15}, and the absolute error between the calculated κ_2 and the exact κ_2 is 1.61×10^{-16}. The method takes about 0.2 s to finish. These results show that our method is efficient.

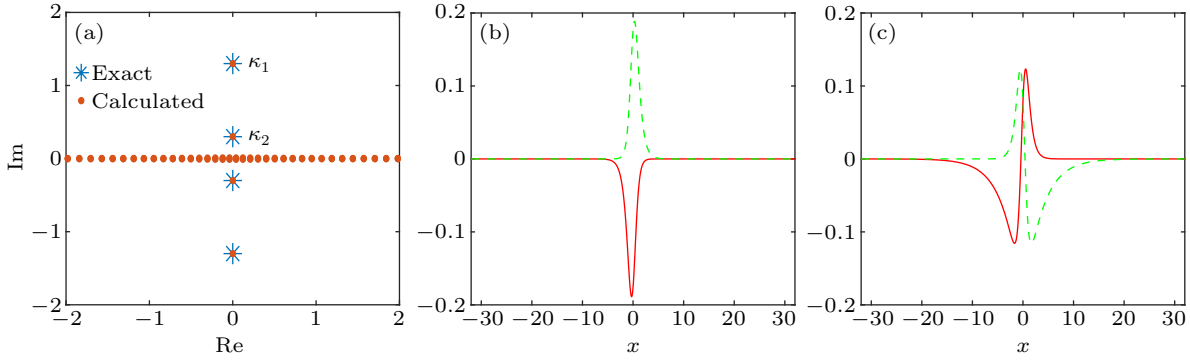


Fig. 2. The calculated results of ZS system with $1.8\mathrm{sech}(x)$ potential. (a) The calculated eigenvalues (red) and exact eigenvalues (blue) of $1.8\mathrm{sech}(x)$ potential. (b) Numerical results of ψ_1 (red line) and ψ_2 (green line) at $k = \kappa_1$. (c) Numerical results of ψ_1 (red line) and ψ_2 (green line) at $k = \kappa_2$.

The stability and convergency of our method need to be analyzed. In area $[a, n] \in [0.1, 0.33] \times [21, 251]$, we calculate the eigenvalues of the ZS system with $q(x) = 1.8\mathrm{sech}(x)$, the absolute error in $k = \kappa_1$ is shown in Fig. 3(a). There are three routes in Fig. 3(a) (blue route 1, black route 2, and green route 3), the convergency of our method is analyzed along the three routes. In the Fourier collocation method, the calculated interval is truncated to $[-25, 25]$. The relationship between the error and n is shown in Fig. 3(b), the red line is the error curve calculated by the Fourier collocation method (FCM), the blue line is the error curve calculated by our method along Fig. 3(a) "route 1", the black line is the error bar calculated by our method along Fig. 3(a) "route 2", and the green line is the error bar calculated by our method along Fig. 3(a) "route 3". Figure 3(b) shows that our method is more accurate than the FCM and the convergence rate of our method is faster than

FCM, so our method is more efficient. Because the error calculated by the FCM decays exponentially with the number of nodes,[19] the error of our method also decays exponentially with the number of nodes and its error decays faster than any power of n^{-1}. Thus the spectral accuracy of the method is confirmed.

