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**Citation:**Chin. Phys. B, 2023, 32 (**9**): 098902. DOI: 10.1088/1674-1056/acd685
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# Percolation transitions in edge-coupled interdependent networks with directed dependency links

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(Received 16 February 2023; revised manuscript received 3 April 2023; accepted manuscript online 18 May 2023)

We propose a model of edge-coupled interdependent networks with directed dependency links (EINDDLs) and develop the theoretical analysis framework of this model based on the self-consistent probabilities method. The phase transition behaviors and parameter thresholds of this model under random attacks are analyzed theoretically on both random regular (RR) networks and Erdös–Rényi (ER) networks, and computer simulations are performed to verify the results. In this EINDDL model, a fraction $\beta$ of connectivity links within network B depends on network A and a fraction $(1-\beta)$ of connectivity links within network A depends on network B. It is found that randomly removing a fraction $(1-p)$ of connectivity links in network A at the initial state, network A exhibits different types of phase transitions (first order, second order and hybrid). Network B is rarely affected by cascading failure when $\beta$ is small, and network B will gradually converge from the first-order to the second-order phase transition as $\beta$ increases. We present the critical values of $\beta$ for the phase change process of networks A and B, and give the critical values of $p$ and $\beta$ for network B at the critical point of collapse. Furthermore, a cascading prevention strategy is proposed. The findings are of great significance for understanding the robustness of EINDDLs.

**Keywords:** edge-coupled interdependent networks with directed dependency links, percolation transitions, cascading failures, robustness analysis

**PACS:** 89.75.–k, 89.75.Fb, 64.60.ah **DOI:** 10.1088/1674-1056/acd685

## 1. Introduction

Many facilities and systems in real life can be abstracted into networks, such as transportation networks, biological networks, financial networks, electric networks, and social networks.[1–9] These networks have brought convenience to human production and life, while the propagation phenomena occurring on the networks have also brought negative effects, and complex networks science provides theories and analytical methods for studying complex systems in the real world.

Early research on complex networks was limited to the topological properties of the single-layer network,[10–13] until it was discovered that the interdependence within networks in the real world posed extremely serious systemic risks, such as large-scale power outages and traffic congestion.[14,15] This is due to the fact that the failures in a single-layer network could spread not only in itself but also in the other networks. Therefore, the study of interdependent networks has attracted much attention of many scholars.

Buldyrev *et al.*[16] first developed the analytical framework of interdependent networks, where there are one-to-one dependent nodes between two networks, and found that interdependence greatly reduces the robustness of the network. Following this research, many results have emerged for the study of interdependent networks. According to the difference of interdependent patterns, Parshani *et al.*[17] analyzed the node-coupled interdependent networks in which a fraction of the nodes are interdependent in both networks, and found that reducing the coupling between the networks leads to a change from a first-order percolation phase transition to a second-order percolation transition at a critical point, i.e., its robustness is better than the fully node-coupled interdependent networks.[16] Shao *et al.*[18] proposed the coupled network systems with multiple support-dependence relations and investigated them analytically and numerically. Buldyrev *et al.*[19] and Parshani *et al.*[20] studied the coupled networks with positively correlated degrees of mutually dependent nodes and found that as the networks become more inter-similar the system would become significantly more robust to random failure. Hu *et al.*[21] found that the phase transition includes the hybrid transition of mixed first and second orders in study of a coupled networks system where both interdependent and interconnected links exist. This hybrid transition phenomenon was also found in the studies of Zhou *et al.*[22] and Zhang *et al.*[23] afterward. Dong *et al.*[24] developed a interacting networks model with feedback-dependency links and found that the feedback-dependency links have no effect on robustness of the system for a weak coupling strength corresponding to the second-order transitions, whereas the system is more vulnerable for strong coupling strength corresponding to first-order or hybrid phase transition. Liu *et al.*[25] found that reducing the coupling strength between interdependent nodes can im-

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prove the system's robustness by studying the interdependent networks with weak node-coupling strength.

In addition to studying interdependent patterns, scholars have proposed numerous network models. Gao *et al.*[26–28] extended interdependent networks to systems with more than two independent networks and analyzed the effect of topology and the number of networks on network robustness. Cellai *et al.*[29] showed that the overlap among layers can improve the robustness of interdependent multiplex systems and change the critical behavior of the percolation phase transition in a complex way. Huang *et al.*[30] studied the effect of clustering on the system robustness of interdependent networks and found that clustering makes the system more vulnerable. Vald *et al.*[31] set the nodes with high degrees as autonomous nodes and found them to be effective in improving the robustness of the network. Liu *et al.*[32] developed a theoretical framework of interdependent directed networks and found that the directionality within the network increases its vulnerability and exhibits hybrid phase transitions.

The above works address the case of node-coupled interdependent networks (NINs), i.e., the failures of nodes in one network lead to the failures of nodes in the other network. Considering that there are also many cases of edge-coupled interdependent networks in the real world, Gao *et al.*[33] developed a new model of edge-coupled interdependent networks (EINs) and analyzed it using the self-consistent probabilities method[34] and found that it is stronger than the NIN.

The existing studies of EINs have only considered the case where both connectivity links and dependency links are undirected. However, there are also some EINs in which the dependency links are directed in the real world. One example is power-transportation interdependent networks. Generally speaking, power system and rail transportation system interdepend on each other. However, by analyzing the interactions between these two networks we found that there are one-way dependencies instead of mutual dependencies for the specific lines. For example, the failures of part power lines can directly affect the functions of some part of rail transportation lines, whereas these transportation lines may not just right affect these power lines. In other words, there are directed dependency relations between transportation lines and power lines.

Therefore, a model of edge-coupled interdependent networks with directed dependency links (EINDDLs) is proposed, and the theoretical framework for the analysis of EINDDLs under random attacks is developed by setting $\beta$ to change the dependence relationship between two networks based on the self-consistent probabilities method. Both theoretical analysis and computer simulations are conducted on RR-RR and ER-ER EINDDLs (RR: random regular, ER: Erdös–Rényi) to explore the relationship between $\beta$ and the phase transition behaviors of networks A and B under random attacks. Some characteristics in the phase transition behaviors of networks A and B are found and the corresponding prevention strategy is proposed.

