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Quantum correlation enhanced bound of the information exclusion principle

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We investigate the information exclusion principle for multiple measurements with assistance of multiple quantum memories that are well bounded by the upper and lower bounds. The lower bound depends on the observables' complementarity and the complementarity of uncertainty whilst the upper bound includes the complementarity of the observables, quantum discord, and quantum condition entropy. In quantum measurement processing, there exists a relationship between the complementarity of uncertainty and the complementarity of information. In addition, based on the information exclusion principle the complementarity of uncertainty and the shareability of quantum discord can exist as an essential factor to enhance the bounds of each other in the presence of quantum memory.

Keywords: quantum correlation, information exclusion principle, entropic uncertainty relation

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1. Introduction

Quantum information theory is an important tool to deal with all kinds of the quantum problem[1] and is widely used in various quantum information processing tasks, such as quantum key distribution, quantum cryptography, and entanglement witness.[2–20] Measurement plays a vital role in quantum theory. It is the interface between the classical world and the quantum world, which is characterized by noncommutativity and superposition. However, for any given measurement, superposition and noncommutativity necessarily lead to uncertainty in the outcomes of measurements. Simultaneously, it produces another critical quantity, information. From the viewpoint of quantum information, Shannon entropy could be used to quantify the amount of uncertainty about the observables. Deutsch[21] quantified uncertainty in terms of Shannon entropy and derived the entropic uncertainty relation for any pair of observables.[22] It describes the limitations of our ability to simultaneously predict the measurement outcomes of non-commuting observables in quantum theory. Later, Kraus conjectured the improved result.[23] Maassen and Uffink[24] gave the tighter entropic uncertainty relations:

$$H(X)+H(Y)\geqslant -\log_2 c, \tag{1}$$

where $H(X)$ $(H(Y))$ is the Shannon entropy of measurement outcomes when a measurement of observable X (Y) is performed on a state ρ, and $c=\max_{i,j}|\langle x_i|y_j\rangle|^2$ quantifies the complementarity of the non-degenerate observables X and Y with $|x_i\rangle, |y_j\rangle$ denoting their eigenvectors, respectively. Subsequently, the entropy uncertainty relation has been extensively studied.[25–39] Based on the accessible information about a quantum system represented by an ensemble of states, Hall first generalizes the bound of the information exclusion principle.[40] Suppose that for the quantum ensemble $\{p_i,\rho_i\}$, the corresponding quantum state formula is $\rho_\varepsilon=\sum_i p_i\rho_i$, subject to the mutual information corresponding to the measurement of A on the quantum state ρ_ε, then we have $I(A|\varepsilon)=S(A|\rho_\varepsilon)-\sum_i p_i S(A|\rho_i)$, where $S(A|\rho)$ is the associated entropy defined as $S(A|\rho)=-\int \mathrm{d}a p(a|\rho)\log p(a|\rho)$. For observables $\Pi_i,(i=1,2,\ldots,N)$, Hall proposed the general form inequalities for the information exclusion principle

$$\sum_{i=1}^{N} I(\Pi_1|\rho_\varepsilon)\leq J(\Pi_1,\Pi_2,\ldots,\rho_\varepsilon), \tag{2}$$

with the nontrivial quantum bound J. It implies that for any selected observables the information gain can be maximized at the expense of the observables' complementary.

Acting as the best-known information exclusion principle, it has not been studied as much as quantum uncertainty relations.[41,42] Subsequently, Grudka *et al.* considered the observables X and Y that are performed on system A, and Z is a classical register that may be correlated to A. They improved the bound and conjectured a stronger information exclusion relation[43]

$$I(X:Z)+I(Y:Z)\leq \log_2\Big(d\sum_{d\ \text{largest}}|\langle x_i|y_j\rangle|^2\Big), \tag{3}$$