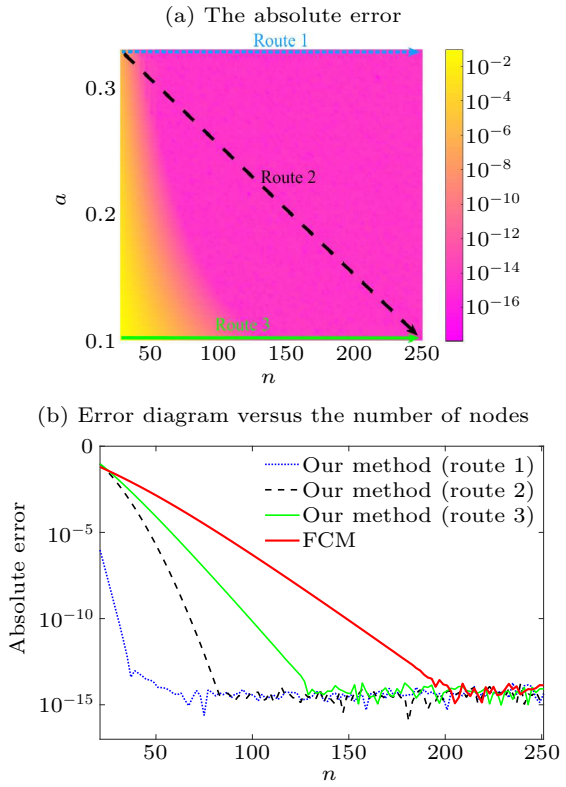


Fig. 3. The absolute error in area $[a,n] \in [0.1,0.33] \times [21,251]$. (a) The absolute error picture ($k = \kappa_1$). (b) The error diagram along route 1 (blue line), the error diagram along route 2 (black line), the error diagram along route 3 (green line), and the error diagram of the Fourier collocation method (red line).

We also explored the computational efficiency of our method and compared it with the FCM. We calculated the discrete eigenvalues of the $1.8\mathrm{sech}(x)$ potential at different numbers of nodes and the time is shown in Table 1.

Table 1. The time required to complete the program at different nodes n.

n	50	100	150	200	250	300	350	400	450
Time (FCM) (s)	0.20	0.26	0.43	0.60	0.82	1.10	1.49	2.07	2.79
Time (our method) (s)	0.07	0.10	0.12	0.17	0.22	0.33	0.41	0.55	0.71

The minimum error generated by our method is about 10^{-15} level. This method can achieve machine accuracy. Since the calculated accuracy of the mathematical software is 16 significant figures, there will be an error of about 10^{-15} level in the calculation process. Our method can greatly improve the calculated accuracy, especially when the number of Chebyshev nodes is small.

3.2. The Y-shape potential

Bronski computed the eigenvalues of the $\mathrm{sech}(2\epsilon x)\,\mathrm{e}^{\mathrm{i}\,\mathrm{sech}(2\epsilon x)/\epsilon}$ potential and found that the shape of the discrete eigenvalues is "Y".[9] Setting $n = 400$ and $a = 0.02$, our method is used to compute the eigenvalues of the $\mathrm{sech}(2\epsilon x)\,\mathrm{e}^{\mathrm{i}\,\mathrm{sech}(2\epsilon x)/\epsilon}$ potential with $\epsilon = 0.2$, $\epsilon = 0.1$ and $\epsilon = 0.05$, and the calculated results are shown in Fig. 4. The calculations are finished within 0.6 s.

When $\epsilon = 0.2$, there are three discrete eigenvalues in $\mathbb{C}_+$; when $\epsilon = 0.1$, there are six discrete eigenvalues in $\mathbb{C}_+$; and when $\epsilon = 0.05$, there are 12 discrete eigenvalues in $\mathbb{C}_+$. The calculated results are consistent with Bronski's results (Ref. [9], p. 385, Table 1). The calculated discrete eigenvalues become Y-shaped with the decrease of ϵ, which are consistent with the theoretical results.

There are six discrete eigenvalues in $\mathbb{C}_+$ in Fig. 4(b), and their values are shown in Table 2.

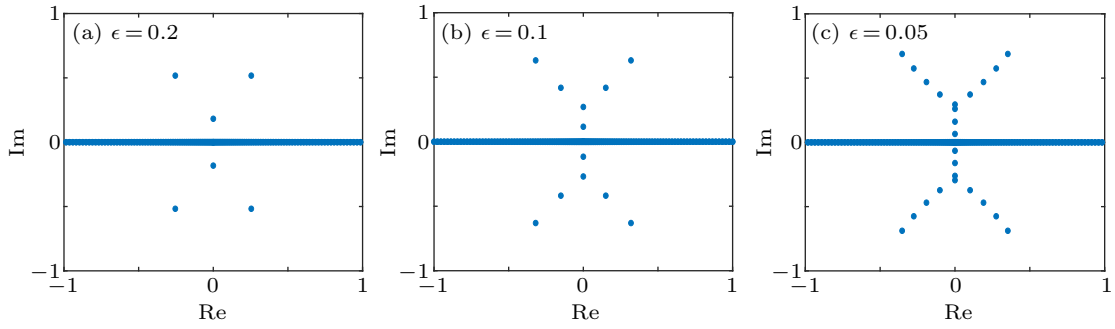


Fig. 4. The calculated eigenvalues of ZS system with $\mathrm{sech}(2\epsilon x)\,\mathrm{e}^{\mathrm{i}\,\mathrm{sech}(2\epsilon x)/\epsilon}$ potential ($a = 0.02$).