## 2. Model description

The model of edge-coupled interdependent networks with directed dependency links (EINDDLs) in this paper refers to two-layer edge-coupled interdependent networks (EINs) consisting of networks A and B, where the dependency links are one direction instead of mutual direction. We assume that the number of nodes and the average degree of both networks are the same, so the total numbers of edges in both networks are equal. A connectivity link in network A is randomly selected to establish a directed dependency link with a connectivity link in network B, as shown by the red and blue dashed lines in Fig. 1(a). Figure 1 gives the process of cascading failures for the EINDDL. Dependency links make the system follow such cascading rules, e.g., failure of $a_1 \leftrightarrow a_8$ causes failure of $b_2 \leftrightarrow b_3$ while the failure of $b_2 \leftrightarrow b_3$ cannot lead to failure of $a_1 \leftrightarrow a_8$. Here, $\beta$ is defined to control the ratio of two directed types of dependency links, i.e., network A has a $1-\beta$ ratio of connectivity links depending on network B, and network B has a $\beta$ ratio of connectivity links depending on network A. The dependencies of the two networks will also be changed by changing the value of $\beta$.

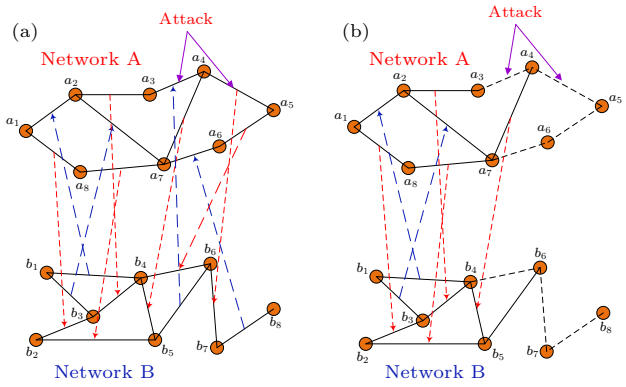



**Fig. 1.** Cascading failures of the EINDDL.

According to the bond percolation theory, when a network is under attacks, it may be broken into many connected clusters. The biggest one referred as giant component (GC) is usually considered as functional part of a network. We randomly remove the $1-p$ fraction of connectivity links in network A in the initial state, this will result in a portion of nodes failing due to being out of GC, while the edges connected to these failed nodes will also fail. Since there are connectivity links in network B that depends on network A, it causes a part of the connectivity links in network B to fail and makes a part of the nodes fail by leaving the GC, which subsequently leads to the failure of connectivity links in network A that depends on network B and causes further damage to network A. Cascading failures repeatedly propagate between two networks

as the above process until no more new connectivity links or nodes fail in both networks, which is the steady state. In terms of interdependent networks, the remainder of the network after the cascading failure composes an extensive component, referred as mutually connected giant component (MCGC). The size of the MCGC at a steady state reflects the robustness of the network.

The solid black lines in Fig. 1 represent the connectivity links within the network, the dashed black lines represent the failed connectivity links, and the dashed red and blue lines are the dependency links between two networks, where $\beta$ is 0.6. Due to the initial state $a_3 \leftrightarrow a_4$ and $a_4 \leftrightarrow a_5$ in network A failed, as shown in Fig. 1(a). According to the cascading failure rules, $b_5 \leftrightarrow b_6$ in network B will not fail and $b_6 \leftrightarrow b_7$ will fail. Finally, a steady state is reached, as shown in Fig. 1(b).

## 3. Theoretical analysis framework

### 3.1. Single-layer network

The degree distribution $p(k)$ is defined as the probability that a node of degree $k$ is randomly selected in the network. The condition that a node of degree $k$ does not belong to the GC is that all $k$ connectivity links of the node do not lead to the GC. We introduce an auxiliary parameter $R$, defined as the probability that a randomly selected link leads to the GC along one direction, then the probability that the node does not belong to the GC can be expressed as $(1-R)^k$. For any node with degree $k$ satisfying the degree distribution $p(k)$, the probability that a node does not belong to the GC is obtained by multiplying the probability $(1-R)^k$ with the degree distribution $p(k)$ and summing up,[34–38] where $S$ denotes the probability that a randomly selected node in the network belongs to the GC,

$$1-S=\sum_k p(k)\cdot(1-R)^k. \tag{1}$$

From the generating function theory,[36,37] it is known that the network degree distribution generating function is $\Phi_0(x)=\sum_k p(k)x^k$ and the excess-degree distribution generating function is $\Phi_1(x)=\sum_k q(k)x^{k-1}$, where $q(k)$ denotes the excess-degree distribution of the nodes, i.e., $q(k)=p(k)\cdot k/\langle k\rangle$, and $\langle k\rangle$ is the average degree of the network, thus Eq. (1) can be rewritten as

$$S=1-\Phi_0(1-R). \tag{2}$$

Apart from Eq. (2), in order to solve $S$ or $R$, it is also necessary to build a consistent equation of the probability $R$ that a connectivity link leads to the GC, i.e., at least one of the remaining links of the node reached by this link leads to the GC, i.e.,

$$R=\sum_k q(k)\sum_{i=1}^{k-1}\binom{k-1}{i}R^i(1-R)^{k-i-1}. \tag{3}$$

Thus Eq. (3) can reduce to

$$R=1-\sum_k q(k)(1-R)^{k-1}. \tag{4}$$

Using the generating function $\Phi_1(x)$, Eq. (4) can be rewritten as

$$R=1-\Phi_1(1-R). \tag{5}$$

If a fraction $1-p$ of connectivity links is randomly removed from the network (bond-percolation), the probability of randomly selecting a node in the network that belongs to the GC remains unchanged since no node is removed from the network, and the expression of the order parameter $S$ remains as Eq. (2). Since only a fraction $p$ of connectivity links remains in the network, the probability of randomly selecting a link to reach a node with degree $k$ becomes $p$ times the original one, i.e., $p\cdot q(k)$. Thus, the consistent equation for the auxiliary parameter $R$ is

$$R=p\cdot[1-\Phi_1(1-R)]. \tag{6}$$

The order parameter $S$ and the auxiliary parameter $R$ can be solved by combining Eqs. (2) and (6).

