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Trajectory equation of a lump before and after collision with other waves for generalized Hirota–Satsuma–Ito equation

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Based on the Hirota bilinear and long wave limit methods, the hybrid solutions of m-lump with n-soliton and n-breather wave for generalized Hirota–Satsuma–Ito (GHSI) equation are constructed. Then, by approximating solutions of the GHSI equation along some parallel orbits at infinity, the trajectory equation of a lump wave before and after collisions with n-soliton and n-breather wave are studied, and the expressions of phase shift for lump wave before and after collisions are given. Furthermore, it is revealed that collisions between the lump wave and other waves are elastic, the corresponding collision diagrams are used to further explain.

Keywords: Hirota bilinear method, long wave limit, hybrid solutions, trajectory equation

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1. Introduction

Nonlinear partial differential equations (NLPDEs) play an important role in fluid mechanics, plasma, quantum field theory, and so on.[1–3] The exact solution of NLPDEs can describe various physical properties exhibited by the equation, therefore, the search for effective methods of solving NLPDEs has received a lot of attention. Until now, many useful methods have been proposed, such as Darboux transformation,[4,5] Bäcklund transformation,[7] inverse scattering transformation,[8] variable separation method,[9,10] Hirota bilinear method,[11–13] etc. Among them, Hirota bilinear method is most studied by many scholars owing to its simplicity and directness. In Refs. [14–19], the lump solutions to many NLPDEs were presented through Hirota bilinear method. Lump wave is a special rational function wave, which is localized in all directions of space. Manakov *et al*. gave the analytical expression of the lump solution when he studied the rational solution of the KP equation[20] firstly. In recent years, the interaction between lumps and solitons, breathers, rogue waves, etc. has become a research focus in the soliton theory. Li *et al*. investigated the high-order lumps, breathers, and hybrid solutions of the (2+1)-dimensional fifth-order KdV equation.[21] Wang constructed lump and interaction solutions to the (2+1)-dimensional Burgers equation in Ref. [22]. Zhou *et al*. studied n-lump and interaction solutions of localized waves to the (2+1)-dimensional generalized KDKK equation in Ref. [23]. In Ref. [24], Zhang *et al*. provided the trajectory equation and phase expression of a lump before and after collisions with line and breather waves based on the hybrid solutions of the KP equation. In Ref. [25], Zhang *et al*. derived a new class of nonlinear superposition between lump waves and other waves of KPI equation. Xia *et al*. discussed the trajectory equation of a lump before and after collisions with n-line and n-breather waves for (2+1)-dimensional Sawada–Kotera equation in Ref. [26].

In this paper, we will study the following (2+1)-dimensional GHSI equation

$$\rho_1(3(v_xv_t)_x+v_{xxxt})+\rho_2v_{yt}+\rho_3v_{xx}+\rho_4v_{xt}=0, \tag{1}$$

where ρ_i ($i=1,\ldots,4$) are arbitrary constants. Equation (1) is an important model in shallow water wave theory, which is proposed by Hirota and Satsuma via a Bäcklund transformation of the Boussinesq equation, it is an extension of the Hirota–Satsuma shallow water wave equation. In Refs. [27–31], the authors have studied Eq. (1) from different perspectives, mainly including multi-soliton solution, lump solution, breather solution, hybrid solutions, etc. However, there is no relevant research on the trajectory equation before and after the interactions of the GHSI equation with other nonlinear waves until now. This will be the main work of our paper.

The rest of this paper is organized as follows: In Section 2, the n-soliton solution of Eq. (1) is constructed via the Hirota bilinear method. In Section 3, by employing the long wave limit method, the hybrid solutions consisting of m-lump and n-soliton wave, m-lump, and n-breather wave for Eq. (1) are obtained. In Section 4, the trajectory equation and phase

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change of the lump wave before and after collisions with n-soliton and n-breather wave are given. The last section contains a discussion and summary.

2. The N-soliton solution for generalized Hirota–Satsuma–Ito equation

Through the dependent variable transformation

$$v = 2(\ln\tau)_x, \tag{2}$$

equation (1) can be converted into the following bilinear equation:

$$(\rho_1 D_x^3 D_t + \rho_2 D_y D_t + \rho_3 D_x^2 + \rho_4 D_x D_t)\tau\cdot\tau = 0, \tag{3}$$

where $\tau = \tau(x,y,t)$, the D is the bilinear differential operator defined by

$$\begin{aligned} &D_x^r D_t^s \tau\cdot f \\ &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^r \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^s \tau(x,t) f(x',t')\big|_{x'=x,t'=t}. \end{aligned} \tag{4}$$

That is, if τ is a solution to Eq. (3), then $v = 2(\ln\tau)_x$ is a solution to Eq. (1).

Based on the Hirota bilinear method and transformation (2), the n-soliton solution of Eq. (1) is

$$\begin{aligned} &\tau = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N}\mu_i\zeta_i + \sum_{i<j}^{N}\mu_i\mu_j H_{ij}\right), \\ &\zeta_i = p_i x + q_i y + \omega_i t + \varphi_i, \\ &\omega_i = -\frac{\rho_3 p_i^2}{\rho_1 p_i^3 + \rho_4 p_i + \rho_2 q_i}, \end{aligned} \tag{5}$$

$$\begin{aligned} \exp(H_{ij}) =\ & (\rho_1^2 p_i^2 p_j^2 (p_i^2 - p_i p_j + p_j^2)(p_i - p_j)^2 \\ &+ \rho_1 p_i p_j (p_i - p_j)(p_i(3\rho_4 p_i p_j + \rho_2 p_i q_j - 3\rho_4 p_j^2) \\ &+ \rho_2 p_j (2p_i q_i - 2p_i q_j - p_j q_i)) \\ &+ \rho_2^2 (p_i q_j - p_j q_i)^2)/(\rho_1^2 p_i^2 p_j^2 (p_i^2 + p_i p_j + p_j^2) \\ &\times (p_i + p_j)^2 + \rho_1 p_i p_j (p_i + p_j)(p_i \\ &\times (3\rho_4 p_i p_j + \rho_2 p_i q_j + 3\rho_4 p_j^2) \\ &+ \rho_2 p_j (2p_i q_i + 2p_i q_j + p_j q_i)) \\ &+ \rho_2^2 (p_i q_j - p_j q_i)^2), \end{aligned}$$

where p_i, q_i, ω_i, and φ_i are arbitrary constants, $\sum_{\mu=0,1}$ represents the sum of all possible combinations of $\mu_i = 0,1$ $(i = 1,\ldots,N)$.