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with the quantum mutual information $I(X(Y):Z)$ and the sum over the largest d terms of the matrix $|\langle x_i|y_j\rangle|^2$. In what follows, we describe a stronger version of the information exclusion principle, proved by Coles and Piani, that can also be extended to the much more general case of quantum memory.[44]

$$I(X:B)+I(Y:B)\leq r-H(A|B), \tag{4}$$

$$r=\min\{r(X,Y),r(Y,X)\}, \tag{5}$$

$$r(\Pi,\tilde{\Pi})=\log_2(d\sum_i \max_j |\langle\pi_i|\tilde{\pi}_j\rangle|^2), \tag{6}$$

where $H(A|B)=H(\rho_{AB})-H(\rho_B)$ is the quantum conditional entropy and $(\Pi,\tilde{\Pi})$ stands for $(X(Y),Y(X))$. In addition, they further explained the distinction between the complementarity of uncertainty and the complementarity of information. Namely, to obtain a state-independent bound for uncertainty relations. It must consider the subspace with the least complementarity whilst concerning the various subspaces that compose the space, i.e., it is the overall complementarity, and it formulates the information exclusion relations. In other words, for uncertainty relations, a trivial bound will be generated whilst the information exclusion principle will be a nontrivial bound. Considering the multiple measurements, Zhang *et al.* gave the quantum memory-assisted information exclusion principle as[45]

$$\sum_{i=1}^{N} I(\Pi_i:B)\leqslant \log_2 d^N-(N-1)H(A|B)-\mathcal{B}_{\mathcal{L}}, \tag{7}$$

where

$$\mathcal{B}_{\mathcal{L}}=\max_{\varepsilon}\{\ell_{\varepsilon}^{U}\}, \tag{8}$$

$$\ell_{\varepsilon}^{U}=-\sum_{\alpha_N} p_{\varepsilon_N^{\alpha_N}}\log\sum_{\alpha_k,N\geqslant k>1}\max_{\alpha_1}\prod_{n=1}^{N-1}|\langle\varepsilon_n^{\alpha_n}|\varepsilon_{n+1}^{\alpha_{n+1}}\rangle|^2, \tag{9}$$

with $p_{\varepsilon_N^{\alpha}}=\mathrm{Tr}(|\varepsilon_N^{\alpha}\rangle\langle\varepsilon_N^{\alpha}|\otimes\mathbf{I})\rho_{AB}$, and $|\varepsilon_i^{\alpha}\rangle$ the αth eigenvector of Π_{ε_i} which can be understood as ith measurement in the ε order. Here ε can be understood as rearranging the measurements $\{\Pi_i,i=1,2,\ldots,N\}$ in a new order. This bound was enhanced by Xiao *et al.* subsequently.[46]

However, the above model focuses on the bipartite quantum state for multiple measurements. It means that one part is the measured subsystem which is measured by multiple measurements, and the other part is the quantum memory which can store measurement information. Thus, it will be interesting to find out the information exclusion principle in multiple quantum states for multiple measurements. Inspired by the ideas of monogamy or decoupling which is close to quantum cryptography, in this paper we would like to split the quantum memory into N parts compared to the bipartite quantum state, and present multiple quantum memory-assisted information exclusion principle for multiple measurements. It is found that the upper bound includes the complementarity of the observables, quantum discord, and quantum condition entropy, while the lower bound consists of the complementarity of the observables and the complementarity of uncertainty. Thus, there exists a relationship between the complementarity of uncertainty and the complementarity of information. In addition, the bounds of the complementarity of uncertainty and the bounds of the quantum discord can exist as essential factors to enhance the bounds of each other.

The paper is organized as follows. Section 2 gives the bounds of the multiple quantum memory-assisted entropic information exclusive principle for multiple measurements and the bounds of the tripartite quantum memory-assisted entropic information exclusive principle for multiple measurements, and in Section 3, the application is given to examine the lower and upper bounds. Finally, the conclusion is drawn.

2. The multiple quantum memory assisted information exclusion principle for multiple measurements

Firstly, we briefly describe an information gain game among $N+1$ participants, Alice (A), and her N friends $(B_i, i\in\mathcal{D}:=\{1,2,\ldots,N\})$. Suppose that Alice's friends prepare a composite quantum state $\rho_{AB_1B_2\ldots B_N}$ in $\otimes^{N+1}d$-dimensional Hilbert space and then send the particle A to Alice. Alice and her friends agree on a group of measurements $\{\Pi_i\}, i\in\mathcal{D}$ with $|i_\alpha\rangle$ denoting αth eigenvector of the Π_i. Subsequently, Alice randomly chooses one measurement Π_i and performs it on herself. She will obtain the corresponding measurement outcomes. Then, she announces her choice to her friend B_i. They try to predict the information gained about Alice's measurement outcomes.