### 3.2. Edge-coupled interdependent networks with directed dependency links

Following the single-layer network method, the EINDDL is a two-layer interdependent networks, where we introduce two auxiliary parameters $R_{\rm A}$ and $R_{\rm B}$, which denote the probabilities that randomly selected link in network A and network B leads to the MCGC along one direction, respectively. Network A has $1-\beta$ fraction of connectivity links depending on network B, and network B has $\beta$ fraction of connectivity links depending on network A. For simplicity, assuming that both networks have the same degree distribution and the same number of nodes, their generating function expressions are also the same. In this case, when a connectivity link is randomly selected in network A, whether the node reached by this link belongs to MCGC depends on the following two factors.

(i) If a randomly selected connectivity link does not depend on network B, i.e., the corresponding link in network B will not cause this link to fail. The fraction of this part of connectivity links is $\beta$. It is only necessary to consider whether the end nodes of these links in network A belong to the GC.

(ii) If a randomly selected connectivity link depends on network B, i.e., the failure of the corresponding connectivity link in network B will cause this link to fail. The ratio of this part of connectivity links is $1-\beta$. It is necessary to consider the probability that the end nodes of this part of links in network A belong to GC, but also the probability that the end nodes of the corresponding links in network B belong to GC.

Therefore, based on Eq. (5), the probability $R_{\rm AA}$ that a fraction of $\beta$ connectivity links leads to the GC is

$$R_{\rm AA}=\beta\cdot[1-\Phi_1(1-R_{\rm A})]. \tag{7}$$

Since a $1-\beta$ fraction of connectivity links depends on network B, at least one of the nodes to which the corresponding connectivity links the network B lead must be in the GC, as shown in Fig. 2.

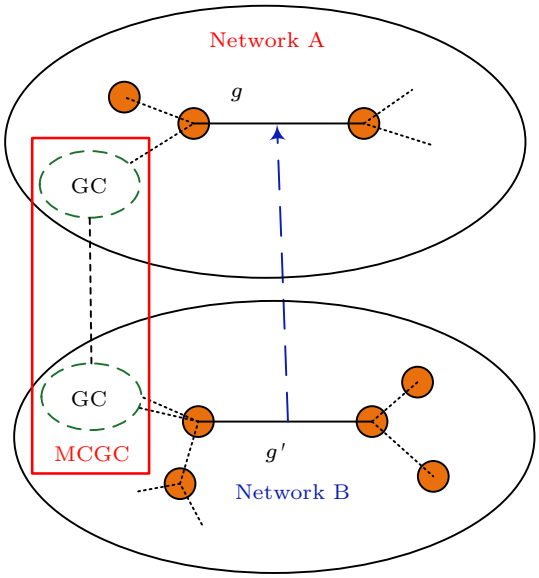



**Fig. 2.** The survival of connectivity link dependent on network B.

The probability that one end of $g'$ not in the GC is $\Phi_1(1-R_B)$ and the probability that both ends are not in the GC is $[\Phi_1(1-R_B)]^2$. Thus the probability $R_{LB}$ that at least one end of the connectivity link $g'$ in the GC is

$$R_{LB} = 1-[\Phi_1(1-R_B)]^2. \tag{8}$$

Similarly, the node to which the $1-\beta$ ratio of connectivity links leads has to be in the GC of network A, and its probability $R_{AB}$ is known according to Eq. (5),

$$R_{AB} = (1-\beta)\cdot[1-\Phi_1(1-R_A)]. \tag{9}$$

Accordingly, the probability that the nodes to which the connectivity links of the $1-\beta$ fraction lead are in the MCGC has to satisfy the conditions in both networks A and B, i.e., $R_{AB}\cdot R_{LB}$.

Combining the two aspects considered above, the auxiliary parameter $R_A$ can be expressed as

$$R_A = R_{AA}+R_{AB}\cdot R_{LB}. \tag{10}$$

Similarly, whether the nodes reached by a randomly selected connectivity link in network B belong to the MCGC requires consideration of the above two aspects. Using the same analysis, we can obtain that the $\beta$ fraction of connectivity links in network B leading to the GC needs to satisfy both constraints in networks A and B, and the probability is

$$R_{BA}\cdot R_{LA} = \beta\cdot[1-\Phi_1(1-R_B)]\cdot\left\{1-[\Phi_1(1-R_A)]^2\right\}. \tag{11}$$

The $1-\beta$ ratio of connectivity links in network B leading to the GC only needs to meet the constraint in network B leading to the GC, and the probability is

$$R_{BB} = (1-\beta)\cdot[1-\Phi_1(1-R_B)]. \tag{12}$$

Thus, the expression for the auxiliary parameter $R_B$ can be expressed as

$$R_B = R_{BB}+R_{BA}\cdot R_{LA}. \tag{13}$$

Following the above formulas, if the $1-p$ fraction of the connectivity links in network A are randomly removed in the initial state, the fraction of remaining connectivity links in network A is $p$, and the connectivity links of $\beta$ and $1-\beta$ fractions become $p\cdot\beta$ and $p\cdot(1-\beta)$, respectively. From Eq. (10), the expression of the auxiliary parameter $R_A$ at this time is

$$R_A = p\cdot(R_{AA}+R_{AB}\cdot R_{LB}). \tag{14}$$

Since there are no attacks in network B, what affects network B is the failed links in the $\beta$ scaled connectivity links. Therefore, the surviving connectivity links in network B are only related to the connectivity links of the $1-\beta$ scale and the connectivity links of the $p\cdot\beta$ scale. From Eq. (13), the expression of the auxiliary parameter $R_B$ is

$$R_B = R_{BB}+p\cdot R_{BA}\cdot R_{LA}. \tag{15}$$

In addition, after randomly removing the $1-p$ fraction of connectivity links in network A, the order parameter $S$ will not be affected by the deletion of connectivity links because the nodes between the two layers are not directly related. From Eq. (2), the expressions for the order parameters $S_A$ and $S_B$ of networks A and B are, respectively,

$$S_A = 1-\Phi_0(1-R_A), \tag{16}$$

$$S_B = 1-\Phi_0(1-R_B). \tag{17}$$

Given the values of $\beta$ and $p$, combing Eqs. (14)–(17), we can solve the values of the auxiliary parameters $R_A$, $R_B$ and the order parameters $S_A$ and $S_B$.