3. The M-lump solution and hybrid solutions for generalized Hirota–Satsuma–Ito equation

The long wave limit method is an effective method to construct lump solution for nonlinear differential equations. In this section, we will explore this method to the GHSI equation.

3.1. The M-lump solution for generalized Hirota–Satsuma–Ito equation

Taking the long wave limit for the n-soliton solution (5)

$$\begin{aligned} &q_i = Q_i\varepsilon, \quad p_i = P_i\varepsilon, \ \varepsilon\to 0, \\ &\varphi_i = I\pi, \quad i = 1,2,\ldots,N, \quad N = 2M, \quad P_1 = \overline{P_2}, \\ &P_3 = \overline{P_4}, \quad \ldots, \quad P_{2M-1} = \overline{P_{2M}}, \\ &Q_1 = \overline{Q_2}, \quad Q_3 = \overline{Q_4}, \ldots, \quad Q_{2M-1} = \overline{Q_{2M}}, \end{aligned} \tag{6}$$

we can get the m-lump solution of Eq. (1) as

$$\begin{aligned} v =\ & 2(\ln\tau)_x, \\ \tau =\ & \prod_{i=1}^{N}\theta_i + \frac{1}{2}\sum_{i,j}^{N} K_{ij}\prod_{l\neq i,j}^{N}\theta_l + \cdots + \frac{1}{M!2^M} \\ &\times \sum_{i,j,\ldots,m,n}^{N} \overbrace{K_{ij}K_{kl}\cdots K_{mn}}^{M} \prod_{p\neq i,j,\ldots,m,n}\theta_p + \cdots, \end{aligned} \tag{7}$$

with

$$\begin{cases} \theta_i = P_i x + Q_i y - \dfrac{\rho_3 P_i^2 t}{\rho_4 P_i + \rho_2 Q_i}, \\ K_{ij} = -\dfrac{6\rho_1 P_i^2 P_j^2 (2\rho_4 P_i P_j + \rho_2 P_i Q_j + \rho_2 P_j Q_i)}{\rho_2^2 (P_i Q_j - P_j Q_i)^2}. \end{cases} \tag{8}$$

When $N = 2, M = 1$ for Eq. (6) and Eq. (7), the following 1-lump solution can be derived:

$$v = 2(\ln\tau)_x, \quad \tau = \theta_1\theta_2 + K_{12}. \tag{9}$$

When $N = 4$, $M = 2$, the 2-lump solution of Eq. (1) is

$$\begin{aligned} v =\ & 2(\ln\tau)_x, \\ \tau =\ & \theta_1\theta_2\theta_3\theta_4 + K_{12}\theta_3\theta_4 + K_{13}\theta_2\theta_4 + K_{14}\theta_2\theta_3 \\ &+ K_{23}\theta_1\theta_4 + K_{24}\theta_1\theta_3 + K_{34}\theta_1\theta_2 \\ &+ K_{12}K_{34} + K_{13}K_{24} + K_{14}K_{23}. \end{aligned} \tag{10}$$

3.2. Hybrid solutions for generalized Hirota–Satsuma–Ito equation

When we take the following partial long wave limit on Eq. (5), that is constraints

$$\begin{aligned} &N = 2M + n, \quad q_i = Q_i\varepsilon, \quad p_i = P_i\varepsilon, \quad \varepsilon\to 0, \\ &\varphi_i = I\pi, \quad i = 1,2,\ldots,2M, \quad P_1 = \overline{P_2}, \\ &P_3 = \overline{P_4}, \ldots, P_{2M-1} = \overline{P_{2M}}, \\ &Q_1 = \overline{Q_2}, \quad Q_3 = \overline{Q_4}, \ldots, \quad Q_{2M-1} = \overline{Q_{2M}}, \end{aligned} \tag{11}$$

the hybrid solution between m-lump and n-soliton of Eq. (1) can be derived.

For example, when $m = 1, n = 1$,

$$\begin{aligned} v =\ & 2(\ln\tau)_x, \\ \tau =\ & (K_{13}K_{23} + K_{13}\theta_2 + K_{23}\theta_1 + \theta_1\theta_2 + K_{12})e^{\zeta_3} \\ &+ K_{12} + \theta_1\theta_2, \end{aligned} \tag{12}$$

equation (12) denotes the hybrid solutions of 1-lump and 1-solitons, where

$$K_{ij}=\begin{cases}\dfrac{-6\rho_1P_i^2P_j^2(2\rho_4P_iP_j+\rho_2P_iQ_j+\rho_2P_jQ_i)}{\rho_2^2(P_iQ_j-P_jQ_i)^2}, & (j<3),\\ \dfrac{-6\rho_1P_i^2p_j^2(\rho_1P_ip_j^3+2\rho_4P_ip_j+\rho_2P_iq_j+\rho_2Q_ip_j)}{\rho_1^2P_i^2p_j^6+3\rho_1\rho_4P_i^2p_j^4+2\rho_1\rho_2P_i^2p_j^3q_j+\rho_1\rho_2P_iQ_ip_j^4+\rho_2^2P_i^2q_j^2-2\rho_2^2P_iQ_ip_jq_j+\rho_2^2Q_i^2p_j^2}, & (j\geq 3).\end{cases} \tag{13}$$

When $m=1$, $n=2$, we can yield the hybrid solutions of 1-lump and 2-solitons as

$$\begin{aligned}v &= 2(\ln\tau)_x,\\ \tau &= \mathrm{e}^{H_{34}}(K_{13}K_{23}+K_{13}K_{24}+K_{13}\theta_2+K_{14}K_{23}\\ &\quad+K_{14}K_{24}+K_{14}\theta_2+K_{23}\theta_1+K_{24}\theta_1+\theta_1\theta_2+K_{12})\mathrm{e}^{\zeta_3+\zeta_4}\\ &\quad+(K_{13}K_{23}+K_{13}\theta_2+K_{23}\theta_1+\theta_1\theta_2+K_{12})\mathrm{e}^{\zeta_3}\\ &\quad+(K_{14}K_{24}+K_{14}\theta_2+K_{24}\theta_1\\ &\quad+\theta_1\theta_2+K_{12})\mathrm{e}^{\zeta_4}+\theta_1\theta_2+K_{12},\end{aligned} \tag{14}$$

where K_{ij} satisfies Eq. (13).