During the processing of measurement, when Alice puts the measurement $\{\Pi_i\}$ on herself, it will destroy the quantum correlation between her and her friend B_i. The quantum correlation (quantum discord) can be defined as the difference between the total correlation and classical correlation, that is,[47,48]

$$\mathcal{Q}_A(\rho_{AB_m})=I(A:B_m)-\mathcal{J}_A(\rho_{AB_m}),\ m\in\mathcal{D}, \tag{10}$$

where $I(A:B_m)=H(\rho_A)+H(\rho_{B_m})-H(\rho_{AB_m})$ is the total correlation, and $\mathcal{J}_A(\rho_{AB_m})$ is the classical correlation that can be defined as

$$\mathcal{J}_A(\rho_{AB_m})=\max_{\{\Omega_i\}} I(\Omega:B_m), \tag{11}$$

with the mutual information $I(\Omega:B_m)=H(\rho_\Omega)+H(\rho_{B_m})-H(\rho_{\Omega B_m})$ of the post-measurement state

$$\rho_{\Omega B_m}=\mathrm{Tr}_{B_1\ldots B_{m-1}B_{m+1}\ldots B_N}\times\Big[\sum_i(|\Omega_i\rangle\langle\Omega_i|\otimes\mathbb{I})\rho_{AB_1\ldots B_N}(|\Omega_i\rangle\langle\Omega_i|\otimes\mathbb{I})\Big].$$

According to these rules of the game, based on the entropic uncertainty relation for multiple measurements, we obtain the multiple quantum memory-assisted information exclusion principle for multiple measurements.

Theorem 1 The subsystem A of the composite quantum state $\rho_{AB_1B_2\ldots B_N}$ is measured by the set of observables $\{\Pi_i\}, i \in \mathcal{D} := \{1,2,\ldots,N\}$. The multiple quantum memory-assisted information exclusion principle for multiple measurements will be restricted by the observables' complementarity, the conditional von Neumann entropy, and the quantum discord.

$$\mathcal{L}_{\mathcal{B}} \leq \sum_{i=1}^{N} H(\Pi_i : B_i) \leq \mathcal{U}_{\mathcal{B}}, \tag{12}$$

where the upper bound and lower bound are, respectively,

$$\mathcal{U}_{\mathcal{B}} = \log_2 d^N - \mathcal{B}_{\mathcal{L}} - \sum_{i=1}^{N} H(A|B_i) - \sum_{i=1}^{N} \mathcal{Q}_A(\rho_{AB_i}) + H(\rho_A), \tag{13}$$

$$\mathcal{L}_{\mathcal{B}} = \mathcal{B}_{\mathcal{L}} + (N-1)H(\rho_A) - \sum_{i=1}^{N} H(\Pi_i|B_i), \tag{14}$$

with $\mathcal{B}_{\mathcal{L}}$ given by Eq. (8).

Proof Based on the definitions of the quantum mutual information $I(A:B) = H(\rho_A) + H(\rho_B) - H(\rho_{AB})$ and the quantum condition entropy $H(A|B) = H(\rho_{AB}) - H(\rho_B)$, one can obtain

$$H(A|B) = H(\rho_A) - I(A:B). \tag{15}$$

Based on the relation (15), the uncertainties of the multiple measurements in the presence of the multiple quantum memory can be expressed as

$$\begin{aligned}
&\sum_{i=1}^{N} H(\Pi_i|B_i) \\
=&\sum_{i=1}^{N} H(\Pi_i) - \sum_{i=1}^{N} I(\Pi_i : B_i) \\
\geq&\sum_{i=1}^{N} H(\Pi_i) - \sum_{i=1}^{N} \mathcal{J}_A(\rho_{AB_i}) \\
\geq&\mathcal{B}_{\mathcal{L}} + (N-1)H(\rho_A) - \sum_{i=1}^{N} \mathcal{J}_A(\rho_{AB_i}) \\
=&\mathcal{B}_{\mathcal{L}} + \sum_{i\in\mathcal{D}-\{k\}}^{N} [H(A|B_i) + I(A:B_i)] - \sum_{i=1}^{N} \mathcal{J}_A(\rho_{AB_i}) \\
=&\mathcal{B}_{\mathcal{L}} + \sum_{i=1}^{N} H(A|B_i) + \sum_{i=1}^{N} \mathcal{Q}_A(\rho_{AB_i}) - H(\rho_A).
\end{aligned} \tag{16}$$

In the last equality, while we add the quantum mutual information $I(A:B_k)$ and delete it, the equality holds.