In the percolation theory,[16,35] a fraction $1-p$ of nodes or links are randomly removed which may trigger cascading failures due to the dependency between networks. An abrupt collapse may occur where the size of GC changes to zero discontinuously upon a removal of a critical fraction $1-p_c$ of nodes/links. This discontinuous percolation transition is named as the fist-order phase transition and $p_c$ is known as the percolation threshold. If the size of GC changes to zero continuously that is typical on single-layer networks, we call it the second-order phase transition.

The type of phase transition and the phase transition threshold $p_c$ of the EINDDL can be investigated by comparing $S$ and $p$. The percolation behaviors of the networks vary with

different values of the parameter $\beta$. If the network undergoes a first-order phase transition, the curve of Eq. (14) tangentially will meet the curve represented by Eq. (15). If the network undergoes a second-order phase transition, the value of the auxiliary parameter $R_A$ or $R_B$ will be 0, and the second-order phase transition threshold can be found by the above equations.

### 3.3. Theoretical framework verification

Figure 3 gives the plots of $S_A$ and $S_B$ versus $p$ for the RR-RR EINDDL and the ER-ER EINDDL with different $\beta$ for the number of nodes $N_A = N_B = 100000$ and the average degree $\langle k_A \rangle = \langle k_B \rangle = 4$, where the solid lines are the theoretical results and the dots are the simulated values.

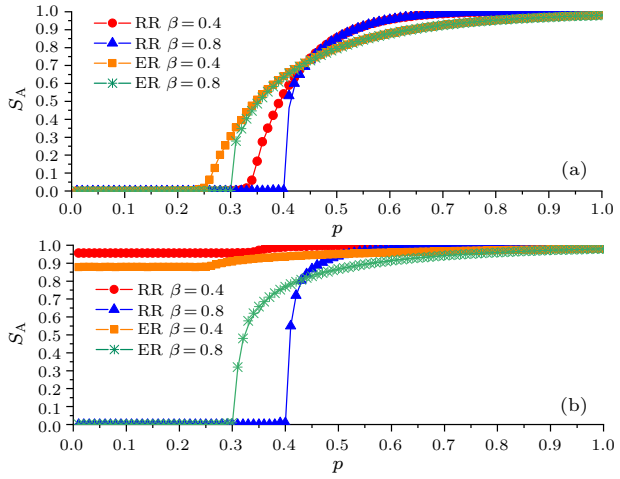



**Fig. 3.** Plots of $S_A$ and $S_B$ versus $p$ with different $\beta$.

It can be seen from Fig. 3 that the theoretical values can match well with the computer simulation values. The theoretical framework can be verified. We apply this framework on the RR and ER networks to analyze their phase transition behaviors.

## 4. Robustness analysis of the EINDDL

### 4.1. The RR-RR EINDDL

In the RR network, all node degrees are $\langle k \rangle$. Therefore, the degree distribution is

$$p(k) = \begin{cases} 1, & k = \langle k \rangle, \\ 0, & k \neq \langle k \rangle. \end{cases} \tag{18}$$

Assuming that the numbers of nodes of networks A and B are the same and the average degree is $\langle k \rangle$, the generating function of the degree distribution of the RR network is $\Phi_0(x) = x^{\langle k \rangle}$ and the generating function of the excess degree distribution is $\Phi_1(x) = x^{\langle k \rangle - 1}$ from the theory of generating functions. According to Eqs. (14) and (15), the expressions for the auxiliary parameters $R_A$ and $R_B$ of the RR-RR EINDDL are

$$R_A = p \cdot [1 - (1 - R_A)^{\langle k \rangle - 1}] \cdot \{\beta + (1 - \beta) \times [1 - (1 - R_B)^{2(\langle k \rangle - 1)}]\}, \tag{19}$$

$$R_B = [1 - (1 - R_B)^{\langle k \rangle - 1}] \cdot \{(1 - \beta) + p \cdot \beta \times [1 - (1 - R_A)^{2(\langle k \rangle - 1)}]\}. \tag{20}$$

From Eqs. (16) and (17), the expressions of the order parameters $S_A$ and $S_B$ are

$$S_A = 1 - (1 - R_A)^{\langle k \rangle}, \tag{21}$$

$$S_B = 1 - (1 - R_B)^{\langle k \rangle}. \tag{22}$$

From Eqs. (19)–(22), the values of the order parameters $S_A$ and $S_B$ are determined by the auxiliary parameters $R_A$ and $R_B$, and only non-zero values of $R_A$ and $R_B$ can be obtained for non-zero values of $S_A$ and $S_B$. Obviously, Eqs. (19) and (20) possess a trivial solution $R_A = R_B = 0$, corresponding to the case in which the GC is 0 in the network.

The first-order phase transition threshold $p_c^I$ of the system can be obtained when the curves represented by Eqs. (19) and (20) are tangent. The GC of network A in the critical case will be 0 if the system undergoes a second-order phase transition. At this time, the auxiliary parameter $R_A \to 0$. The nontrivial solution $R_{B0}$ of the auxiliary parameter $R_B$ can be found by substituting it into Eq. (20), and then the second-order phase transition threshold $p_c^{II}$ is obtained as follows:

$$p_c^{II} = \frac{1}{\langle k \rangle - 1} \cdot \frac{1}{1 - (1 - \beta) \cdot (1 - R_{B0})^{2(\langle k \rangle - 1)}}. \tag{23}$$

Assuming that the number of nodes in both network A and network B in the RR-RR EINDDL is $N = 100000$ and the average degree is the same, Fig. 4 gives a plot of the order parameters $S_A$ and $S_B$ versus $p$ at different $\beta$ for $\langle k \rangle = 3$, 4, and 5.