Similarly, taking the following partial long wave limit on n-soliton solution (5):

$$\begin{aligned}&N=2(M+n),\quad q_i=Q_i\varepsilon,\quad p_i=P_i\varepsilon,\quad \varepsilon\to 0,\\ &\varphi_i=I\pi,\quad i=1,2,\ldots,2M,\quad P_1=\overline{P_2},\\ &P_3=\overline{P_4},\ldots,\quad P_{2M-1}=\overline{P_{2M}},\\ &Q_1=\overline{Q_2},\quad Q_3=\overline{Q_4},\ldots,\quad Q_{2M-1}=\overline{Q_{2M}},\\ &p_{2M+1}=\overline{p_{2M+2}},\ldots,\quad p_{2M+2n-1}=\overline{p_{2M+2n}},\\ &q_{2M+1}=\overline{q_{2M+2}},\ldots,\quad q_{2M+2n-1}=\overline{q_{2M+2n}},\\ &\varphi_{2M+1}=\overline{\varphi_{2M+2}},\ldots,\quad \varphi_{2M+2n-1}=\overline{\varphi_{2M+2n}},\end{aligned} \tag{15}$$

the hybrid solutions between m-lump and n-breather wave of Eq. (1) can be obtained.

For example, when $m=1,n=1$, the hybrid solutions of 1-lump and 1-breather are

$$\begin{aligned}v &= 2(\ln\tau)_x,\\ \tau &= \mathrm{e}^{H_{34}+\zeta_3+\zeta_4}(K_{13}K_{23}+K_{13}K_{24}+K_{13}\theta_2\\ &\quad+K_{14}K_{23}+K_{14}K_{24}+K_{14}\theta_2+K_{23}\theta_1\\ &\quad+K_{24}\theta_1+\theta_1\theta_2+K_{12})\\ &\quad+\mathrm{e}^{\zeta_3}(K_{13}K_{23}+K_{13}\theta_2+K_{23}\theta_1+\theta_1\theta_2+K_{12})\\ &\quad+\mathrm{e}^{\zeta_4}(K_{14}K_{24}+K_{14}\theta_2+K_{24}\theta_1+\theta_1\theta_2+K_{12})\\ &\quad+\theta_1\theta_2+K_{12}.\end{aligned} \tag{16}$$

It is easy to found that although equations (14) and (16) have the same expression form, they represent completely different hybrid solutions due to different constraints conditions (11) and (15).

4. Trajectory equation of a lump before and after collision with other waves for generalized Hirota–Satsuma–Ito equation

According to Eq. (7), the 1-lump solution of the (2+1)-dimensional GHSI equation can be expressed as

$$v=2(\ln(\theta_1\theta_2+K_{12}))_x, \tag{17}$$

where θ_i and K_{ij} are given by Eq. (8). Through the extreme value theorem of binary functions, we can obtain the parameter equation of the wave peak and valley of the 1-lump solution for Eq. (1) as follows:

$$\begin{aligned}x &= \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\quad+\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)},\\ y &= -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)},\end{aligned} \tag{18}$$

$$\begin{aligned}x &= \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\quad-\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)},\\ y &= -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}.\end{aligned} \tag{19}$$

the parametric equations (18) and (19) can be converted into the following general form

$$\begin{aligned}y &= -\frac{\rho_2P_1P_2x}{\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1}\\ &\quad+\frac{\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}P_1P_2}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)},\end{aligned} \tag{20}$$

$$\begin{aligned}y &= -\frac{\rho_2P_1P_2x}{\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1}\\ &\quad-\frac{\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}P_1P_2}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}.\end{aligned} \tag{21}$$

From Eqs. (20) and (21), we know that the lump wave moves along a straight line, and the wave speed is

$$v=\sqrt{\frac{\rho_3^2(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)^2}{(\rho_4P_2+\rho_2Q_2)^2(\rho_4P_1+\rho_2Q_1)^2}+\frac{\rho_2^2\rho_3^2P_1^2P_2^2}{(\rho_4P_2+\rho_2Q_2)^2(\rho_4P_1+\rho_2Q_1)^2}}. \tag{22}$$

The height of peak and the depth of valley are respectively

$$h=-\frac{\rho_2(P_1Q_2-P_2Q_1)\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{3\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}, \tag{23}$$

$$d=\frac{\rho_2(P_1Q_2-P_2Q_1)\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{3\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}. \tag{24}$$

It is well known that the research on the motion trajectory of the lump wave usually refers to study its peak motion. In order to better understand the motion behavior of the 1-lump wave, we select the parameters $\rho_1=2$, $\rho_2=2$, $\rho_3=3$, $\rho_4=3$, $P_1=1+(3/4)I$, $P_2=1-(3/4)I$, $Q_1=-1/2+I$, $Q_2=-(1/2)-I$, the following image simulations are shown in Fig. 1.

