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The Reissner–Nordström black hole surrounded by quintessence may not be destroyed

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In the study of weak cosmic censorship conjectures (WCCC), some research finds that the Reissner–Nordström black hole might be destroyed by a test particle with particular mass and charge under some conditions, which means that the naked singularity of the black hole could be observed. This is not allowed in WCCC. We have never observed such naked singularities which should not exist in theory, so we need to find a proper way to protect the black hole from being destroyed by such particles. In this paper, we study a Reissner–Nordström black hole that is surrounded by quintessence (RN-Q) and find that the black hole would be stable and safe because of the effective potential barrier induced by the quintessence term. This result may also show in a sense that the quintessence might have more potential value.

Keywords: weak cosmic censorship conjectures (WCCC), black hole, quintessence, naked singularity

PACS: 04.70.Bw, 04.20.Dw, 97.60.Lf **DOI:** 10.1088/1674-1056/acd620

1. Introduction

Penrose proposed the weak cosmic censorship conjectures (WCCC), which mainly assume that the singularity of a black hole is always located inside the event horizon.[1] One could not observe the naked singularity in the universe unless it comes out from the event horizon or the event horizon disappears. This is not allowed in theory. Many papers have studied this important issue.[2–15] Some of them find that by throwing a test particle with particular mass and charge into a black hole, under some idealized conditions such as ignoring the radiation and self-force effects, might cause the disappearance of the event horizon, which would mean that the naked singularity could be observed.[8–12]

For instance, in Ref. [16], the Reissner–Nordström (RN) black hole could be destroyed by a test particle with particular mass and charge. Yet no naked singularity has ever been observed. Although the authors said that they ignore the radiation and self-force effects, which might protect the black hole from being destroyed, we would like to study the protection scheme in a different way.

As is well known, the expansion of our universe is accelerating, which is supported by some cosmological observations such as those of Type Ia supernovae, the anisotropy of the cosmic microwave background radiation and some x-ray experiments.[17–19] Astrophysicists and cosmologists consider a missing energy component with negative pressure called 'dark energy' to be responsible for this accelerating expansion. There are many different candidates for dark energy,[20] such as the cosmological constant,[21] quintessence,[22] chameleons,[23] k-essence,[24] tachyons,[24] phantoms,[26] dilatons[27] and quintoms.[28,29] The differences between these models relate to the magnitude of ω_q, which is the ratio of pressure to energy density of dark energy. Black holes surrounded by dark energy are believed to play a crucial role in cosmology.

Of those candidates for the dark energy model, quintessence is defined as an ordinary scalar field coupled to gravity, and by considering Einstein's field equations for a black hole, charged or not, and surrounded by quintessence, a new solution can be derived related to ω_q.[30–35] As an ordinary scalar field, the effect on the metric is simpler than for other models. The metric is defined as a scalar field with the equations of a negative state parameter, which is the ratio of pressure and density. Kiselev[35] studied the RN black hole surrounded by quintessence and described some of its properties. Thus, we would like to study a RN black hole surrounded by quintessence to check the WCCC.

This paper is organized as follows. In Section 2, we will calculate the proper mass and charge of the test particle to destroy the black hole, which may cause the disappearance of the event horizon. In Section 3 we will examine the effective potential that might prevent the test particle from falling in. A summary of the work will be given in Section 4.

2. The proper mass and charge of the test particle

For a RN black hole that is surrounded by quintessence (RN-Q), the metric is given as[35]

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2, \quad (1)$$

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with $f(r)=1-\frac{2M}{r}+\frac{Q^2}{r^2}-\frac{c}{r^{3\omega_q+1}}$, and the last term $-\frac{c}{r^{3\omega_q+1}}$ is the quintessence term, where M is the mass of the black hole, Q is its charge, ω_q is the quintessential state parameter, which has the range $-1<\omega_q<-1/3$, c is the positive normalization factor related to $\rho_q=-\frac{c}{2}\frac{3\omega_q}{r^{3(\omega_q+1)}}$ and ρ_q is the density of quintessence, which is always positive.