From Table 2, we learn that the $\mathrm{sech}(0.2x)\,\mathrm{e}^{10\mathrm{i}\,\mathrm{sech}(0.2x)}$ potential has two pure imaginary eigenvalues and four complex discrete eigenvalues in $\mathbb{C}_+$. Thus, the $q(x,0) = \mathrm{sech}(0.2x)\,\mathrm{e}^{10\mathrm{i}\,\mathrm{sech}(0.2x)}$ initial profile will evolve into a second-order breather and four solitons for the NLS equation. The Fourier spectral method[18] is used to calculate the evolution of the NLS equation with $q(x,0) = \mathrm{sech}(0.2x)\,\mathrm{e}^{10\mathrm{i}\,\mathrm{sech}(0.2x)}$. The density of the calculated result is shown in Fig. 5. In Figs. 5(a) and 5(b), the initial profile $q(x,0) = \mathrm{sech}(0.2x)\,\mathrm{e}^{10\mathrm{i}\,\mathrm{sech}(0.2x)}$ evolves into four solitons and a second-order breather, which is consist with Fig. 4(b).

The correctness of the calculated result is verified by ana-

lyzing the convergence of the method. When $n = 400$ and $a = 0.02$, we obtain the eigenvalue $\kappa_1 = -1.78524894765016 \times 10^{-15} + 0.116148026898534\mathrm{i}$ of the $\mathrm{sech}(0.2x)\,\mathrm{e}^{10\mathrm{i}\,\mathrm{sech}(0.2x)}$ potential. Under different n Chebyshev nodes, we calculate the Cauchy error for the $q(x,0) = \mathrm{sech}(0.2x)\,\mathrm{e}^{10\mathrm{i}\,\mathrm{sech}(0.2x)}$ potential at κ_1. The calculated result is shown in Fig. 6. The Cauchy error in Fig. 6 is defined by

$$\text{Cauchy error} = |\kappa_1(n) - \kappa_1|,$$

where $\kappa_1(n)$ is the calculated κ_1 under n Chebyshev nodes and $\kappa_1 = -1.78524894765016 \times 10^{-15} + 0.116148026898534\mathrm{i}$.

Table 2. The values for the calculated discrete eigenvalues of the ZS system with $\mathrm{sech}(0.2x)\,\mathrm{e}^{10\mathrm{i}\,\mathrm{sech}(0.2x)}$ potential.

No.	The discrete eigenvalues
κ_1	$-1.78524894765016 \times 10^{-15} + 0.116148026898534\mathrm{i}$
κ_2	$5.16823894592694 \times 10^{-15} + 0.269496534408172\mathrm{i}$
κ_3	$0.150457991591637 + 0.418161274246707\mathrm{i}$
κ_4	$-0.150457991591641 + 0.418161274246702\mathrm{i}$
κ_5	$0.319248334509386 + 0.630381427554910\mathrm{i}$
κ_6	$-0.319248334509384 + 0.630381427554907\mathrm{i}$

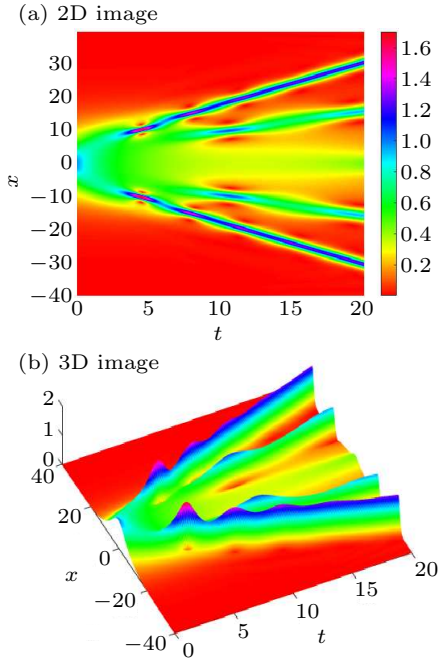


Fig. 5. The evolution of the initial profile $q(x,0) = \mathrm{sech}(0.2x)\,\mathrm{e}^{10\mathrm{i}\,\mathrm{sech}(0.2x)}$ for the NLS equation.