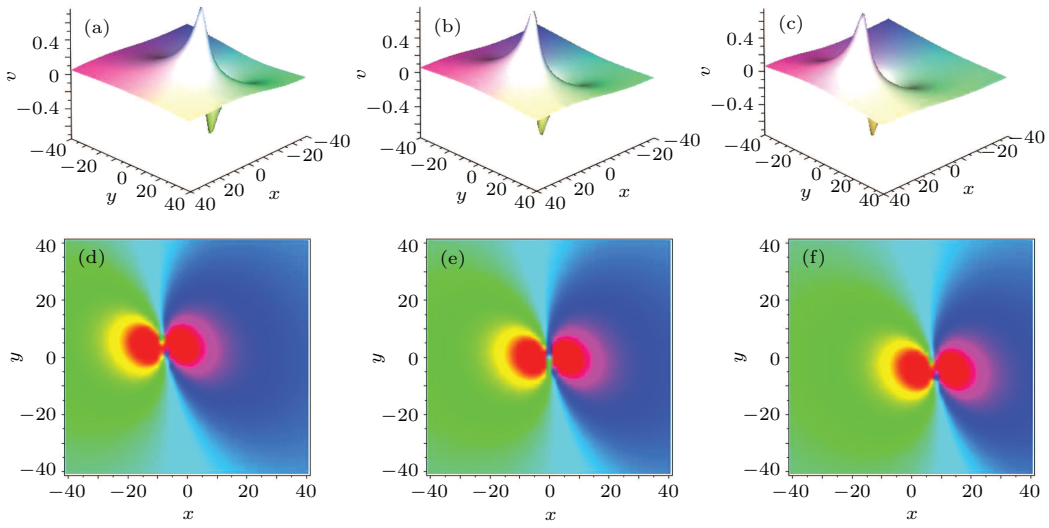


Fig. 1. Evolution behaviors of the 1-lump solution for Eq. (1) at $t=-10$, 0, 10. (a)–(c) Three-dimensional plots; (d)–(f) density plots.

4.1. Trajectory equation of 1-lump before and after collisions with n-soliton

In this subsection, with the help of the motion trajectory of a single lump, we mainly study the trajectory equation of 1-lump before and after collisions with n-soliton. In order to give a concise form of the trajectory equation before and after collisions, we refer to the following two function tokens:

$$h_-(x)=\begin{cases}1, & (x<0),\\ 0, & (x\geq 0),\end{cases}\quad h_+(x)=\begin{cases}0, & (x\leq 0),\\ 1, & (x>0).\end{cases} \tag{25}$$

4.1.1. Trajectory equation of 1-lump before and after collisions with 1-soliton

To proceed, with the help of expression (12), we propose the trajectory equation of 1-lump before and after collisions with 1-soliton for Eq. (1) by Theorem 1, and the related proof is given.

Theorem 1 If the hybrid solution between 1-lump and 1-soliton for Eq. (1) can be represented as Eq. (12), and

$$\begin{aligned}\lambda_3 =& -\frac{\rho_3p_3^2}{\rho_1p_3^3+\rho_4p_3+\rho_2q_3}\\ &+\frac{\rho_3p_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &-\frac{\rho_2\rho_3P_1P_2q_3}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\neq 0,\end{aligned} \tag{26}$$

then, before and after the collisions of 1-lump and 1-soliton, the trajectory equations for the lump wave are

$$\begin{aligned}x_- =& \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\pm\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)}\\ &+h_-(\lambda_3)\beta_3,\\ y_- =& -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}+h_-(\lambda_3)\gamma_3,\\ x_+ =& \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\pm\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)}\\ &+h_+(\lambda_3)\beta_3,\\ y_+ =& -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}+h_+(\lambda_3)\gamma_3,\end{aligned} \tag{27}$$

with

$$\beta_i=\frac{Q_1K_{2i}-Q_2K_{1i}}{P_1Q_2-P_2Q_1},\quad \gamma_i=\frac{P_2K_{1i}-P_1K_{2i}}{P_1Q_2-P_2Q_1},\quad (i=3) \tag{28}$$

the phase change of the lump wave is

$$\Delta=\mathrm{sign}(\lambda_3)\frac{P_2^2K_{13}(\rho_4P_1+\rho_2Q_1)-P_1^2K_{23}(\rho_4P_2+\rho_2Q_2)}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}. \tag{29}$$

However, before and after the collisions, the height and depth of the lump wave do not change, they still satisfy Eqs. (23) and (24). Where sign(x) is a symbolic function, x_-, y_- and x_+, y_+ are the trajectory equations of the lump wave before and after collisions with solitons respectively.

Proof Applying the following constraints:

$$\begin{aligned} x =& \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} \\ &\pm\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)}+c_1, \end{aligned}$$

$$y = -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}+c_2, \tag{30}$$

to the second equation of Eq. (12), it can be transformed into

$$\begin{aligned} \tau =& \chi_1\chi_2+K_{12}+\exp(\lambda_3t+\eta_3) \\ &\times(K_{13}K_{23}\pm\chi_2K_{13}\pm\chi_1K_{23}+\chi_1\chi_2+K_{12}), \end{aligned} \tag{31}$$

where

$$\begin{aligned} \chi_i =& \frac{\pm(c_1P_i+c_2Q_i)\rho_2(P_1Q_2-P_2Q_1)+\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}P_i}{\rho_2(P_1Q_2-P_2Q_1)}, \\ \eta_i =& \left(\pm\frac{\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)}+c_1\right)p_i+c_2q_i+\varphi_i. \end{aligned} \tag{32}$$

For Eq. (31), we first discuss the case of $\lambda_3>0$.