If $c=0$, Eq. (1) will reduce to the RN black hole, which contains the two solutions $r_\pm=M\pm\sqrt{M^2-Q^2}$. According to Refs. [36–38], c is the normalization factor related to the density of quintessence around the black hole. Meanwhile, in Ref. [38], the authors showed that the parameter c in the quintessence term should be smaller than 0.001, which means that the contribution of the quintessence term to the RN metric could be regarded as a perturbation. Actually, c may be much smaller than 0.001 and hence the influence on gravitational perturbation could be neglected. Thus, we could get the radii of RN-Q black hole $R_\pm$ for a very small value of c as

$$R_\pm=r_\pm+\varepsilon_\pm, \tag{2}$$

where $\varepsilon_\pm$ take the form

$$\varepsilon_\pm\simeq\frac{cr_\pm^{1-3\omega_q}}{r_\pm-r_\mp}=\frac{c(M\pm\sqrt{M^2-Q^2})^{1-3\omega_q}}{2\sqrt{M^2-Q^2}}. \tag{3}$$

Then the two horizons of the RN-Q black hole would be

$$R_+=M+\sqrt{M^2-Q^2}+\frac{c(M+\sqrt{M^2-Q^2})^{1-3\omega_q}}{2\sqrt{M^2-Q^2}}, \tag{4}$$

$$R_-=M-\sqrt{M^2-Q^2}-\frac{c(M-\sqrt{M^2-Q^2})^{1-3\omega_q}}{2\sqrt{M^2-Q^2}}. \tag{5}$$

For simplicity, we choose $\omega_q=-2/3$ in this paper and set $c=0.0005<0.001$ based on Ref. [38]. The choice of $c=0.0005$ could make the two horizons much closer to each other in order to achieve the near-extreme condition. Then we have

$$f(r)=1-\frac{2M}{r}+\frac{Q^2}{r^2}-cr, \tag{6}$$

and

$$R_+=M+\sqrt{M^2-Q^2}+\frac{0.00025(M+\sqrt{M^2-Q^2})^3}{\sqrt{M^2-Q^2}},$$
$$R_-=M-\sqrt{M^2-Q^2}-\frac{0.00025(M-\sqrt{M^2-Q^2})^3}{\sqrt{M^2-Q^2}}. \tag{7}$$

By solving the equation $r_+=r_-$, one could get the extreme condition of the RN black hole as $M=Q$. However, for the RN-Q black hole, we could not get the extreme condition anymore. This means that the extreme RN-Q black hole does not exist rather than that the RN black hole. So we could not study the extreme RN-Q black hole here.

However, because of the small value of c in Eq. (7), the two horizons R_- and R_+ would be very close to each other when M is small enough. We can see this result in Fig. 1. Based on this point, we could regard the black hole as a near-extreme black hole when M is small enough.

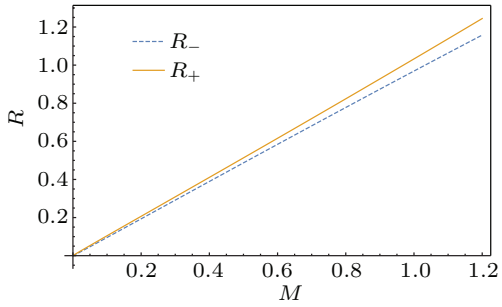


Fig. 1. Plot of R_- and R_+ with mass M. R_+ is just a little bigger than R_- when M is small enough.

For a near-extreme RN-Q black hole, we could assume $Q=M-\delta$, where $\delta\ll M$. This choice ensures that R_- and R_+ could both be valid and be close to each other. Then Eq. (7) could be written as

$$\begin{aligned}R_+ &= M+\sqrt{M^2-(M-\delta)^2}\\ &\quad+\frac{0.00025(M+\sqrt{M^2-(M-\delta)^2})^3}{\sqrt{M^2-(M-\delta)^2}},\\ R_- &= M-\sqrt{M^2-(M-\delta)^2}\\ &\quad-\frac{0.00025(M-\sqrt{M^2-(M-\delta)^2})^3}{\sqrt{M^2-(M-\delta)^2}}.\end{aligned} \tag{8}$$

Now we could set

$$R_{\rm ne}=R_+=M+\sqrt{M^2-Q^2}+\frac{0.00025(M+\sqrt{M^2-Q^2})^3}{\sqrt{M^2-Q^2}}.$$