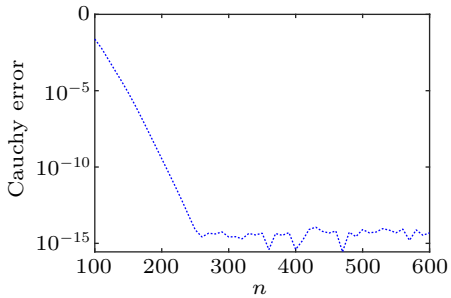


Fig. 6. The Cauchy error for the $\mathrm{sech}(2\epsilon x)\,\mathrm{e}^{\mathrm{i}\,\mathrm{sech}(2\epsilon x)/\epsilon}$ in κ_1.

In Fig. 6, the method gradually converges as n increases, and generates an error of 10^{-15} level.

3.3. The solitonic potential

Optical solitons have very important applications in all optical networks, optical communication, and optical logic devices. The soliton solution of the NLS equation is considered to be a potential solution for fiber optic transmission, so it is important to calculate its discrete eigenvalues. In this subsection, our method is used to calculate the eigenvalues of $q_{\mathrm{so}} = \exp(-\mathrm{i}x)\mathrm{sech}(x)$. The initial value of the optical solitons can describe the propagation of signals in optical fibers.

As we all know, q_{so} has a discrete eigenvalue $\kappa_1 = 0.5 + 0.5\mathrm{i}$ in $\mathbb{C}_+$.[10] Setting $n = 200$ and $a = 0.1$, our method is used to compute the spectrum of $q_{\mathrm{so}} = \exp(-\mathrm{i}x)\mathrm{sech}(x)$, the calculated result is shown in Fig. 7. The absolute error between the calculated κ_1 and the exact κ_1 is 7.77×10^{-16}, and the absolute error between the calculated κ_2 and the actual κ_2 is 6.31×10^{-15}. The calculation is finished within 0.3 s. Our method is more accurate and faster than the NFT method.[13]

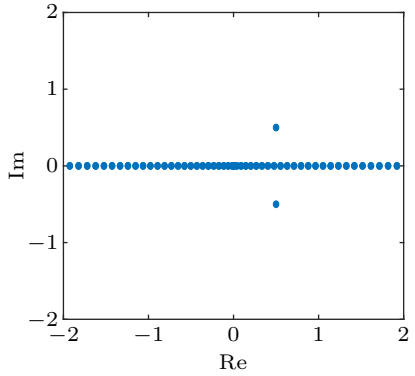


Fig. 7. The calculated eigenvalues of the $q_{\mathrm{so}} = \exp(-\mathrm{i}x)\mathrm{sech}(x)$ potential.

4. Conclusion

A numerical method is proposed to calculate the discrete eigenvalues for the ZS system. The tools that are used are Chebyshev polynomials and $\tanh(ax)$ mapping. We can effectively identify the key information of the given function with the help of $\tanh(ax)$ mapping and realize the high-efficiency calculation.

Our method has following advantages. First, we do not need to truncate the calculated region for analytical potentials, so our method will not produce truncation error when using Chebyshev polynomials to appropriate the given function. Second, the method can calculate the discrete eigenvalues for the ZS system with spectral accuracy. This method is high-precision and efficient. We calculate the discrete eigenvalues of the Satsuma–Yajima potential and compare the method with the Fourier collocation method, and find that the convergence rate of our method is faster than the Fourier collocation method. For the complex $\mathrm{sech}(2\epsilon x)\,\mathrm{e}^{\mathrm{i}\,\mathrm{sech}(2\epsilon x)/\epsilon}$ potential, our

method still converges quickly. It is worth mentioning that our method can be further extended to solve other linear eigenvalue problems.

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