When $\lambda_3>0$ and $t\to-\infty$, equation (31) can be reduced to the following form:

$$\tau_-=\chi_1\chi_2+K_{12}. \tag{33}$$

When $t\to\infty$, due to the invariance of the solution $v=2(\ln\tau)_x$, we multiply the expression $\exp(-\lambda_3t-\eta_3)$ to the right side of Eq. (31) will get

$$\tau_+=K_{13}K_{23}\pm\chi_2K_{13}\pm\chi_1K_{23}+\chi_1\chi_2+K_{12}. \tag{34}$$

Solving Eq. (30), c_1 and c_2 can be obtained below:

$$\begin{aligned} c_1 =& -\frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t-(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)x}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\mp\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)}, \\ c_2 =& \frac{\rho_2\rho_3P_1P_2t+(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)y}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}. \end{aligned} \tag{35}$$

Substituting Eq. (35) into Eqs. (33) and (34) leads to

$$\begin{aligned} \tau_- &= \theta_1\theta_2+K_{12}, \\ \tau_+ &= K_{13}K_{23}+\theta_2K_{13}+\theta_1K_{23}+\theta_1\theta_2+K_{12}. \end{aligned} \tag{36}$$

Here, both τ_- and τ_+ satisfy the bilinear equation (3). Therefore, before and after collisions, the trajectory equations of the lump wave can be written as

$$\begin{aligned} x_- =& \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} \\ &\pm\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)}, \\ y_- =& -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}, \\ x_+ =& \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} \\ &\pm\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)} \\ &+\frac{Q_1K_{23}-Q_2K_{13}}{P_1Q_2-P_2Q_1}, \end{aligned}$$

$$\begin{aligned} y_+ =& -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} \\ &+\frac{P_2K_{13}-P_1K_{23}}{P_1Q_2-P_2Q_1}. \end{aligned} \tag{37}$$

The general forms of Eq. (37) are

$$\begin{aligned} y_- =& -\frac{\rho_2P_1P_2x_-}{\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1} \\ &\pm\frac{\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}P_1P_2}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}, \\ y_+ =& -\frac{\rho_2P_1P_2x_+}{\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1} \\ &\pm\frac{\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}P_1P_2}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)} \\ &+\frac{P_2^2K_{13}(\rho_4P_1+\rho_2Q_1)-P_1^2K_{23}(\rho_4P_2+\rho_2Q_2)}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}. \end{aligned} \tag{38}$$

Substituting x_-, y_- and x_+, y_+ into the solution $v=2(\ln\tau)_x$, it can be verified that the height and depth of the lump wave remain unchanged. For the case of $\lambda_3<0$, the proof process is similar to the case of $\lambda_3>0$, it is omitted here.

From Theorem 1, we find that the collision between 1-lump and 1-soliton is elastic when $\lambda_3 \neq 0$. In order to better characterize Theorem 1, let the parameters $\rho_1 = 2$, $\rho_2 = 2$, $\rho_3 = 3$, $\rho_4 = 3$, $P_1 = -1/2$, $P_2 = -1/2$, $Q_1 = I$, $Q_2 = -I$, $p_3 = 1/3$, $q_3 = -5/6$, $\varphi_3 = 0$, the corresponding diagrams plotted in Fig. 2.

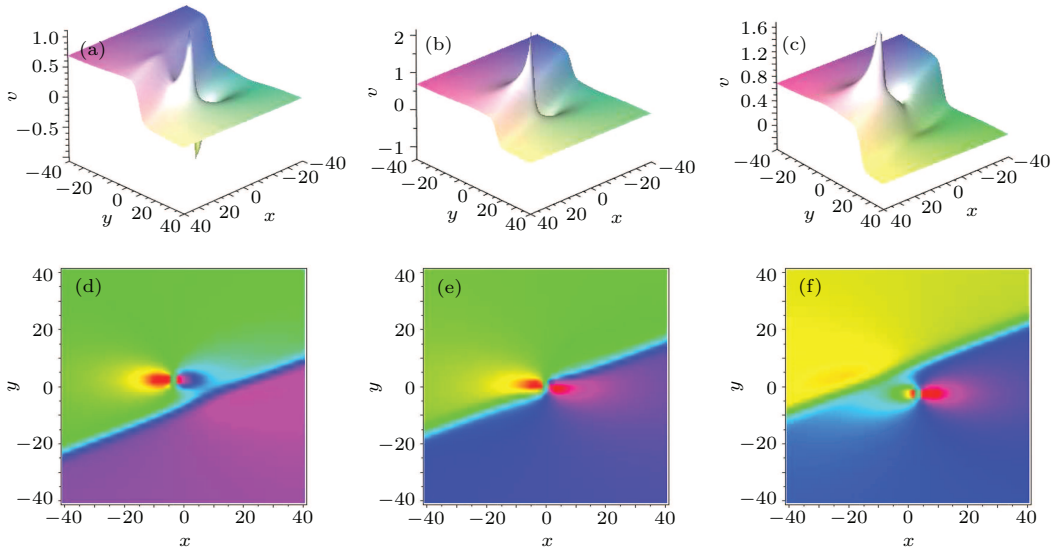


Fig. 2. Evolution behaviors of hybrid solution of 1-lump and 1-soliton for Eq. (1) at $t = -10, 0, 10$: (a)–(c) three-dimensional (3D) plots; (d)–(f) density plots.

4.1.2. Trajectory equation of 1-lump before and after collisions with 2-soliton

In this part, the trajectory equation of 1-lump before and after the collisions with 2-soliton is discussed from the 4-soliton solution, and the trajectory equation of 1-lump colliding with n-soliton is given.

Theorem 2 If the hybrid solution between 1-lump and 2-solitons for Eq. (1) can be represented as Eq. (14), and

$$\begin{aligned}\lambda_i =& -\frac{\rho_3 p_i^2}{\rho_1 p_i^3 + \rho_4 p_i + \rho_2 q_i} \\ &+ \frac{\rho_3 p_i(\rho_4 P_1 P_2 + \rho_2 P_1 Q_2 + \rho_2 P_2 Q_1)}{(\rho_4 P_2 + \rho_2 Q_2)(\rho_4 P_1 + \rho_2 Q_1)} \\ &- \frac{\rho_2 \rho_3 P_1 P_2 q_i}{(\rho_4 P_2 + \rho_2 Q_2)(\rho_4 P_1 + \rho_2 Q_1)} \neq 0, \quad (i = 3, 4), \end{aligned} \tag{39}$$

then, before and after collisions, the trajectory equations of the lump wave are given below:

$$\begin{aligned} x_- =& \frac{\rho_3(\rho_4 P_1 P_2 + \rho_2 P_1 Q_2 + \rho_2 P_2 Q_1)t}{(\rho_4 P_2 + \rho_2 Q_2)(\rho_4 P_1 + \rho_2 Q_1)} \\ &\pm \frac{\sqrt{-6\rho_1 P_2 P_1(2\rho_4 P_1 P_2 + \rho_2 P_1 Q_2 + \rho_2 P_2 Q_1)}}{\rho_2(P_1 Q_2 - P_2 Q_1)} \\ &+ \sum_{i=3}^{4} h_-(\lambda_i)\beta_i, \\ y_- =& -\frac{\rho_2 \rho_3 P_1 P_2 t}{(\rho_4 P_2 + \rho_2 Q_2)(\rho_4 P_1 + \rho_2 Q_1)} + \sum_{i=3}^{4} h_-(\lambda_i)\gamma_i, \\ x_+ =& \frac{\rho_3(\rho_4 P_1 P_2 + \rho_2 P_1 Q_2 + \rho_2 P_2 Q_1)t}{(\rho_4 P_2 + \rho_2 Q_2)(\rho_4 P_1 + \rho_2 Q_1)} \\ &\pm \frac{\sqrt{-6\rho_1 P_2 P_1(2\rho_4 P_1 P_2 + \rho_2 P_1 Q_2 + \rho_2 P_2 Q_1)}}{\rho_2(P_1 Q_2 - P_2 Q_1)} \\ &+ \sum_{i=3}^{4} h_+(\lambda_i)\beta_i, \\ y_+ =& -\frac{\rho_2 \rho_3 P_1 P_2 t}{(\rho_4 P_2 + \rho_2 Q_2)(\rho_4 P_1 + \rho_2 Q_1)} + \sum_{i=3}^{4} h_+(\lambda_i)\gamma_i, \end{aligned} \tag{40}$$

where β_i and γ_i are determined by Eq. (28). Before and after collisions, the phase change of the lump is

$$\begin{aligned} \Delta =& \sum_{i=3}^{4} \mathrm{sign}(\lambda_i)\sigma_i, \\ \sigma_i =& \frac{P_2^2 K_{1i}(\rho_4 P_1 + \rho_2 Q_1) - P_1^2 K_{2i}(\rho_4 P_2 + \rho_2 Q_2)}{(P_1 Q_2 - P_2 Q_1)(\rho_4 P_1 P_2 + \rho_2 P_1 Q_2 + \rho_2 P_2 Q_1)}. \end{aligned} \tag{41}$$

But the height and depth of the lump do not change and are consistent with Eqs. (23) and (24).

Proof By employing the condition (30), the second equation of Eq. (14) can be converted into

$$\begin{aligned} \tau =& (K_{13}K_{23} + K_{13}K_{24} \pm K_{13}\chi_2 + K_{14}K_{23} \\ &+ K_{14}K_{24} \pm K_{14}\chi_2 \pm K_{23}\chi_1 \pm K_{24}\chi_1 \\ &+ \chi_1\chi_2 + K_{12})\,\mathrm{e}^{H_{34}+\lambda_3 t+\lambda_4 t+\eta_3+\eta_4} \\ &+ (K_{13}K_{23} \pm K_{13}\chi_2 \pm K_{23}\chi_1 + \chi_1\chi_2 + K_{12})\,\mathrm{e}^{\lambda_3 t+\eta_3} \\ &+ (K_{14}K_{24} \pm K_{14}\chi_2 \pm K_{24}\chi_1 + \chi_1\chi_2 + K_{12}) \\ &\times \mathrm{e}^{\lambda_4 t+\eta_4} + \chi_1\chi_2 + K_{12}, \end{aligned} \tag{42}$$

where χ_i and η_i satisfy Eq. (32).

For Eq. (42), we need to consider the following cases $\lambda_3 > 0$, $\lambda_4 > 0$; $\lambda_3 > 0$, $\lambda_4 < 0$; $\lambda_3 < 0$, $\lambda_4 > 0$, and $\lambda_3 < 0$, $\lambda_4 < 0$. Here, we only discuss the first two cases, and the others can be discussed in the same way.

(i) When $\lambda_3>0,\ \lambda_4>0$, according to the method in Theorem 1, when $t\to-\infty$ and $t\to\infty$, equation (42) can be written as

$$\begin{aligned}\tau_- &= \chi_1\chi_2+K_{12},\\ \tau_+ &= K_{13}K_{23}+K_{13}K_{24}\pm K_{13}\chi_2+K_{14}K_{23}\\ &\quad +K_{14}K_{24}\pm K_{14}\chi_2\pm K_{23}\chi_1\\ &\quad \pm K_{24}\chi_1+\chi_1\chi_2+K_{12},\end{aligned}\tag{43}$$

the trajectory equations of the lump wave before and after collisions are given below:

$$\begin{aligned}x_- &= \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\quad \pm\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)},\\ y_- &= -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)},\\ x_+ &= \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\quad \pm\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)}\\ &\quad +\frac{Q_1(K_{23}+K_{24})-Q_2(K_{13}+K_{14})}{P_1Q_2-P_2Q_1},\\ y_+ &= -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\quad +\frac{P_2(K_{13}+K_{14})-P_1(K_{23}+K_{24})}{P_1Q_2-P_2Q_1}.\end{aligned}\tag{44}$$

The general forms of Eq. (44) are as follows:

$$\begin{aligned}y_- &= -\frac{\rho_2P_1P_2x_-}{\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1}\\ &\quad \pm\frac{\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}P_1P_2}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)},\\ y_+ &= -\frac{\rho_2P_1P_2x_+}{\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1}\\ &\quad \pm\frac{\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}P_1P_2}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}\\ &\quad +\frac{K_{13}P_2^2(\rho_4P_1+\rho_2Q_1)-K_{23}P_1^2(\rho_4P_2+\rho_2Q_2)}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}\\ &\quad +\frac{K_{14}P_2^2(\rho_4P_1+\rho_2Q_1)-K_{24}P_1^2(\rho_4P_2+\rho_2Q_2)}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}.\end{aligned}\tag{45}$$

By Eq. (45), we can yield the value of phase change in Eq. (41). Substituting Eq. (44) into $v=2(\ln\tau)_x$, the height and depth of the lump wave do not change.