Let us consider a freely falling test particle that moves from infinity to the black hole with mass $m=E$ and charge q. From Ref. [3], in order to reach the horizon of a black hole, the energy of the test particle should at least satisfy

$$E=\frac{-g_{t\phi}L-qA_tg_{\phi\phi}+qA_\phi g_{t\phi}}{g_{\phi\phi}}\pm\frac{1}{g_{\phi\phi}}\sqrt{(g_{t\phi}^2-g_{\phi\phi}g_{tt})[L^2-2qLA_\phi+q^2A_\phi^2+m^2g_{\phi\phi}(1+g_{rr}\dot{r}^2+g_{\theta\theta}\dot{\theta}^2)]}, \tag{9}$$

where L is the angular momentum of the particle and A_t and A_ϕ are electromagnetic potentials. For the RN-Q black hole, we have[39]

$$\begin{cases} g_{tt} = -f(r), \\ g_{rr} = f(r)^{-1}, \\ g_{\theta\theta} = r^2, \\ g_{\phi\phi} = r^2 \sin^2\theta, \\ g_{t\phi} = 0, \\ A_t = Q/r, \\ A_\phi = 0. \end{cases} \tag{10}$$

By putting Eq. (10) into Eq. (9) and calculating the energy at the event horizon $R_{\rm ne}$, one could get the minimum energy of the test particle as

$$\begin{aligned} E_{\rm min} &= qQ/R_{\rm ne} \\ &= \frac{q(M-\delta)}{M+\sqrt{M^2-(M-\delta)^2}+\frac{0.00025(M+\sqrt{M^2-(M-\delta)^2})^3}{\sqrt{M^2-(M-\delta)^2}}}. \end{aligned} \tag{11}$$

We can see that $E_{\rm min}$ will always be positive.

If the black hole absorbs the test particle with mass $m = E$ and charge q, the new horizons will be

$$\begin{aligned} R_{+\rm after} &= (M+E)+\sqrt{(M+E)^2-(Q+q)^2} \\ &\quad + \frac{c[(M+E)+\sqrt{(M+E)^2-(Q+q)^2}]^3}{2\sqrt{(M+E)^2-(Q+q)^2}}, \\ R_{-\rm after} &= (M+E)-\sqrt{(M+E)^2-(Q+q)^2} \\ &\quad - \frac{c[(M+E)-\sqrt{(M+E)^2-(Q+q)^2}]^3}{2\sqrt{(M+E)^2-(Q+q)^2}}. \end{aligned} \tag{12}$$

We can see that if $(M+E)^2-(Q+q)^2 < 0$, both $R_{-\rm after}$ and $R_{+\rm after}$ would vanish, which means the black hole would be destroyed by the test particle. So, in order to destroy the black hole, we need to find the proper mass m and charge q of the test particle.

Since $M+E>0$ and $Q+q>0$, the mass and charge of the test particle need to satisfy $E < Q+q-M$. If we still choose $Q = M-\delta$, then we find

$$E < q-\delta, \tag{13}$$

so the maximum energy of the test particle $E_{\rm max}$ could be regarded as

$$E_{\rm max} = q-\delta. \tag{14}$$

Therefore, as long as $\delta < q$, $E_{\rm max}$ will always be positive. Then we have

$$\begin{aligned} \Delta E &\equiv E_{\rm max} - E_{\rm min} \\ &= q-\delta-\frac{q(M-\delta)}{M+\sqrt{M^2-(M-\delta)^2}+\frac{0.00025(M+\sqrt{M^2-(M-\delta)^2})^3}{\sqrt{M^2-(M-\delta)^2}}}. \end{aligned} \tag{15}$$

If require $\Delta E > 0$, we should choose the proper energy (or mass) of the test particle between $E_{\rm min}$ and $E_{\rm max}$. Based on this requirement, we could get

$$\begin{aligned} q &> \frac{\delta}{1-\frac{M-\delta}{M+\sqrt{M^2-(M-\delta)^2}+\frac{0.00025(M+\sqrt{M^2-(M-\delta)^2})^3}{\sqrt{M^2-(M-\delta)^2}}}} \\ &\equiv q_{\rm min}. \end{aligned} \tag{16}$$

So, we could set $q = q_{\rm min}+\varepsilon$, where ε is a very small positive value. Meanwhile, we also require $\delta < q_{\rm min}+\varepsilon$. One can easily see that as long as Eq. (14) is valid, any positive ε would be satisfied as well.

Thus, we could rewrite the expressions for $E_{\rm min}$ and $E_{\rm max}$. Since the energy of the particle should be between $E_{\rm min}$ and $E_{\rm max}$, so we could set the energy as

$$E_{\rm min} < m = E = \frac{E_{\rm min}+E_{\rm max}}{2} < E_{\rm max}. \tag{17}$$

Then, putting the proper $m = (E_{\rm min}+E_{\rm max})/2$ into Eq. (12), one can find that there are no real roots anymore, which means that the black hole has been destroyed.