Regrouping the terms yields

$$\sum_{i=1}^{N} I(\Pi_i : B_i) \leq \sum_{i=1}^{N} H(\Pi_i) + H(\rho_A) - \mathcal{B}_{\mathcal{L}} - \sum_{i=1}^{N} H(A|B_i) - \sum_{i=1}^{N} \mathcal{Q}_A(\rho_{AB_i}). \tag{17}$$

Due to the inequality $H(\Pi) \leq \log_2 d$, the upper bounds of the information exclusion principle $\mathcal{U}_{\mathcal{B}}$ holds.

For the lower bounds of the information exclusion principle, we also substitute the relation (15) into the uncertainties of the multiple measurements in the presence of the multiple quantum memory, it will arrive at

$$\begin{aligned}
\sum_{i=1}^{N} H(\Pi_i|B_i) &= \sum_{i=1}^{N} H(\Pi_i) - \sum_{i=1}^{N} I(\Pi_i : B_i) \\
&\geq \mathcal{B}_{\mathcal{L}} + (N-1)H(\rho_A) - \sum_{i=1}^{N} I(\Pi_i : B_i).
\end{aligned} \tag{18}$$

Rearranging the relation (18), we have

$$\sum_{i=1}^{N} I(\Pi_i : B_i) \geq \mathcal{B}_{\mathcal{L}} + (N-1)H(\rho_A) - \sum_{i=1}^{N} H(\Pi_i|B_i). \tag{19}$$

The proof is finished.

It is worth noting that the information gain will be constrained by the lower and upper bounds. The lower bound of Eq. (12) includes the measurement incompatibility $\mathcal{B}_{\mathcal{L}}$, the total information of the subsystem ρ_A, and the sum of the uncertainty of the measurements $\sum_{i=1}^{N} H(\Pi_i|B_i)$. It is obvious that if the observables measure the quantum state more accurately, the uncertainties of the measurements will be smaller, and the greater amount of information will be gained. Whilst the upper bound of Eq. (12) includes four terms. The first term $\mathcal{B}_{\mathcal{L}}$ describes measurement incompatibility. The second term $\sum_{i=1}^{N} H(A|B_i)$ describes the effects of quantum entanglement on the information exclusion principle and $\sum_{i=1}^{N} \mathcal{Q}_A(\rho_{AB_i})$ that depends on the quantum discord between the measured quantum state A and the unmeasured quantum state B_i, it describes the effects of quantum correlation on the information exclusion principle. It is shown that the quantum correlation restricts the information exclusion principle, namely, if the quantum state has fewer quantum correlations (including quantum discord and quantum entanglement), the quantum state will be closer to the classical state. When the quantum measurements are performed on the quantum state, it gains more information. From the perspective of quantum correlation, it reveals the shareability of quantum discord that can also be restricted by the measurement incompatibility $\mathcal{B}_{\mathcal{L}}$, the sum of the uncertainty of the measurements $\sum_{i=1}^{N} H(\Pi_i|B_i)$ and the conditional entropy $\sum_{i=1}^{N} H(A|B_i)$ which can be given by the following corollary.

Corollary 1 The shareability of quantum discord among different parties of the $N+1$ composite quantum system can be constrained by the entropic uncertainty relation

$$\sum_{i=1}^{N} \mathcal{Q}_A(\rho_{AB_i}) \leq \delta + H(\rho_A), \tag{20}$$

with $\delta=\sum_{i=1}^{N}H(\Pi_i|B_i)-\mathcal{B}_{\mathcal{L}}-\sum_{i=1}^{N}H(A|B_i)$.

For the tripartite pure quantum state ρ_{ABC} or the tripartite mixed quantum state $\rho_{ABC}=|\psi\rangle_{A(BC)^L}\otimes|\psi\rangle_{A(BC)^R}$, the shareability of quantum discord among different parties can be given by

$$\mathcal{Q}_A(\rho_{AB})+\mathcal{Q}_A(\rho_{AC})\leq\tilde{\delta}+\mathcal{Q}_A(\rho_{A:BC}),\tag{21}$$

with $\tilde{\delta}=\mathcal{B}_{\mathcal{L}}+H(X|B)+H(Y|C)-H(A|B)-H(A|C)$.