### 4.2. The ER-ER EINDDL

The degree distribution of a Erdös–Rényi (ER) network is Poisson distribution, i.e., $p(k) = \frac{\langle k \rangle^k}{k!} \cdot e^{-\langle k \rangle}$. Similarly, assuming that the number of nodes of networks A and B are the same and the average degree is $\langle k \rangle$, the generation function of the degree distribution of the ER network and the generation function of the excess degree distribution are the same from the theory of generating functions, that is, $\Phi_0(x) = \Phi_1(x) = e^{\langle k \rangle (x-1)}$. According to Eqs. (14) and (15), the expressions for the auxiliary parameters $R_A$ and $R_B$ are, respectively,

$$R_A = p \cdot (1 - e^{-\langle k \rangle R_A}) \cdot [1 - (1 - \beta) \cdot e^{-2\langle k \rangle R_B}], \tag{24}$$

$$R_B = (1 - e^{-\langle k \rangle R_B}) \cdot [(1 - \beta) + p \cdot \beta \cdot (1 - e^{-2\langle k \rangle R_A})]. \tag{25}$$

From Eqs. (16) and (17), the expressions of the order parameters $S_A$ and $S_B$ are

$$S_A = 1 - e^{-\langle k \rangle R_A}, \tag{26}$$

$$S_B = 1 - e^{-\langle k \rangle R_B}. \tag{27}$$

Given the values of $\beta$ and the average degree $\langle k\rangle$, Eqs. (24) and (25) can be rewritten as

$$\begin{cases} R_{\rm A} = \omega_{\rm A}(p, R_{\rm B}), \\ R_{\rm B} = \omega_{\rm B}(p, R_{\rm A}). \end{cases} \tag{28}$$

Eliminating the auxiliary parameter $R_{\rm B}$ from the system of Eq. (28), we obtain

$$R_{\rm A} = \omega_{\rm A}[p, \omega_{\rm B}(p, R_{\rm A})]. \tag{29}$$

When Eq. (29) satisfies the following conditions

$$1 = \frac{{\rm d}\omega_A[p, \omega_{\rm B}(p, R_{\rm A})]}{{\rm d}R_{\rm A}} \cdot \frac{{\rm d}\omega_{\rm B}(p, R_{\rm A})}{{\rm d}R_{\rm A}}\Big|_{R_{\rm A}=R_{\rm Ac}, p=p_{\rm c}}, \tag{30}$$

Equations (24) and (25) appear as a nontrivial solution, corresponding to the critical case when the system undergoes a first-order phase transition, where $p$ is the first-order phase transition threshold $p_{\rm c}^{\rm I}$.

If the system undergoes a second-order phase transition, the same analytical method of substituting $R_{\rm A} \to 0$ into Eq. (25) can give the solution $R_{\rm B0}$ of the auxiliary parameter $R_{\rm B}$, which leads to the second-order phase transition threshold $p_{\rm c}^{\rm II}$,

$$p_{\rm c}^{\rm II} = \frac{1}{\langle k\rangle} \cdot \frac{1}{1-(1-\beta)\cdot {\rm e}^{-2\langle k\rangle R_{B0}}}. \tag{31}$$

Figure 5 gives a plot of the order parameters $S_{\rm A}$ and $S_{\rm B}$ versus $p$ at different $\beta$ for $\langle k\rangle = 3$, 4 and 5.

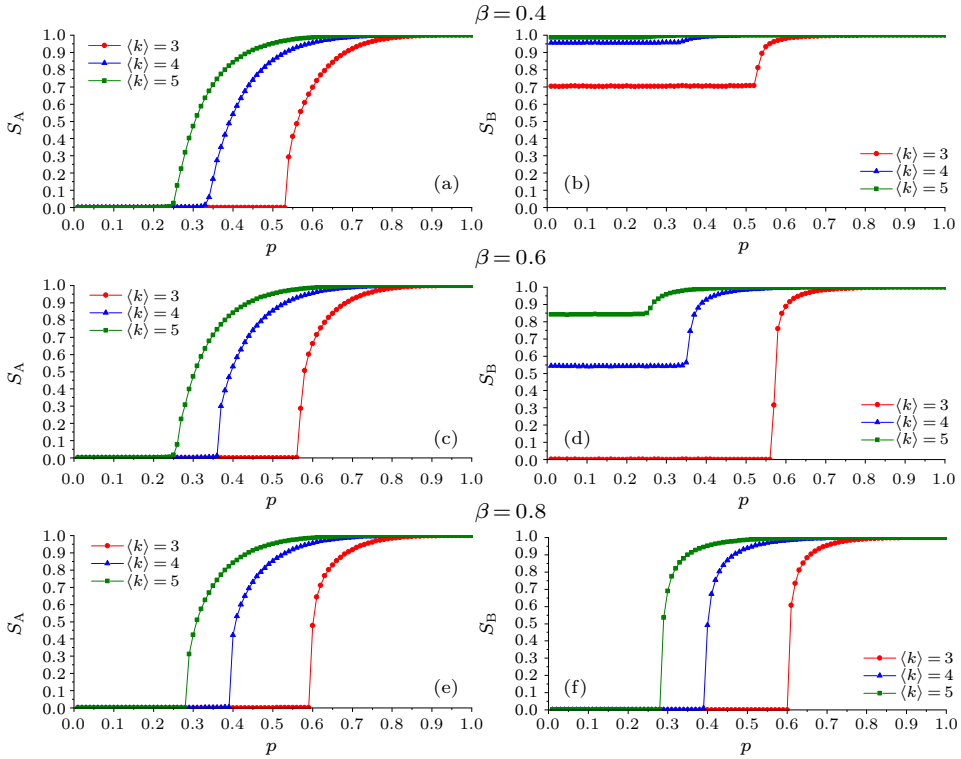



**Fig. 4.** Plots of $S_{\rm A}$ and $S_{\rm B}$ versus $p$ for the RR-RR EINDDL at different $\beta$ and $\langle k\rangle$.

### 4.3. Robustness analysis

#### 4.3.1. Robustness analysis of network A

It can be seen that, no matter whether it is the RR-RR EINDDL or the ER-ER EINDDL, network A gradually converges to the first-order phase transition from the second-order phase transition as $\beta$ increases, as seen in Figs. 4 and 5. Network A will change from the first-order to the hybrid phase transition and then from the hybrid to the second-order phase transition when $\beta$ keeps increasing, as shown in Fig. 6.

This process can be explained by the graphical method. $S_{\rm A}$ is a monotonically increasing function about $R_{\rm A}$ when $k$ and $\beta$ are inevitable, so the value of $R_{\rm A}$ can directly reflect the phase transition. Taking the RR-RR EINDDL $k = 4$ as an example, make the plot of the curve represented by Eqs. (19) and (20) and label the solution of Eq. (19) (i.e., the value of $R_{\rm A}$), as shown in Fig. 7. It can be seen that $R_{\rm A}$ changes from zero to non-zero value continuously as $p$ increases in Fig. 7(a). Therefore, the second-order phase transition occurs in network A. $R_{\rm A}$ jumps from zero to a non-zero value as $p$ increases in Fig. 7(b), and the first-order phase transition occurs in network A. In Fig. 7(c), $R_{\rm A}$ changes continuously from zero to non-zero values and jumps when the two curves are tangent so that network A undergoes the hybrid phase transition. In Fig. 7(d), $R_{\rm A}$ still changes from zero to non-zero continuously as $p$ increases, and $R_{\rm A}$ does not jump because the tangent (intersection) point of the two curves is precisely on the $R_{\rm A}$ axis. Thus, network A undergoes the second-order phase transition.