(ii) When $\lambda_3>0,\ \lambda_4<0$, we have

$$\begin{aligned}\tau_- &= K_{14}K_{24}\pm K_{14}\chi_2\pm K_{24}\chi_1+\chi_1\chi_2+K_{12},\\ \tau_+ &= K_{13}K_{23}\pm K_{13}\chi_2\pm K_{23}\chi_1+\chi_1\chi_2+K_{12},\\ x_- &= \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\quad \pm\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)}\\ &\quad +\frac{Q_1K_{24}-Q_2K_{14}}{P_1Q_2-P_2Q_1},\\ y_- &= -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\quad +\frac{P_2K_{14}-P_1K_{24}}{P_1Q_2-P_2Q_1},\\ x_+ &= \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\quad \pm\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)}\\ &\quad +\frac{Q_1K_{23}-Q_2K_{13}}{P_1Q_2-P_2Q_1},\\ y_+ &= -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\quad +\frac{P_2K_{13}-P_1K_{23}}{P_1Q_2-P_2Q_1}.\end{aligned}\tag{47}$$

The general equations corresponding to Eq. (47) are

$$\begin{aligned}y_- &= -\frac{\rho_2P_1P_2x_-}{\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1}\\ &\quad \pm\frac{\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}P_1P_2}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}\\ &\quad +\frac{K_{14}P_2^2(\rho_4P_1+\rho_2Q_1)-K_{24}P_1^2(\rho_4P_2+\rho_2Q_2)}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)},\\ y_+ &= -\frac{\rho_2P_1P_2x_+}{\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1}\\ &\quad \pm\frac{\sqrt{-6\rho_1P_1P_2(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}P_1P_2}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}\\ &\quad +\frac{K_{13}P_2^2(\rho_4P_1+\rho_2Q_1)-K_{23}P_1^2(\rho_4P_2+\rho_2Q_2)}{(P_1Q_2-P_2Q_1)(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}.\end{aligned}\tag{48}$$

The values of Eq. (41) of the phase change can be obtained by Eq. (48). Substituting Eq. (47) into $v=2(\ln\tau)_x$, we can verify that the height and depth of the lump wave do not change.

To better explain the Theorem 2, by taking a set of parameters $\rho_1=2$, $\rho_2=2$, $\rho_3=3$, $\rho_4=3$, $P_1=I/2$, $P_2=-I/2$, $Q_1=1+I/4$, $Q_2=1-I/4$, $p_3=1/4$, $p_4=1/7$, $q_3=-3/4$, $q_4=-2$, $\varphi_3=0$, $\varphi_4=0$, the collision diagrams of a lump wave and two-kink soliton can be given in Fig. 3.

By further generalizing Theorem 1 and Theorem 2, the trajectory equations before and after collisions of 1-lump and n-solitons can be obtained below.

Theorem 3 If $\lambda_i\neq0$, $(i=3,\ 4,\ \ldots,\ 2+n)$, then before and after 1-lump wave collidings with n-solitons, the trajectory equations of the lump wave are as follows:

$$\begin{aligned}x_- &= \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}\\ &\quad \pm\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)}\\ &\quad +\sum_{i=3}^{2+n}h_-(\lambda_i)\beta_i,\\ y_- &= -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}+\sum_{i=3}^{2+n}h_-(\lambda_i)\gamma_i,\end{aligned}$$

$$x_+ = \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} \pm \frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)} + \sum_{i=3}^{2+n} h_+(\lambda_i)\beta_i,$$

$$y_+ = -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} + \sum_{i=3}^{2+n} h_+(\lambda_i)\gamma_i. \quad (49)$$

The phase shift of lump wave is

$$\Delta = \sum_{i=3}^{2+n} \mathrm{sign}(\lambda_i)\sigma_i. \quad (50)$$

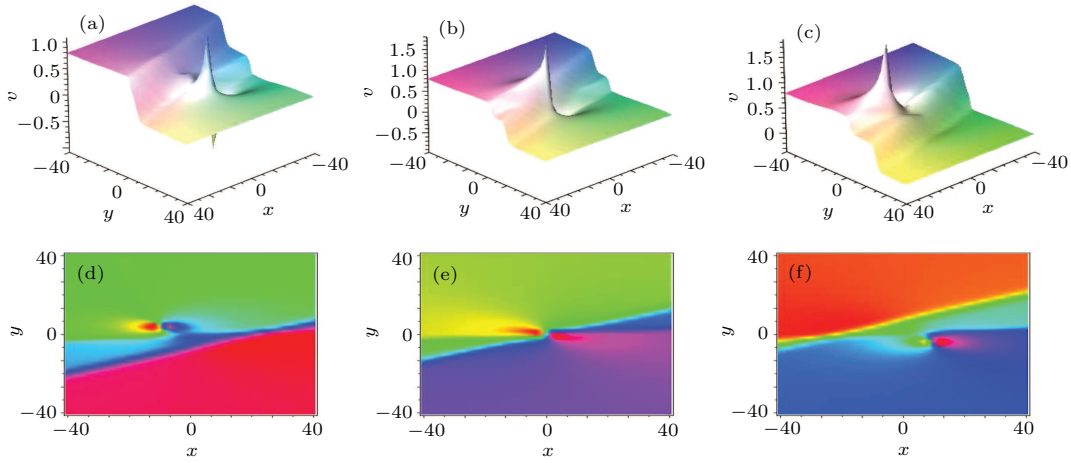


Fig. 3. Evolution behaviors of hybrid solution of 1-lump and 2-soliton for Eq. (1) at $t=-30, 0, 30$: (a)–(c) the 3D plots; (d)–(f) density plots.