To demonstrate this, we could choose some values to test our result. If we set $\delta = 0.0001$, one can get the plot of ΔE with M and q in Fig. 2.

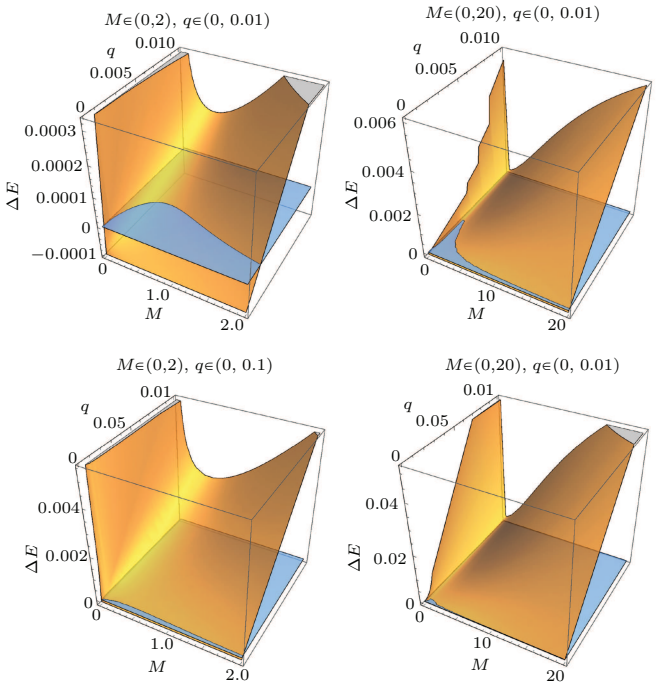


Fig. 2. Plot of ΔE with M and q. The blue surface is $\Delta E = 0$.

In Fig. 2, one could actually choose many values of M and q to ensure $\Delta E > 0$. This means that for a specific black hole of mass M, we could always find a proper q to make sure $\Delta E > 0$. In other words, we could always find a test particle that satisfies the requirements of energy and charge.

For instance, if we set $\delta = 0.0001$ and $M = 1.0001$, then $Q = M-\delta = 1$ and $q = q_{\rm min}+\varepsilon = 0.0031594299+\varepsilon$. Then we have

$$
\begin{cases}
E_{\min} = 0.96834872(0.0031594299+\varepsilon), \\
E_{\max} = 0.0030594299+\varepsilon, \\
\Delta E = 0.0030594299+\varepsilon-0.96834872(0.0031594299+\varepsilon), \\
R_{+} = 1.0326858, \\
R_{-} = 0.96901457, \\
R_{+\text{after}} = \dfrac{0.00025\left(\sqrt{(m+1.0001)^2-(\varepsilon+1.00316)^2}+m+1.0001\right)^3}{\sqrt{(m+1.0001)^2-(\varepsilon+1.00316)^2}}+\sqrt{(m+1.0001)^2-(\varepsilon+1.00316)^2}+m+1.0001, \\
R_{-\text{after}} = -\dfrac{0.00025\left(-\sqrt{(m+1.0001)^2-(\varepsilon+1.00316)^2}+m+1.0001\right)^3}{\sqrt{(m+1.0001)^2-(\varepsilon+1.00316)^2}}-\sqrt{(m+1.0001)^2-(\varepsilon+1.00316)^2}+m+1.0001.
\end{cases}
$$

By solving $\Delta E > 0$, one could get $\varepsilon > -1.28043\times10^{-11}$, which means that any small positive ε would ensure $\Delta E > 0$. The charge q should satisfy $q > 0.00315943$. By choosing several values of ε to test the result, one can get Table 1.

Table 1. Different choices of ε could all cause the horizons to be destroyed.

ε	q	$E_{\min}$	$E_{\max}$	$m=E$	$R_{-\text{after}}$	$R_{+\text{after}}$
0.001	0.0041594299	0.0040277786	0.0040594299	0.0040436042	1.0048998+0.039256682i	1.0048998−0.039256682i
0.002	0.0051594299	0.0049961273	0.0050594299	0.0050277786	1.0058855+0.023843047i	1.0058855−0.023843047i
0.003	0.0061594299	0.005964476	0.0060594299	0.006011953	1.0068711+0.01626755i	1.0068711−0.01626755i
0.004	0.0071594299	0.0069328248	0.0070594299	0.0069961273	1.0078568+0.011313937i	1.0078568−0.011313937i
0.005	0.0081594299	0.0079011735	0.0080594299	0.0079803017	1.0088424+0.0076357775i	1.0088424−0.0076357775i
0.006	0.0091594299	0.0088695222	0.0090594299	0.008964476	1.0098281+0.0047009621i	1.0098281−0.0047009621i

From the results for $R_{-\text{after}}$ and $R_{+\text{after}}$, one can see that we can find the proper test particles to destroy the black hole.