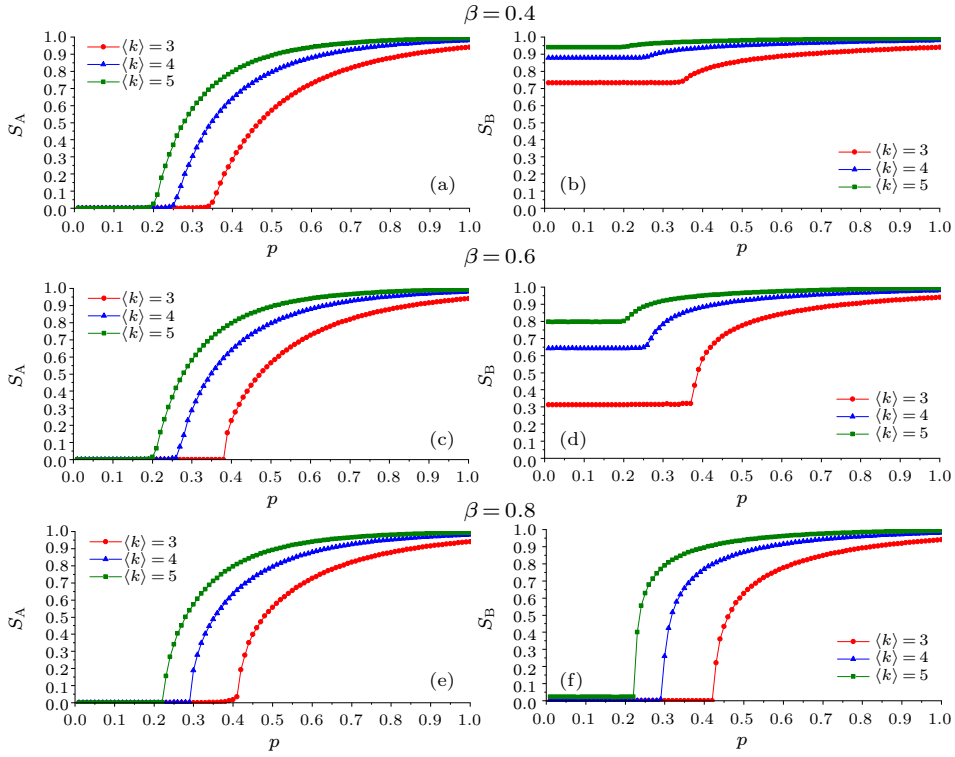



**Fig. 5.** Plots of $S_{\rm A}$ and $S_{\rm B}$ versus $p$ for the ER-ER EINDDL at different $\beta$ and $\langle k\rangle$.

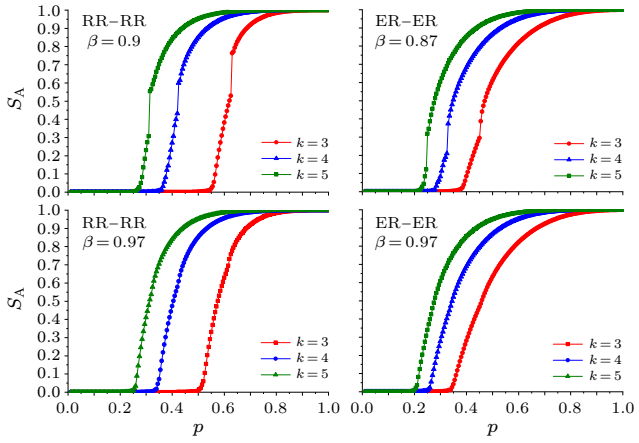



**Fig. 6.** Plots of $S_{\rm A}$ versus $p$ for the EINDDL when $\beta > 0.8$.

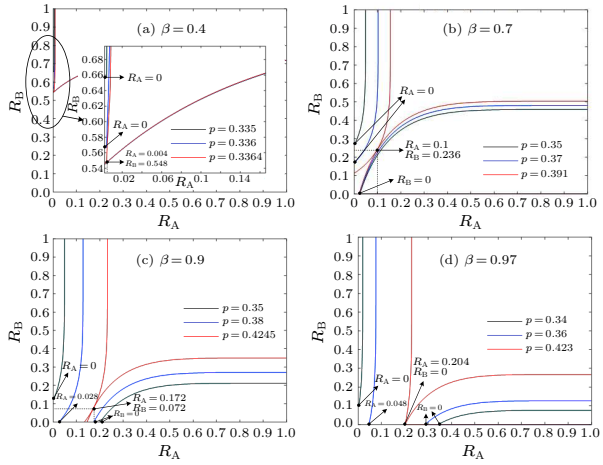



**Fig. 7.** Plots of curves represented by Eqs. (19) and (20).

The relationship between $\beta$ and the phase transition threshold $p_{\rm c}^{\rm A}$ of network A is explored below. The first-order phase transition thresholds $p_{\rm cI}^{\rm A}$ and second-order phase transition thresholds $p_{\rm cII}^{\rm A}$ of the network A can be obtained according to the equations in Subsections 4.1 and 4.2. When the curve represented by Eq. (14) passes the $(0,0)$, the $p$ at this point is the phase transition threshold $p_{\rm cII-H}^{\rm A}$ when $S_{\rm A}$ transitions is from zero to a non-zero value in the hybrid phase transition. When the two curves are tangent, the $p$ at this point is the phase transition threshold $p_{\rm cH-I}^{\rm A}$ when $S_{\rm A}$ jumps in the hybrid transition. The relation between the phase transition threshold $p_{\rm c}^{\rm A}$ and $\beta$ of the network A is shown in Fig. 8, the phase transition types change at the points of $\beta_{\rm c(II-I)}^{\rm A}$, $\beta_{\rm c(I-H)}^{\rm A}$, and $\beta_{\rm c(H-II)}^{\rm A}$.