4.2. Trajectory equation of 1-lump before and after collisions with *n*-breather

In this subsection, the motion trajectories and phase expressions of the lump wave in the hybrid solutions of 1-lump and 1-breather, 1-lump and n-breather will be investigated.

Theorem 4 If $\Re(\lambda_3),\ \Re(\lambda_4)\neq 0$, then the trajectory equations of the collisions between 1-lump and 1-breather are

$$x_- = \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} \pm \frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)} + \sum_{i=3}^{4} h_-(\Re(\lambda_i))\beta_i,$$

$$y_- = -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} + \sum_{i=3}^{4} h_-(\Re(\lambda_i))\gamma_i,$$

$$x_+ = \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} \pm \frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)} + \sum_{i=3}^{4} h_+(\Re(\lambda_i))\beta_i,$$

$$y_+ = -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} + \sum_{i=3}^{4} h_+(R(\lambda_i))\gamma_i. \quad (51)$$

and the phase shift of the lump wave is

$$\Delta = \sum_{i=3}^{4} \mathrm{sign}(\Re(\lambda_i))\sigma_i. \quad (52)$$

Comparing Theorem 2 and Theorem 4, their proof procedures are similar except for different constraint conditions, so the proof process will not be given here. In order to better describe the motion trajectory of 1-lump and 1-breather before and after collisions, the collision diagrams can be given in Fig. 4 by selecting parameters $\rho_1=2$, $\rho_2=2$, $\rho_3=3$, $\rho_4=3$, $P_1=3+I$, $P_2=3-I$, $Q_1=1+6I$, $Q_2=1-6I$, $p_3=3I$, $p_4=-3I$, $q_3=5/2$, $q_4=5/2$, $\varphi_3=0$, $\varphi_4=0$.

Next, we further extend Theorem 4 to the case of 1-lump collidings with n-breather. The specific content is shown in the following Theorem 5.

Theorem 5 If $\Re(\lambda_i)\neq 0, (i=3,4,\ldots,2+2n)$, then, before and after collisions of a lump and n-breather the trajectory equations of the lump wave are

$$x_- = \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} \pm \frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)} + \sum_{i=3}^{2+2n} h_-(\Re(\lambda_i))\beta_i,$$

$$y_- = -\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)} + \sum_{i=3}^{2+2n} h_-(\Re(\lambda_i))\gamma_i,$$

$$x_+ = \frac{\rho_3(\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}$$

$$\pm\frac{\sqrt{-6\rho_1P_2P_1(2\rho_4P_1P_2+\rho_2P_1Q_2+\rho_2P_2Q_1)}}{\rho_2(P_1Q_2-P_2Q_1)}+\sum_{i=3}^{2+2n}h_+(\Re(\lambda_i))\beta_i,$$

$$y_+=-\frac{\rho_2\rho_3P_1P_2t}{(\rho_4P_2+\rho_2Q_2)(\rho_4P_1+\rho_2Q_1)}+\sum_{i=3}^{2+2n}h_+(R(\lambda_i))\gamma_i, \tag{53}$$

and the phase shift of the lump wave is

$$\Delta=\sum_{i=3}^{2+2n}\mathrm{sign}(\Re(\lambda_i))\sigma_i. \tag{54}$$

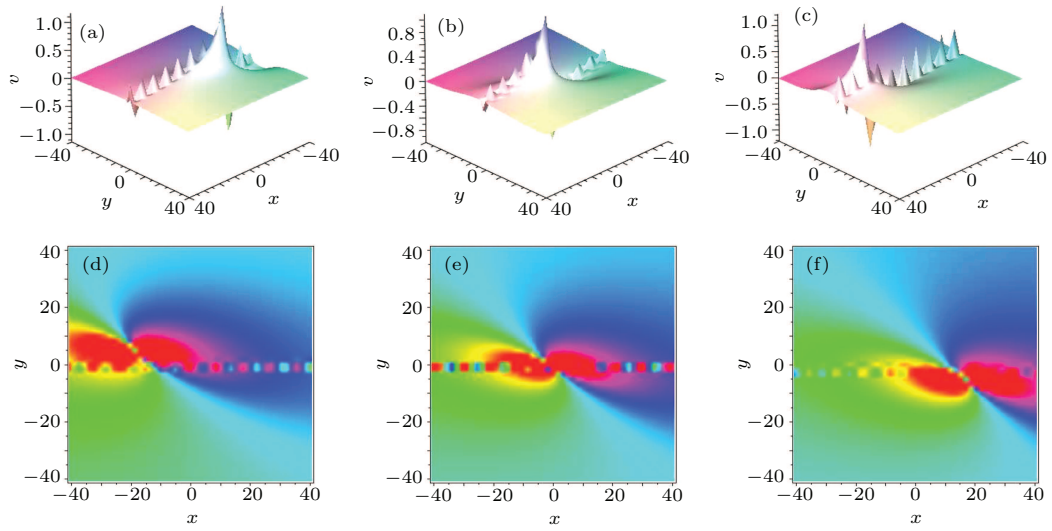


Fig. 4. Evolution behaviors of hybrid solution of 1-lump and 1-breather for Eq. (1) at $t=-30, 0, 30$: (a)–(c) the 3D plots; (d)–(f) density plots.

5. Summary and discussion

In this paper, the trajectory equations of a lump before and after collisions with other waves for generalized Hirota–Satsuma–Ito equation are studied. Firstly, we start from the multi-soliton solution of the (2+1)-dimensional GHSI equation, many different types of mixed solutions of the GHSI equation are derived, such as the interaction between the lump solution and the line-soliton solution, the lump solution, and the breather solution. Secondly, under the constraint conditions $\lambda_i\neq 0$, $\Re(\lambda_i)\neq 0$ the trajectory equations of the lump wave before and after collisions with n-soliton (Theorm 3) and n-breather (Theorm 5) are proposed, and we have proved that the collision between lump and other waves are elastic, and the height and depth of the lump wave do not change. However, what does it mean when $\lambda_i=0$, $\Re(\lambda_i)=0$? It will be shown in our subsequent research.

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