Proof The proof is evident. We rearrange the elements of inequality in Theorem 1, and it will arrive at Eq. (20). For the tripartite pure quantum state ρ_{ABC} or the tripartite mixed quantum state $\rho_{ABC}=|\psi\rangle_{A(BC)^L}\otimes|\psi\rangle_{A(BC)^R}$, it means that there exist two observables X and Y operated on the subsystem A, the result will be reduced to

$$\begin{aligned}&H(X|B)+H(Y|C)\\&\geq\mathcal{B}_{\mathcal{L}}+H(A|B)+H(A|C)+\mathcal{Q}_A(\rho_{AB})+\mathcal{Q}_A(\rho_{AC})-H(\rho_A).\end{aligned}\tag{22}$$

We can obtain

$$\mathcal{Q}_A(\rho_{AB})+\mathcal{Q}_A(\rho_{AC})\leq\tilde{\delta}+H(\rho_A),\tag{23}$$

with $\tilde{\delta}=H(X|B)+H(Y|C)-H(A|B)-H(A|C)-\mathcal{B}_{\mathcal{L}}$.

Notice that there exists the relation $H(\rho_A)=\mathcal{Q}_A(\rho_{A:BC})$ for all tripartite pure quantum state $|\psi\rangle_{ABC}$. For the mixed quantum state ρ_{ABC}, there exists a factorization $\mathcal{H}_{BC}=\mathcal{H}_{(BC)^L}\otimes\mathcal{H}_{(BC)^R}$ in the Hilbert space $\mathcal{H}_{BC}$, thus the mixed quantum state $\rho_{ABC}=|\psi\rangle_{A(BC)^L}\otimes|\psi\rangle_{A(BC)^R}$, the relation $\mathcal{Q}_A(\rho_{A:BC})=\mathcal{Q}_A(|\psi\rangle_{A(BC)^L})=H(\rho_A)$ holds.[49] It will arrive at the relation (21). The proof is completed.

It is worth noting that, from this relation, one can find that the sum of the quantum discord $\mathcal{Q}_A(\rho_{AB})$ and $\mathcal{Q}_A(\rho_{AC})$ of the tripartite quantum state ρ_{ABC} will be restricted by the $\mathcal{Q}_A(\rho_{A:BC})$ and the parameter $\tilde{\delta}$ which contains the uncertainties of the measurements X and Y and the incompatibility of the measurements $\mathcal{B}_{\mathcal{L}}$ excluding the effects of quantum entanglement $H(A|B)$ and $H(A|C)$. However, in Ref. [50], the shareability of quantum discord among different parties of a composite system is given by $\mathcal{Q}_A(\rho_{AB})+\mathcal{Q}_A(\rho_{AC})\leq\delta+\mathcal{Q}_A(\rho_{A:BC})$ with $\delta=\mathcal{B}_{\mathcal{L}}+H(X|B)+H(Y|C)-H(A|B)$. It is evident that $\tilde{\delta}\leq\delta$, thus the bound of Corollary 2 is tighter than the result of Ref. [50]. Notably, the shareability of quantum discord among different parties of the tripartite quantum system is consistent with the recent results in Ref. [51].

From the lower bounds of the multiple quantum memory-assisted information exclusion principle for multiple measurements, it can be found that the complementarity of uncertainty can be described as $\mathcal{C}_{\mathcal{U}}(\Pi)=\sum_{i=1}^{N}H(\Pi_i|B_i)$ while the complementarity of information can be described as $\mathcal{C}_{\mathcal{I}}(\Pi)=\sum_{i=1}^{N}I(\Pi_i:B_i)$. Thus, there exists a complementarity relation given in the following.

Corollary 2 The relation between the complementarity of uncertainty and the complementarity of information

$$\mathcal{C}_{\mathcal{U}}(\Pi)+\mathcal{C}_{\mathcal{I}}(\Pi)\geq\mathcal{B}_{\mathcal{L}}+(N-1)H(\rho_A).\tag{24}$$

Proof The proof is evident. We rearrange the elements of inequality in Theorem 1, and it will arrive at Eq. (24).