### 4.3.2. Robustness analysis of network B

From Figs. 4 and 5, it can be found that $S_{\rm B}$ in network B gradually changes from a more significant non-zero value to zero when $p < p_{\rm c}^{\rm A}$ as $\beta$ increases, because network A fails completely when $p < p_{\rm c}^{\rm A}$ and network B is equivalent to the failure of a single network with the initial removal $\beta$ fraction of connectivity links.

Next, we investigate the conditions, on which network B completely collapses. When $S_{\rm B}$ is non-zero, Eq. (20) intersects the $R_{\rm B}$ axis in the plane $(R_{\rm A}, R_{\rm B})$ as shown in Fig. 9(a), where the red line indicates Eq. (19) and the blue line indicates Eq. (20). As $\beta$ gradually increases to a certain value, Eq. (20) gradually goes from intersecting the $R_{\rm B}$ axis to passing the $(0,0)$ point, when the order parameter $S_{\rm B}$ is exactly 0

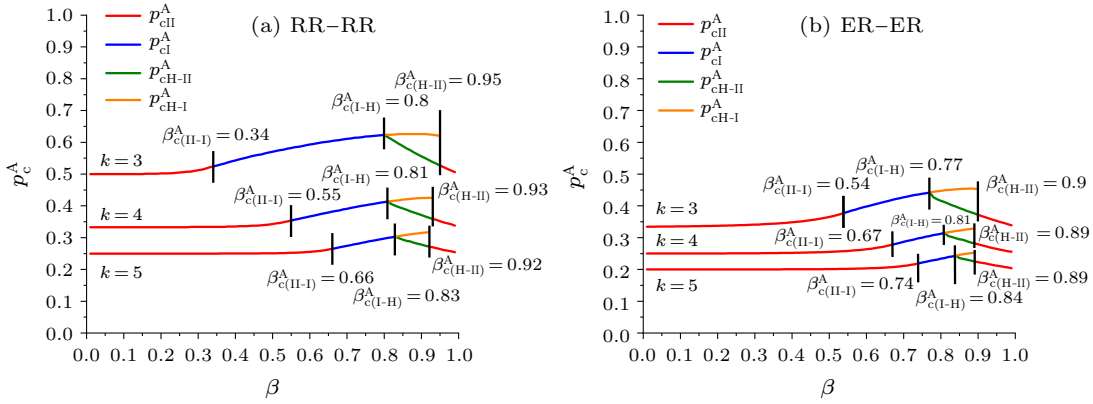



**Fig. 8.** Plots of phase transition threshold $p_c^A$ versus $\beta$ for network A.

at a certain $p$, as shown in Fig. 9(b), we define the $\beta$ and $p$ at this case as the critical values $\beta_c^B$ and $p_c^B$ when network B collapses completely. Subsequently, at $\beta \geq \beta_c^B$ and $p \leq p_c^B$, the value of the order parameter $S_B$ is always 0. Equation (20) no longer intersects with the $R_B$ axis, as shown in Fig. 9(c).

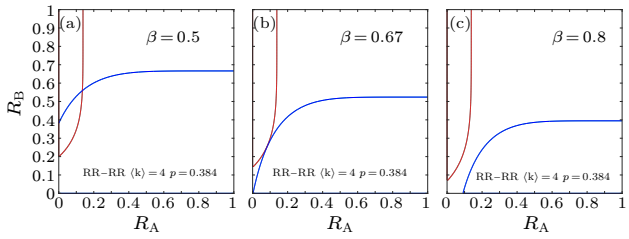



**Fig. 9.** Plots of Eqs. (19) and (20) with identical $p$ and different $\beta$.

It follows that in RR-RR EINDDL, the critical values at the completely collapses of network B are $\beta_c^B = 0.51$, $p_c^B = 0.572$; $\beta_c^B = 0.67$, $p_c^B = 0.384$; $\beta_c^B = 0.76$, $p_c^B = 0.288$ for $\langle k\rangle = 3$, 4, and 5. In the ER-ER EINDDL, the critical values at the complete collapse of network B are $\beta_c^B = 0.67$, $p_c^B = 0.418$; $\beta_c^B = 0.75$, $p_c^B = 0.3$; $\beta_c^B = 0.8$, $p_c^B = 0.236$ for $\langle k\rangle = 3$, 4 and 5.

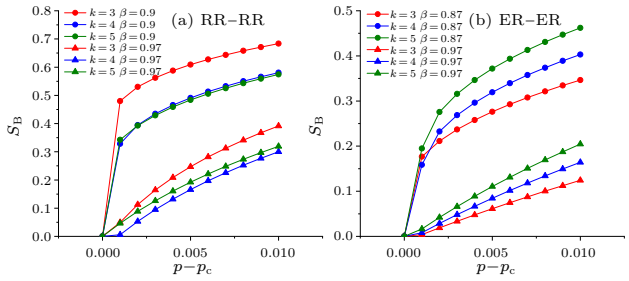



**Fig. 10.** Plots of $S_B$ as a function of $p - p_c$ for the EINDDL when $\beta > 0.8$.

Then we study what type of phase transition occurs when network B is able to completely collapses. The plots of $S_B$ as a function of $p - p_c$ are made when $\beta \geq \beta_c^B$, as shown in Fig. 10. It can be seen that $S_B$ turns from discontinuous to continuous with the increase of $\beta$ from Fig. 10. Hence, as $\beta$ increases, network B turns from the first-order to second-order phase transition when $\beta \geq \beta_c^B$.

The process is similarly explained graphically by labeling the solution of Eq. (20) (i.e., the value of $R_B$) in Figs. 7(b)–7(d). As $\beta$ increases, the value of $R_B$ changes from discontinuously to a non-zero value to continuously to a non-zero value. Hence, network B changes from the first-order to the second-order phase transition. Also, from Fig. 7(d), it can be seen that when the tangent (intersection) point of the two curves is exactly on the $R_A$ axis, the values of $R_A$ and $R_B$ no longer jump, and the second-order phase transition occurs in both network A and network B at this time.

It is clear that the first-order phase transition occurs in network B when $\beta_c^B \leq \beta < \beta_{c(H-II)}^A$, and the second-order phase transition occurs in network B when $\beta \geq \beta_{c(H-II)}^A$ from the above study.