However, there is one additional thing we need to consider, which is to check the effective potential. If the effective potential forms a potential barrier that is higher than the energy of the test particle, the particle might not be able to fall into the black hole and thus could not destroy it.

3. The potential barrier

For an axisymmetric metric, there exist orbits lying entirely in the equatorial plane $\theta = \pi/2$. For such an orbit in the RN-Q metric, one could solve Eq. (9) for $\dot{r}$ and obtain the corresponding effective potential $V_{\text{eff}}(r)$. The effective potential can be regarded as a barrier that lies ahead of the falling particle. The particle cannot fall into the black hole if the effective potential forms a peak that is higher than the energy of the particle.[3] According to Ref. [3], one can express the effective potential $V_{\text{eff}}(r)$ as

$$
V_{\text{eff}}(r) = -\dot{r}^2 = \frac{1}{1+\frac{Q^2}{r^2}-\frac{2M}{r}-cr^{-3(1+\omega_{\text{p}})}} - \frac{(m+qA_t)^2}{m^2}. \tag{18}
$$

If we still choose the same parameters $c = 0.0005$, $\omega_{\text{p}} = -2/3$, $Q = M-\delta$, $q = q_{\min}+\varepsilon$, and set $M = 1.0001$, $\delta = 0.0001$, $Q = M-\delta = 1$ and $q = q_{\min}+\varepsilon = 0.0031594299+\varepsilon$, then we have

$$
V_{\text{eff}}(r) = \frac{1}{\frac{1}{r^2}-\frac{2.0007}{r}+1} - \frac{4\left(\frac{1}{2}(\varepsilon+0.968349(\varepsilon+0.0031594299)+0.00305943)-\frac{(\varepsilon+0.0031594299)^2}{r}\right)^2}{(\varepsilon+0.968349(\varepsilon+0.0031594299)+0.00305943)^2}. \tag{19}
$$

Putting the six different values of ε and m into Eq. (19), we can obtain Fig. 3.

One can see that none of the test particles can fall into the black hole. This means that although we could find test particles with the proper mass and charge, there exists a very sharp barrier that stops them from falling in.

According to Ref. [16], the RN black hole could be destroyed by choosing a proper test particle with the same method. This means that the quintessential term plays an important role, perhaps even a key role, here. This might be an important result of the function of the quintessence term.

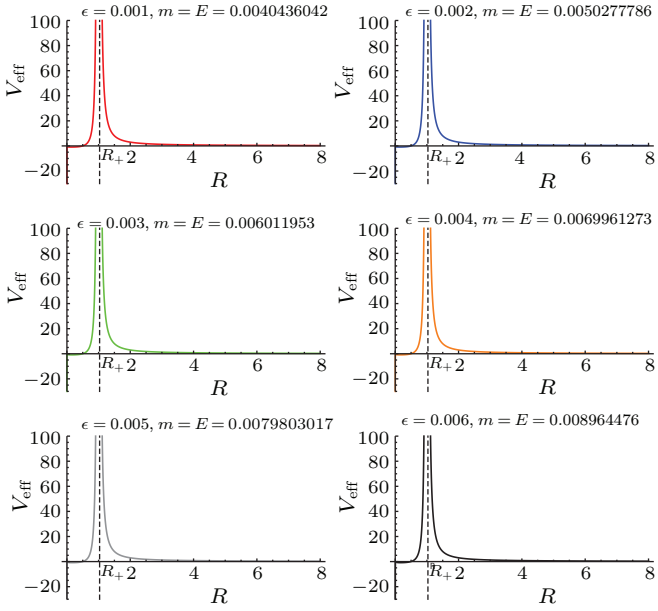


Fig. 3. Plot of V_{eff} with different ε and m. There exists a sharp potential barrier that stops the particle from falling into the black hole.

4. Summary

In summary, by introducing the quintessence term into the RN black hole, we find that a black hole that might otherwise be destroyed may now be 'protected' by the quintessence that surrounds the black hole. This result might show that the quintessence could have more physical meaning than we know of. This is also an interesting effect and result about the WCCC. In order to verify whether this result is universal or not, we would like to study some different black holes in future work.

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