Next, we derive the tripartite quantum memory-assisted information exclusion principle for multiple measurements. For tripartite quantum system ρ_{ABC}, the multiple measurements N can be randomly divided into two parts N_B and N_C, which satisfy $N_B+N_C=N$. Suppose that N_B measurements are performed on subsystem A, Alice tells her choice to Bob, B. In contrast, N_C measurements are performed on subsystem A; Alice announces her choice to Charlie, C. Subsequently, Bob and Charlie try to quantify the information gained from Alice's measurement outcomes. There exists the following information exclusion principle.

Theorem 2 For any tripartite quantum state ρ_{ABC} measured by the set of observables $\{\Pi_i,i=1,2,\ldots,N\}$, the quantum memory-assisted information exclusion principle for multiple measurements will be given by

$$\tilde{\mathcal{L}}_{\mathcal{B}}\leq\sum_{\Xi=B,C}\sum_{i=1}^{N_\Xi}I(\Pi_i:\Xi)\leq\tilde{\mathcal{U}}_{\mathcal{B}},\tag{25}$$

where the lower and upper bounds are, respectively,

$$\begin{aligned}\tilde{\mathcal{L}}_{\mathcal{B}}&=\mathcal{B}_{\mathcal{L}}+(N-1)H(\rho_A)-\sum_{\Xi=B,C}\sum_{i=1}^{N_\Xi}H(\Pi_i|\Xi),\\\tilde{\mathcal{U}}_{\mathcal{B}}&=\log_2 d^N+H(\rho_A)-\mathcal{B}_{\mathcal{L}}-\sum_{\Xi=B,C}N_\Xi H(A|\Xi)\\&\quad-N_\Xi\mathcal{Q}_A(\rho_{A\Xi}),\end{aligned}$$

with $(\Xi,\tilde{\Xi})$ standing for (B,C) or (C,B). It is equivalent to

$$\begin{aligned}&\sum_{i=1}^{N_B}H(\Pi_i|B)+\sum_{i=1}^{N_C}H(\Pi_i|C)\\&\geq\mathcal{B}_{\mathcal{L}}+\sum_{\Xi=B,C}N_\Xi H(A|\Xi)+N_\Xi\mathcal{Q}_A(\rho_{A\Xi})-H(\rho_A).\end{aligned}\tag{26}$$

Proof For the lower bounds of the tripartite quantum memory-assisted information exclusion principle for multiple measurements, we substitute the relation (15) into the uncertainties of the multiple measurements in the presence of the multiple quantum memory, it will arrive at

$$\begin{aligned}\sum_{\Xi=B,C}\sum_{i=1}^{N_\Xi}H(\Pi_i|\Xi)&=\sum_{i=1}^{N}H(\Pi_i)-\sum_{\Xi=B,C}\sum_{i=1}^{N_\Xi}I(\Pi_i:\Xi)\\&\geq\mathcal{B}_{\mathcal{L}}+(N-1)H(\rho_A)-\sum_{\Xi=B,C}\sum_{i=1}^{N_\Xi}I(\Pi_i:\Xi),\end{aligned}\tag{27}$$

where $(\Xi,\tilde{\Xi})$ stands for (B,C) or (C,B). Rearranging the relation (27), we have

$$\sum_{\Xi=B,C}\sum_{i=1}^{N_\Xi} I(\Pi_i:\Xi) \geq \mathcal{B}_\mathcal{L} + (N-1)H(\rho_A) + \sum_{\Xi=B,C}\sum_{i=1}^{N_\Xi} H(\Pi_i|\Xi). \tag{28}$$

For the upper bounds of the tripartite quantum memory assisted information exclusion principle for multiple measurements, we start from the uncertainties of the multiple measurements in the presence of the tripartite quantum memory in terms of relation (15),