### 4.3.3. The prevention strategy

From Subsections 4.3.1 and 4.3.2, it is clear that the robustness of both network A and network B become weaker and then stronger as $\beta$ increases from 0 to 1 with the phase transition point of view. The mechanism of this phenomenon is the way of spreading the damages influenced by $\beta$. To illustrate this, we define $F$ as the ratio of failed links of network $A(B)$ spreading to network $B(A)$ to the total number of links of cascading failure in the second and third steps, and make a plot of $F$ versus $\beta$ for EINDDL $k = 4, p = 0.3$, as shown in Fig. 11.

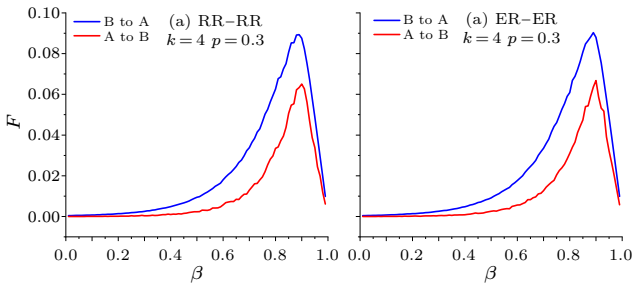



**Fig. 11.** Ratio of failure links of A(B) spreading to B(A) with different $\beta$.

The number of failed links of network A (B) cascade to network B (A) increases and decreases as $\beta$ increases from 0 to 1. The more failed links there are in each cascading failure

step, the more fragile the network is, that is the reason why the robustness of networks A and B decreases and then increases. Therefore, the value of $\beta$ can directly affect the robustness of the EINDDL. Thus the prevention strategy is to find the value of $\beta$ when the EINDDL is most robust.

We use $S_{AB}$ similar to the order parameter $S$ to denote the size of MCGC. Figure 12 shows the plots of $S_{AB}$ as a function of $p$. The robustness of the EINDDL can be measured by the phase transition threshold $p_c$ and the jump value $P_J$ at the transition threshold. In order to find the prevention methods to optimize the network design, we study the relationship between robustness measured by both phase transition threshold $p_c$ and the jump value $P_J$ and $\beta$. As we know, the smaller the $p_c$ and $P_J$ are, the more robust the system is.

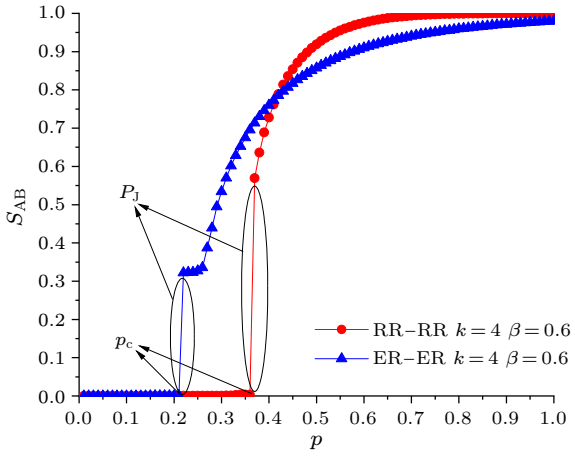



**Fig. 12.** Plots of $S_{AB}$ as a function of $p$ for the EINDDL.

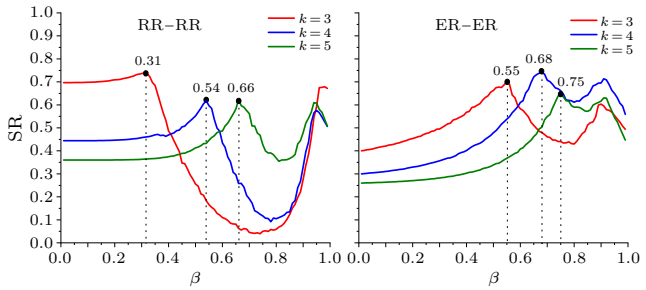



**Fig. 13.** Plots of SR versus $\beta$ for the EINDDL.

For different $\beta$, we get two vectors $p_c'$ and $P_J'$. To evaluate the robustness of networks, we forward transit and normalize the vectors $p_c'$ and $P_J'$ to get $Sp_c$ and $SP_J$, so the comprehensive robustness evaluation metric $SR$ of the EINDDL is expressed in the following, where the weights $\omega_c$ and $\omega_J$ of the two metrics are received by the entropy method.

$$SR = \omega_c \cdot Sp_c + \omega_J \cdot SP_J \tag{32}$$

Due to the forward transition and normalized vectors $p_c'$ and $P_J'$, the larger the $SR$ is, the stronger the networks are. The plots of $SR$ versus $\beta$ for different $k$ is shown in Fig. 13. It can be seen that the robustness of the EINDDL shows an rising-falling-rising trend with the increase of $\beta$, and there exists peaks which reflect the strongest cases. Therefore, to prevent mass cascading failures, the value of $\beta$ should be chosen around the high peaks and away from the low peak.

## 5. Conclusions

In summary, we have established a model for edge-coupled interdependent networks with directed dependency links (EINDDLs), and the directed dependency strength between two networks is determined by $\beta$. The theoretical analysis framework of an EINDDL is developed based on the self-consistent probability method, and the theoretical analysis and simulation of RR-RR and ER-ER networks are carried out to prove the theoretical analysis. The results show that under random attacks, network A exhibits different types of phase transitions (first-order, second-order and hybrid) with the increase of $\beta$, and the critical values of $\beta$ for the whole phase transition type change process of network A are obtained. Network B is rarely affected by cascading failure when $\beta$ is small, and as $\beta$ increases, network B will gradually converge from the first-order to the second-order phase transition. The robustness of both network A and network B becomes weaker and then stronger as $\beta$ increases from 0 to 1 with the phase transition point of view. Finally, by measuring the robustness of the EINDDL with the phase transition threshold $p_c$ and the jump value $P_J$, we obtain the value of $\beta$ where the robustness of the EINDDL is best, and propose the prevention strategy.

## Acknowledgments

Project supported by the National Natural Science Foundation of China (Grant Nos. 61973118, 51741902, 11761033, 12075088, and 11835003), Project in JiangXi Province Department of Science and Technology (Grant Nos. 20212BBE51010 and 20182BCB22009), and the Natural Science Foundation of Zhejiang Province (Grant No. Y22F035316).

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