$$\begin{aligned}
&\sum_{i=1}^{N_B} H(\Pi_i|B) + \sum_{j=1}^{N_C} H(\Pi_j|C) \\
&= \sum_{i=1}^{N} H(\Pi_i) - \sum_{\Xi=B,C}\sum_{i=1}^{N_\Xi} I(\Pi_i:\Xi) \\
&\geq \sum_{i=1}^{N} H(\Pi_i) - \sum_{\Xi=B,C} N_\Xi \mathcal{J}_A(\rho_{A\Xi}) \\
&\geq \mathcal{B}_\mathcal{L} + (N-1)H(\rho_A) - \sum_{\Xi=B,C} N_\Xi \mathcal{J}_A(\rho_{A\Xi}) \\
&= \mathcal{B}_\mathcal{L} + (N_\Xi - 1)[H(A|\Xi) + I(A:\Xi)] - N_\Xi \mathcal{J}_A(\rho_{A\Xi}) \\
&\quad + N_{\tilde{\Xi}}[H(A|\tilde{\Xi}) + I(A:\tilde{\Xi})] - N_{\tilde{\Xi}} \mathcal{J}_A(\rho_{A\tilde{\Xi}}) \\
&= \mathcal{B}_\mathcal{L} + \sum_{\Xi=B,C} N_\Xi H(A|\Xi) + N_\Xi \mathcal{Q}_A(\rho_{A\Xi}) - H(\rho_A).
\end{aligned} \tag{29}$$

Regrouping the terms yields

$$\begin{aligned}
\sum_{\Xi=B,C}\sum_{i=1}^{N_\Xi} I(\Pi_i:\Xi) \leq &\sum_{i=1}^{N} H(\Pi_i) + H(\rho_A) - \mathcal{B}_\mathcal{L} \\
&- \sum_{\Xi=B,C} N_\Xi H(A|\Xi) - N_\Xi \mathcal{Q}_A(\rho_{A\Xi}).
\end{aligned} \tag{30}$$

The proof is completed.

One will find this lower bound is similar to the result in Theorem 1; that is, the incompatibility of the measurements $\mathcal{B}_\mathcal{L}$, the quantum discord between the measured subsystem and each unmeasured subsystem, and the conditional quantum entropy in the lower bound also exist. It is evident that the quantum measurements performed on subsystem A bring the uncertainties of the measurements while the quantum correlation of the quantum system is destroyed simultaneously. This result could reveal that the information gain corresponding to the measurement can be maximized only at the expense of the information gains corresponding to complementary observables and quantum correlations. In other words, the quantum correlation within the composite quantum system is also restricted by the uncertainties of the measurements. According to this division and combination idea, it can be extended to multiple quantum states that the number of measurements is more than the number of quantum memory-assisted systems.

3. Applications for three projective measurements

As applications, in order to verify the effectiveness of the information exclusion principle, we consider three two-dimensional observables $\Pi_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\Pi_2 = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$, $\Pi_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$ measured on the generalized W state which is given by

$$\begin{aligned}
|\psi\rangle_{ABCD} = &\sin\theta\cos\phi|1000\rangle + \sin\theta\sin\phi|0100\rangle \\
&+ \cos\theta\cos\varphi|0010\rangle + \cos\theta\sin\varphi|0001\rangle,
\end{aligned} \tag{31}$$

where $\theta \in [0,2\pi]$, $\phi \in [0,\pi]$, and $\varphi \in [0,\pi]$.

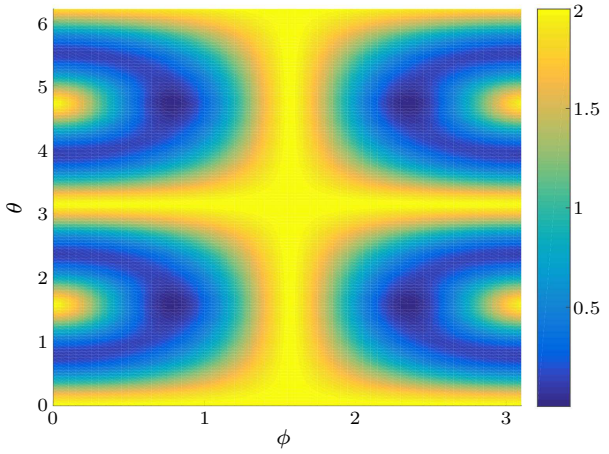


Fig. 1. The difference between the upper bound $\mathcal{U}_\mathcal{B}$ and the complementarity of information $\mathcal{C}_\mathcal{I}(\Pi)$ versus the parameters θ and ϕ of the quantum state.

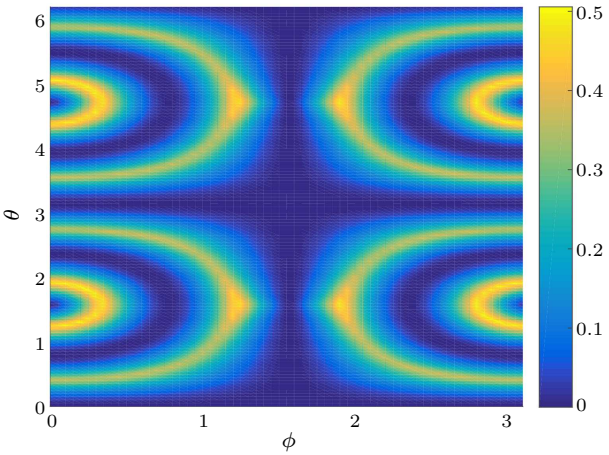


Fig. 2. The difference between the complementarity of information $\mathcal{C}_\mathcal{I}(\Pi)$ and the lower bound $\mathcal{L}_\mathcal{B}$ versus the parameters θ and ϕ of the quantum state.

Figure 1 shows the difference between the complementarity of information of the three two-dimensional observables and the upper bound given by Theorem 1 versus the parameters θ and ϕ of the quantum state with $\varphi = \pi/3$. Figure 2 similarly shows the difference between the complementarity of information and the lower bound given by Theorem 1. The differences are always positive with the increasing parameters θ and ϕ. It is illustrated that, for the generalized W state, the information gained from the observables are well bounded by

the effective upper and lower bounds. Namely, the information gain corresponding to the selected observables Π_i $(i = 1,2,3)$ can be maximized at the expense of the information carried by complementary observables, quantum discord, and the conditional quantum entropy, which are quantities between the observed subsystem and the quantum memory-assisted subsystems. Figure 3 shows the relationship between the complementarity of uncertainty and the complementarity of information versus θ for the quantum state with $\phi = \pi/8$ and $\varphi = \pi/3$. It indicates that the lower bound depends on the complementarity of the observables and the conditional quantum entropy of subsystem ρ_A and shows that the limitations on our ability to simultaneously predict the measurement outcomes and information gained from non-commuting observables.

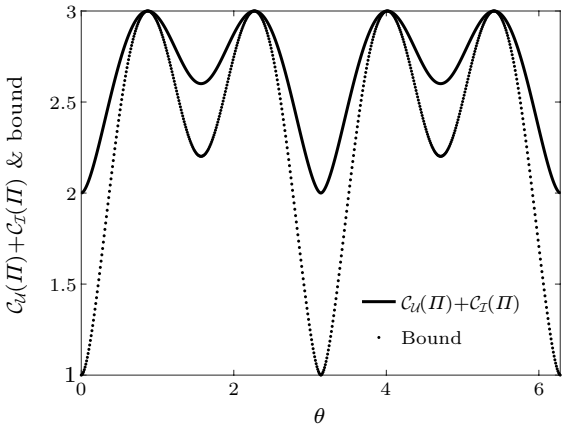


Fig. 3. The sum of the complementarity of uncertainty $\mathcal{C}_{\mathcal{U}}(\Pi)$ and the complementarity of information $\mathcal{C}_{\mathcal{I}}(\Pi)$ with the lower bound versus the parameter θ of the quantum state.

4. Conclusion

As the fundamental features of quantum information theory, we mainly consider the multiple quantum memory-assisted information exclusion principle for multiple measurements. The nontrivial upper bound can be determined by the complementary of measurements, quantum discord, and the conditional quantum entropy, which are quantities between the observed subsystem and the quantum memory-assisted subsystems. They describe the effects of quantum correlation (including quantum discord and quantum entanglement) on the information exclusion principle. They imply that the information gain corresponding to any selected observables can be maximized only at the expense of the information gains corresponding to the observables' complementary and quantum correlations. In addition, there exists a relationship between the complementarity of uncertainty and the complementarity of information which describes the limitations on our ability to simultaneously predict the measurement outcomes and information gained from non-commuting observables in quantum theory. What's more, the shareability of quantum discord among different parties of the composite quantum system is restricted by the complementarity of uncertainty. As a consequence, the results shed new light on the quantum entropic uncertainty relation, information exclusion principle, and quantum correlation.

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